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Rheological Modeling in Solid Mechanics from the Beginning up to Now

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Rheological models are widely used in material modelling and simulation. The first steps were done by Bingham, Reiner and others in the first half of the 20th century. With the increasing use of plastics starting in the fifties of the last century, the method of rheological modelling became more and more popular. The main focus was on the phenomenological description of the material behaviour. However, in the last years, rheological models were used also in the case of microstructural models. The contribution at hand presents a brief introduction to the method of rheological models and some new applications.

Keywords: Rheological models, Material behaviour, Continuum mechanics.

AMS Subject Classification: 74Q15, 74E10, 74E25, 74E30, 74C10

1. Introduction

Figure 1. Vladimir A. Palmov.

This paper is dedicated to Vladimir Alexandrovich Palmov (Fig. 1), the great scientist in the field of continuum mechanics, who was the academic teacher of the first author. Prof. Palmov was born on July 7th, 1934 in Batumi (Soviet Union, now Georgia), where his father served as a military officer. Finally, his family moved to Leningrad (now St. Petersburg), and he finished the secondary school. Since 1952, his life was connected with the Leningrad Polytechnic Institute (now Peter the Great State Polytechnic University), as he was a student of the Faculty of Physics and Mechanics. In 1958, he graduated from the Department of Mechanics and Control Processes (specialization in Dynamics and Strength of Machines), he obtained the degree of Candidate of Science (under supervision of Prof. Anatoly I. Lurie) in Physics and Mathematics in 1963 and the degree of Doctor Science in Physics and Mathematics in 1972. After

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that, he was appointed as a professor in 1974 and since 1974 he was the Head of the Department of Mechanics and Control Processes (he held this position up to 2013). He died on October 15th, 2018 in St. Petersburg (Russian Federation).

His research focus was on mechanics of deformable solids including Cosserat continua, the direct approach in theory of plates, the theory of vibrations of inelastic bodies, the theory of constitutive equations, and general problems of dynamics and strength of machines. In his first fundamental monograph (Russian edition 1976 [1], English translation 1998 [2]), he presented the method of rheological modelling with applications to microplastic materials, which was in that time absolutely new. A special issue of the ZAMM (Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik, 2009, Vol. 89, No. 4) containing papers dedicated to Palmov on the occasion of his 75th birthday was published (edited by H. Altenbach, A. Belyaev and V. Eremeyev). His scientific testament is his last book [3], see Fig. 2.

Figure 2. Palmov's last book.

2. Foundations of Rheology

The term *rheology* comes from ancient Greek, meaning the theory (λογοα) of flow (ρεω). In this sense, it is related to fluids (liquids or gases), but also to "soft" solids or solids under conditions in which they respond with plastic flow (moderate temperature, loads beyond the yield stress) or creep (elevated temperature, loads below the yield stress) to an applied load. The formula "panta rhei" (ancient Greek πανεα ρει, which means everything flows) is an aphorism traced back to the Greek philosopher Heraklit, suggested by Plato (in the dialogue Kratylos), but literally appearing for the first time in the late ancient Neo-Platonist Simplikios to characterize the Heraclitic doctrine [4].

One of the first investigations in the field of rheology was related to the design of water clocks in the 16th century AD. In that time, it was obvious that one had to take into account the temperature dependency of the viscosity of water. Deeper scientific studies of rheological questions started in the 17th century, in particular by Isaac Newton, who defined the viscosity for liquids showing a proportional dependency between shear stresses and the shear strain rates, and by Robert Hooke, who established the law of linear elasticity (Hooke's law). A brief historical survey is presented in [5, 6].

From the 17th century, we distinguish several classes and subclasses of rheological models. The models are at present:

- ideal materials with the subclasses
  - rigid or Euclidian solids (no deformation), which were also in the focus of Isaac Newton.
  - elastic solids (linear and non-linear), firstly studied by Robert Hooke and Robert Boyle, for which was introduced a material parameter characterizing the individual response of a material on a load (Young's modulus, probably introduced

