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REPRESENTING POSITIVE INTEGERS BY BINARY FORMS WITH AN EVEN DISCRIMINANT

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Abstract. Using the formulae for the average number of representations we describe positive integers which can be represented by the quadratic forms with an even discriminant belonging to multi-class genera.

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1 Introduction. Let $r(n; f)$ denote the number of representations of a positive integer by a positive quadratic form $f = ax^2+bxy+cy^2$ with the discriminant $d = b^2-4ac$.

In [1], we worked out a full solution of defining singular series for arbitrary binary quadratic forms and proved that half of a generalized singular series that corresponds to a positive binary form is equal to the average number of representations of a positive integer by the genus containing this form. Moreover, convenient formulas are obtained for calculating of values of this sum. Using these formulas, in [2], we showed that the problem of obtaining formulas for the number of representations of positive integers by positive binary forms belonging to multi-class genera can be easily reduced to the case of one-class genera. This made it possible in [3], to characterize positive integers which can be represented by binary forms with an odd discriminant belonging to multi-class genera. In this paper the similar results are obtained in the case of binary forms with even discriminant belonging to multi-class genera. This is all motivated by a problem in number theory that dates back at least to Fermat: for a given n, characterizing all the primes which can be written $p = x^2 + ny^2$ for some integers x, y. Euler studied the problem extensively, and was able to solve it for $n = 1, 2, 3, 4$ (see, [4]). For example, he proved the following theorem of Fermat: $p = x^2 + 4y^2$ for some integers x, y if and only if $p \equiv 1 \pmod 4$. The similar results in the case of $n = 5, 6, 7, 10, 13, 15, 21, 22, 30$ are given in [4] and [5]. All these binary forms belong to the one-class genera.

2 Basic results. The set of binary forms with discriminant 144 forms two genera, each consisting of two classes with reduced forms

$$f_1 = x^2 + 36y^2, \quad f_2 = 4x^2 + 9y^2$$

and

$$f_3 = 5x^2 + 4xy + 8y^2, \quad f_4 = 5x^2 - 4xy + 8y^2.$$

It follows from the corollary 1 of the paper [2], that for any number $n = 2^\alpha p$ (α is a positive integer, p is a prime number), we have

$$r\,(n; f_1) = r\,(n; f_2)$$

$$= \begin{cases} \left(1 + \left(\frac{1}{p}\right)\right)\left(1 + (1)^\alpha \left(\frac{p}{3}\right)\right) & \text{if } \alpha \neq 1,\ 3\dagger p, \\ 0 & \text{if } \alpha = 1, \text{ or } p = 3. \end{cases}$$

It follows from these formulas, that for any integer $n = 2^\alpha p$ ($\alpha \geq 1$, $\alpha \in N$, p is a prime number, $p > 2$), $r(2^\alpha p; f_1) \neq 0$ and $r(2^\alpha p; f_2) \neq 0$ if and only if $\left(\frac{-1}{p}\right) = 1$, $(1)^\alpha \left(\frac{p}{3}\right) = 1$ and $p \neq 3$.

The argument above yields

Theorem 1. *For a given prime p and positive integer α a number $2^\alpha p$ is represented by the form f_1 and f_2 if and only if $\alpha \geq 2$, $2|\alpha$, $p \equiv 1 (\text{mod } 12)$ or $\alpha \geq 2$, $2\dagger\alpha$, $p \equiv 5$ (mod 12).*

It follows from the corollary 1 of the paper [2], that for any number $n = 2^\alpha p$ (α is a non-negative integer, p is a prime number), we have

$$r\,(n; f_3) = r\,(n; f_4)$$

$$= \begin{cases} \frac{1}{2}\left(1 + \left(\frac{1}{p}\right)\right)\left(1(1)^\alpha \left(\frac{p}{3}\right)\right) & \text{if } \alpha \neq 1,\ \ p \neq 3, \\ 0 & \text{if } \alpha = 1, \text{ or } p = 3. \end{cases}$$

It follows from these formulas, that for any integer $n = 2^\alpha p$ (α is a non-negative integer, p is a prime number), $r(2^\alpha p; f_3) \neq 0$ and $r(2^\alpha p; f_4) \neq 0$ if and only if $\left(\frac{-1}{p}\right) = 1$, $(1)^\alpha \left(\frac{p}{3}\right) = -1$ and $p \neq 3$.

The argument above yields

Theorem 2. *For a given prime number p and non-negative integer α, $n = 2^\alpha p$ is represented by the form f_3 and f_4 if and only if $\alpha \neq 1$, $p \equiv 1$ (mod 12) or $2|\alpha$, $p \equiv 5$ (mod 12).*

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