  by Giordano Riccati 25 years before Thomas Young and discussed also by Leonhard Euler 80 years earlier) and finally a full set of the classical (small strain) elasticity equations assuming isotropy and linear behaviour was presented by Augustin-Louis Cauchy, Charles Augustin de Coulomb, Claude Louis Marie Henri Navier, Siméon Denis Poisson, among others.
  - inviscid or Pascalian fluids (pressure in the fluid is the same in all directions or no resistance to flow). The theory for such fluids without viscosity was developed by Leonhard Euler, Daniel Bernoulli and Giovanni Battista Venturi.
  - Newtonian fluids which have been discussed not only by Isaac Newton, but also by Gotthilf Hagen, Jean Léonard Marie Poiseuille, Maurice Marie Alfred Couette, Claude Louis Marie Henri Navier, George Gabriel Stokes, and others. The last two presented a first set of three-dimensional equations for linear-viscous Newtonian fluids describing experimental data in a proper manner.
- linear viscoelasticity, which is related to material phenomena like creep and relaxation. Such material behaviour cannot be represented by elastic or viscous models alone. Various models have been suggested by James Clerk Maxwell, William Thomson (later Lord Kelvin), and John Henry Poynting. Finally, Ludwig Boltzmann introduced the superposition principle for this material behaviour.
- generalized Newtonian materials, suggested for materials with a more complex behaviour, which was discussed by Theodore Schwedoff (colloids) or Eugene Cook Bingham (paints).
- non-linear viscosity, etc.

It is obvious that many scientists contribute important results to rheology such that many rheological models have been named after these researchers. Examples for this are the Hookean model, the Newtonian model, the Prandtl model, the Kelvin-Voigt model, the Maxwell model, the Schwedoff model, the Bingham model, the Burgers model, among others [7].

3. The Development of Rheology as a Scientific Branch

Using the term "rheology" for the science that deals with the flow and deformation behaviour of matter was only established in the late 1920s by Eugene Cook Bingham, who himself worked in the field of plasticity theory, together with Markus Reiner [8]. The chemist Eugene Cook Bingham was convinced of the necessity of a branch of mechanics/physics that deals with rheological questions and thus has certain intersections with chemistry and engineering. In a discussion with Markus Reiner, he mentioned [8]: *Here you, a civil engineer, and I, a chemist, work together on common problems. With the development of colloid chemistry, such a situation will appear more and more often. We must therefore establish a branch of physics that deals with such problems.* On August 29, 1929, he founded the Society of Rheology in Columbus, Ohio with others [6].

Establishing the new scientific branch, the aim of research was formulated: Rheology is the study of the deformation and flow of matter. However, there were strong restrictions at the beginning. For example, the flow of electrons and heat were excluded by agreement. Later, various models were developed. These models make it possible to describe qualitatively and quantitatively, categorize, and predict different flow and deformation behaviour. In addition to rheology itself, they are used in many areas of technology and science, e.g., in materials science, in geology and in food technology.

From the statements above it is clear that rheology and continuum mechanics are in a close interaction. The constitutive equations suggested in continuum mechanics contain parameters which should be identified. One possible way for establishing constitutive equations is their representation by combining rheological models. In addition, within rheology, experimental methods for characterizing the flow of the materials are established. In [5], the following necessary basics of continuum mechanics for rheology are presented briefly:

(1) conservation of mass,
(2) stress concept,
(3) symmetry of stress tensor,
(4) stress equations of motion, and
(5) energy conservation.

These statement should be slightly extended. First, it is better to use balance equations instead of conservation laws. They are more general and allow to include the interaction from the environment. Second, taking into account the progress with respect to generalized continua, see for example [9–12], the symmetry of the stress tensor cannot be guaranteed. However, the assumption that the symmetry condition for the stress tensor is valid simplifies the derivation of constitutive equations. Note that this assumption plays the role of a restriction, and we can connect rheology with continua based on the assumption that the stress tensor must be symmetrical in a simple manner. Otherwise, even if we can establish constitutive equations for continua with symmetric and antisymmetric stress tensors, the identification effort for the additional constitutive parameters is increasing dramatically. Third, the second law of thermodynamics should be taken into account – it allows to distinguish physical admissible and not admissible constitutive equations. However, we should find the answer to the question: *How can one reflect the individual response of a material on acting loading(s)?*

At present, the rheological modelling is successfully applied to many practical problems, particularly in mechanical and civil engineering. The reason for that is that the method takes a position between the deductive and inductive approach in modelling the constitutive behaviour [13]. In addition, it is straightforward to combine rheological elements by connecting them in parallel or in series to obtain more complex models. Furthermore, the physical admissibility of complex rheological models is guaranteed as long as the individual rheological elements are physically admissible. Further simplifications result from the following axioms of rheology [7]:

(1) Under the action of hydrostatic pressure, all materials behave in the same manner as perfectly elastic body.
(2) Each matter features all rheological properties, but in different degrees. Note that the main rheological properties include elasticity, viscosity, plasticity, etc.
(3) There is a hierarchy of ideal bodies such that the rheological equation of the simpler body, i.e., a body lower in the hierarchy, can be derived by setting one or other of the constants of the rheological equation of the less simple body, i.e., a body higher in the hierarchy, equal to zero.

Finally, it is worthwhile noting that one-dimensional rheological models can be generalized to three dimensions in many cases.

4. Palmov's Method of Rheological Modelling the Simplest Three-Dimensional Case

In the monograph [1], Palmov presented his ideas concerning rheological modelling for the first time. Let us briefly introduce his basic ideas.

As a start, let us assume isotropy for the material behaviour. In addition, we consider the split of the stress tensor $\boldsymbol{\sigma}$ into its hydrostatic part $\sigma_\mathrm{m}\boldsymbol{I}$ and its deviatoric part $\boldsymbol{s} = \boldsymbol{\sigma} - \sigma_\mathrm{m}\boldsymbol{I}$ with the second-order unit tensor $\boldsymbol{I}$ and the hydrostatic stress $\sigma_\mathrm{m} = \frac{1}{3}\mathrm{tr}\,\boldsymbol{\sigma}$. The same is done for the strain tensor $\boldsymbol{\varepsilon}$, whose deviatoric part is denoted by $\boldsymbol{e} = \boldsymbol{\varepsilon} - \frac{1}{3}\epsilon\boldsymbol{I}$ with the volumetric strain $\epsilon = \mathrm{tr}\,\boldsymbol{\varepsilon}$. Let us now formulate the constitutive equations for the stress deviator, the free energy F, and the entropy S for the rheological element α as functions of the temperature Θ, the temperature gradient $\boldsymbol{\nabla}\Theta$, the volumetric strain ϵ, and the strain deviator $\boldsymbol{e}$:

$$\boldsymbol{s}_\alpha = \boldsymbol{s}_\alpha(\Theta, \boldsymbol{\nabla}\Theta, \epsilon, \boldsymbol{e}), \qquad F_\alpha = F_\alpha(\Theta, \boldsymbol{e}), \qquad S_\alpha = S_\alpha(\Theta, \boldsymbol{e}), \tag{1}$$

whereby the subscript α denotes the corresponding rheological element. The connections of n rheological elements are given as:

- connection in parallel:

$$\boldsymbol{s} = \sum_{\alpha=1}^{n} \boldsymbol{s}_\alpha, \quad \boldsymbol{e} = \boldsymbol{e}_1 = \ldots = \boldsymbol{e}_\alpha = \ldots = \boldsymbol{e}_n, \quad F = \sum_{\alpha=1}^{n} F_\alpha, \quad S = \sum_{\alpha=1}^{n} S_\alpha, \tag{2}$$

- connection in series:

$$\boldsymbol{s} = \boldsymbol{s}_1 = \ldots = \boldsymbol{s}_\alpha = \ldots = \boldsymbol{s}_n, \quad \boldsymbol{e} = \sum_{\alpha=1}^{n} \boldsymbol{e}_\alpha, \quad F = \sum_{\alpha=1}^{n} F_\alpha, \quad S = \sum_{\alpha=1}^{n} S_\alpha. \tag{3}$$

Now the basic elements can be introduced:

- Hookean element for linear elasticity with the shear modulus μ as material parameter

$$\boldsymbol{s} = 2\mu\boldsymbol{e}, \tag{4}$$

- Newtonian element for linear viscosity with the viscosity coefficient ν (the dot indicates the partial derivative with respect to time)

$$\boldsymbol{s} = 2\nu\dot{\boldsymbol{e}}, \tag{5}$$

- and the Prandtl-St. Venant element for plasticity with the yield stress σ_y as material parameter:

$$\begin{cases} \dot{\boldsymbol{e}} = \boldsymbol{0} & \text{if } N(\boldsymbol{\sigma}) < \sigma_\mathrm{y}, \\ \dot{\boldsymbol{e}} = \dfrac{1}{\lambda}\boldsymbol{s} & \text{if } N(\boldsymbol{\sigma}) = \sigma_\mathrm{y}, \end{cases} \tag{6}$$

  where $N(\boldsymbol{\sigma})$ is the norm of the stress tensor and the variable λ is referred to as "plasticity factor".

- In addition, for the remaining part, i.e., the relation between the hydrostatic stress $\sigma_{\rm m} = \frac{1}{3}\mathrm{tr}\,\boldsymbol{\sigma}$ and the volumetric strain $\epsilon = \mathrm{tr}\,\boldsymbol{\varepsilon}$ (with the stress tensor $\boldsymbol{\sigma}$ and the strain tensor $\boldsymbol{\varepsilon}$), we make use of the following constitutive equation for pure elastic behaviour, where the parameter K is the bulk modulus:

$$\sigma_{\rm m} = K\epsilon. \tag{7}$$

By connecting several of the basic elements, we can develop complex models, such as:

- viscoelastic models
  - Kelvin-Voigt model: one elastic and on viscous element in parallel,
  - Maxwell model: one elastic and one viscous element in series,
  - Poynting model: a Maxwell model in parallel with a second elastic element,
  - generalized Kelvin-Voigt model: n Kelvin-Voigt elements in series,
  - generalized Maxwell model: n Maxwell elements in parallel,
- plastic models
  - Prandtl model: one elastic and one plastic model in series,
  - Bingham model: one viscous and one plastic element in parallel with an elastic element in series,
  - generalized Prandtl model: n Prandtl models in parallel,
- etc.

Further discussions concerning Palmov's approach with respect to large strains, for example, are given in [14, 15].

Until now, there are some open questions:

- One of the basic assumptions is isotropy. The split of the stress tensor and the strain tensor into volumetric and deviatoric parts is unique. How can we formulate the anisotropic constitutive equations?
- The volumetric part in the constitutive equations is assumed to be purely elastic. There are experimental data indicating that this assumption is not always appropriate, see, for example, [16].
- For rheological modelling, one can use both the Langrangian or the Eulerian description. Which approach is the better choice?

Here we need further research efforts to give proper answers. For example, in many textbooks on continuum mechanics, it is written that the Langrangian description should be used for solids, while the Eulerian approach should be chosen for fluids. However, this is questionable since the plastic flow (plastic behaviour of a material) is similar to the behaviour of fluids (see [17]).

5. Two-Dimensional Rheological Modelling

The first author of this paper has extended Palmov's approach to two-dimensional continua in Ref. [18]. In this paper, the following items were established:

- governing equations with respect to the direct plate theory (see, for example, Ref. [19]),
- basic elements, and
- the statement that plasticity cannot be formulated in the same way as done for

elasticity and viscosity.

Within the framework of a Zhilin-type theory [20], let us introduce a Cosserat plate. The main variables are

- the stress resultants, i.e., the transverse force vector $\boldsymbol{F}$ and the moment tensor $\boldsymbol{M}$,
- the strains, i.e., the transverse shear strain vector $\boldsymbol{\gamma}$ and the tensor of the bending and torsional strains $\boldsymbol{\kappa}$, and
- an energetic variable, i.e., the free energy H.

A new variable can be introduced following Palmov's suggestion in Ref. [21]:

$$\boldsymbol{G} = \boldsymbol{M} \times \boldsymbol{n}, \tag{8}$$

whereby $\boldsymbol{G}$ is referred to as “polar moment tensor” [22]. It can also be obtained based on the stress tensor:

$$\boldsymbol{G} = \langle \boldsymbol{a}_1 \cdot \boldsymbol{\sigma} z \cdot \boldsymbol{a}_1 \rangle , \tag{9}$$

where $\boldsymbol{\sigma}$ is the classical symmetric stress tensor, $\boldsymbol{a}_1 = \boldsymbol{e}_\alpha \boldsymbol{e}_\alpha$ is the first metric tensor (note that we consider the two-dimensional orthonormal coordinate system $\boldsymbol{e}_\alpha,\ \alpha = \{1, 2\}$, and make use of Einstein's sum convention), z is the coordinate in the transverse direction of the plate, and the brackets $\langle \ldots \rangle$ denote the integration over the thickness of the plate-like body. It is obvious that $\boldsymbol{G} = \boldsymbol{G}^{\mathrm{T}}$.

For the introduced moment tensors, the following relation is valid:

$$\boldsymbol{M}^{\mathrm{T}} : \boldsymbol{\kappa} = \boldsymbol{M} : \boldsymbol{\kappa}^{\mathrm{T}} = \boldsymbol{G} : \boldsymbol{\mu} \tag{10}$$

with $\boldsymbol{\mu} = \boldsymbol{\kappa} \times \boldsymbol{n}$. Note that the variable $\boldsymbol{\mu}$ is introduced in analogy to the transformation of the moment tensor $\boldsymbol{M}$ into the polar moment tensor $\boldsymbol{G}$, cf. Eq. (8).

In the following sections, we discuss some basic two-dimensional rheological elements.

5.1. *Elastic Cosserat Plate*

Let us introduce the Helmholtz free energy H as a function of the kinematic variables and calculate the time-derivative:

$$\rho \dot{H} = \frac{\partial \rho H}{\partial \boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}} + \frac{\partial \rho H}{\partial \boldsymbol{\mu}} : \dot{\boldsymbol{\mu}}. \tag{11}$$

Based on the free energy, the vector of the transverse forces $\boldsymbol{F}$ and the polar moment tensor $\boldsymbol{G}$ are obtained as the following derivatives:

$$\boldsymbol{F} = \frac{\partial \rho H}{\partial \boldsymbol{\gamma}}, \qquad \boldsymbol{G} = \frac{\partial \rho H}{\partial \boldsymbol{\mu}}. \tag{12}$$

Here, we limit our considerations to the quadratic form of the Helmholtz free energy:

$$\rho H = \frac{1}{2} \boldsymbol{\gamma} \cdot \boldsymbol{\Gamma} \cdot \boldsymbol{\gamma} + \frac{1}{2} \boldsymbol{\mu} : \boldsymbol{\mathcal{C}} : \boldsymbol{\mu} \tag{13}$$

with the transverse shear stiffness tensor $\boldsymbol{\Gamma}$ and the out-of-plane stiffness tensor $\boldsymbol{\mathcal{C}}$. The first variable is a second rank tensor, while the out-of-plane stiffness tensor is of fourth rank:

$$\boldsymbol{\Gamma} = \Gamma_0 \boldsymbol{a}_1, \quad \boldsymbol{\mathcal{C}} = C_1 \boldsymbol{cc} + C_2(\boldsymbol{a}_2\boldsymbol{a}_2 + \boldsymbol{a}_4\boldsymbol{a}_4), \tag{14}$$

where in the case of the two-dimensional orthonormal coordinate system $\boldsymbol{e}_\alpha,\ \alpha = \{1, 2\}$, the following notations are used:

$$\begin{aligned} \boldsymbol{c} &= \boldsymbol{e}_1\boldsymbol{e}_2 - \boldsymbol{e}_2\boldsymbol{e}_1 = \boldsymbol{a}_3, \\ \boldsymbol{a}_2 &= \boldsymbol{e}_1\boldsymbol{e}_1 - \boldsymbol{e}_2\boldsymbol{e}_2, \\ \boldsymbol{a}_4 &= \boldsymbol{e}_1\boldsymbol{e}_2 + \boldsymbol{e}_2\boldsymbol{e}_1. \end{aligned}$$

Taking into account Eqs. (11)–(14), we obtain

$$\boldsymbol{F} = \Gamma_0 \boldsymbol{\gamma}, \quad \boldsymbol{G} = (C_1 - C_2)(\boldsymbol{a}_1 \boldsymbol{:} \boldsymbol{\mu})\boldsymbol{a}_1 + 2C_2\boldsymbol{\mu}. \tag{15}$$

Finally, assuming linear elastic isotropic material behaviour and solving the boundary-value problems for the two-dimensional continuum and the plate-like body [23], one obtains the stiffness parameters

$$C_2 = \frac{Gh^3}{12}, \quad C_1 = C_2\frac{1+\nu}{1-\nu}, \quad \Gamma_0 = \left(\frac{\pi}{h}\right)^2 C_2. \tag{16}$$

5.2. *Viscous Cosserat Plate*

Now we assume the following expression for the Helmholtz free energy

$$\rho H = \rho H(\dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{\mu}}). \tag{17}$$

The force vector and the polar moment tensor can be estimated as

$$\boldsymbol{F} = \tilde{\Gamma}_0 \dot{\boldsymbol{\gamma}}, \quad \boldsymbol{G} = (\tilde{C}_1 - \tilde{C}_2)(\boldsymbol{a}_1 \boldsymbol{:} \dot{\boldsymbol{\mu}})\boldsymbol{a}_1 + 2\tilde{C}_2\dot{\boldsymbol{\mu}}, \tag{18}$$

where $\tilde{\Gamma}_0, \tilde{C}_1$, and $\tilde{C}_2$ are again material parameters of the two-dimensional continuum. In this case, the time derivative of the Helmholtz free energy is

$$\rho \dot{H} = \frac{\partial \rho H}{\partial \dot{\boldsymbol{\gamma}}} \boldsymbol{\cdot} \ddot{\boldsymbol{\gamma}} + \frac{\partial \rho H}{\partial \dot{\boldsymbol{\mu}}} \boldsymbol{:} \ddot{\boldsymbol{\mu}}. \tag{19}$$

Combining Eqs. (17)–(19) results in

$$\frac{1}{2}\tilde{\Gamma}_0 \dot{\boldsymbol{\gamma}} \boldsymbol{\cdot} \dot{\boldsymbol{\gamma}} + \frac{1}{2}(\tilde{C}_1 - \tilde{C}_2)(\boldsymbol{a}_1 \boldsymbol{:} \dot{\boldsymbol{\mu}})^2 + \tilde{C}_2 \dot{\boldsymbol{\mu}} \boldsymbol{:} \dot{\boldsymbol{\mu}} - \frac{\partial \rho H}{\partial \dot{\boldsymbol{\gamma}}} \boldsymbol{\cdot} \ddot{\boldsymbol{\gamma}} - \frac{\partial \rho H}{\partial \dot{\boldsymbol{\mu}}} \boldsymbol{:} \ddot{\boldsymbol{\mu}} \geq 0 \tag{20}$$

and, finally, the second law of thermodynamics for the two-dimensional continuum yields

$$H = \mathrm{const}(\dot{\boldsymbol{\gamma}}, \dot{\boldsymbol{\mu}}), \quad \tilde{\Gamma}_0 \geq, \quad \tilde{C}_1 - \tilde{C}_2 \geq 0, \quad \tilde{C}_2 \geq 0. \tag{21}$$

Assuming linear viscous isotropic behaviour, the constitutive parameters are

$$\tilde{C}_2 = \frac{\eta h^3}{12}, \quad \tilde{C}_1 = \tilde{C}_2 \frac{1+\kappa}{1-\kappa}, \quad \tilde{\varGamma}_0 = \left(\frac{\pi}{h}\right)^2 \tilde{C}_2. \tag{22}$$

Note that in this case the computation is similar to Ref. [23]. An alternative approach of modelling viscoelastic behaviour is presented in Ref. [24].

5.3. *Plastic Cosserat Plate*

The last rheological model introduced here is the plastic two-dimensional continuum. It is obvious that we have to introduce "yield conditions" for both the transverse force vector and the polar moment tensor:

$$\begin{cases} N(\boldsymbol{F}) < F_{\mathrm{y}}, & \dot{\boldsymbol{\gamma}} = \boldsymbol{0}, \\ N(\boldsymbol{F}) = F_{\mathrm{y}}, & \dot{\boldsymbol{\gamma}} = \alpha \boldsymbol{F}, \quad \alpha \geq 0, \\ N(\mathrm{dev}\boldsymbol{G}) < G_{\mathrm{y}},\, \dot{\boldsymbol{\mu}} = \boldsymbol{0}, \\ N(\mathrm{dev}\boldsymbol{G}) = G_{\mathrm{y}},\, \dot{\boldsymbol{\mu}} = \beta \boldsymbol{a}_1 \boldsymbol{:} \mathrm{dev}(\boldsymbol{G}), \quad \beta \geq 0 \end{cases} \tag{23}$$

N denotes the norm, which is defined for a vector and a second-rank tensor in different ways:

$$N(\boldsymbol{F}) = \|\boldsymbol{F}\|, \quad N(\mathrm{dev}\boldsymbol{G}) = \sqrt{\frac{1}{2}\mathrm{dev}\boldsymbol{G} \boldsymbol{:} \mathrm{dev}\boldsymbol{G}}. \tag{24}$$

Note that dev$\boldsymbol{G}$ is the two-dimensional deviator of $\boldsymbol{G}$. It seems that there is no solution in the general case of a yield condition. Nevertheless, Palmov has provided a solution for rigid plastic materials, cf. Ref. [25]. Assuming that the stress tensor can be represented as follows

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 \mathrm{sign}(z) + \boldsymbol{n}\boldsymbol{\tau} + \boldsymbol{\tau}\boldsymbol{n} \tag{25}$$

with the plane stress tensor $\boldsymbol{\sigma}_0$ and the stress vector $\boldsymbol{\tau}$ in the "thickness" direction, we estimate

$$\boldsymbol{G} = \frac{h^2}{4}\boldsymbol{\sigma}_0, \quad \boldsymbol{F} = h\boldsymbol{\sigma}. \tag{26}$$

Finally, we get the following yield condition

$$\begin{cases} N(\boldsymbol{G})^2 + \dfrac{h^2}{16} N(\boldsymbol{F})^2 < G_{\mathrm{y}}, & \dot{\boldsymbol{\gamma}} = \boldsymbol{0},\ \dot{\boldsymbol{\mu}} = \boldsymbol{0}, \\ N(\boldsymbol{G})^2 + \dfrac{h^2}{16} N(\boldsymbol{F})^2 = G_{\mathrm{y}}, & \dot{\boldsymbol{\gamma}} = A\alpha \boldsymbol{F},\ \dot{\boldsymbol{\mu}} = \beta \boldsymbol{a}_1 \boldsymbol{:} \mathrm{dev}(\boldsymbol{G}). \end{cases} \tag{27}$$

To conclude Sect. 5, let us give some final comments on literature. Palmov presented his pioneering work concerning two-dimensional continua in Ref. [25]. The problems arising with the plastic model are due to the "no thickness" assumption in the case of the two-dimensional continuum. At the same time, the plastic material behavior is developing w.r.t the thickness direction (exceptional case is the rigid-plastic material behaviour - but this is only a rough approximation and idealization of the real material behaviour). The actual state of the art is given in Ref. [27]. Because of the problems with the formulation of the plastic model in Ref. [26], the theory for viscoelastic plates was presented in the paper at hand.

6. Application: Advanced Rheological Models

The aim of this section is to present the development of phenomenological constitutive equations that describe inelastic material behaviour at elevated temperature and relatively small loads, i.e., loads far below the yield limit. Such behaviour is often named "creep". It is well-known that a creep process can be split into three stages: primary (hardening is dominant), secondary (equilibrium between hardening and softening), and tertiary (softening including damage) creep [29]. To characterize hardening, recovery, and softening processes, a fraction model with creep-hard and creep-soft constituents [31, 32] is introduced because it is obvious from micrographs of alloys under consideration that such alloys contain two phases (Fig. 3). In the case of heat-resistant steels, the volume fraction of the creep-hard constituent is assumed to decrease towards a saturation value. Such an approach describes well the primary creep as a result of stress redistribution between constituents and tertiary creep as a result of softening (decrease of the volume fraction). To describe the whole creep curve, a damage parameter in the sense of continuum damage mechanics [30] is introduced. The material parameters and the response functions in the model are calibrated against published experimental curves for X20CrMoV12-1 steel [33]. The model is verified based on experimental creep curves under varying stress load and the stress-strain curves of high-temperature tensile tests under constant strain rate. The consideration of both softening and damage processes is necessary to characterize the long-term strength in a wide stress and temperature range. The one-dimensional model can be generalized to the case of multiaxial stress states taking into account the material parameters of the calibration.

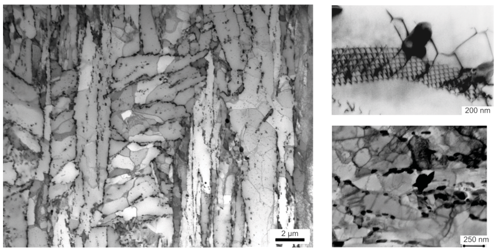


Figure 3. Microstructure of advanced Chromium steels: complex microstructure with grains, precipitates, subgrains and dislocations [34].

The constitutive model should account for some experimental observations shown in Figs. 4 and 5. From Fig. 4, it becomes obvious that the width of the stress-strain loop increases under cyclic loading and that the stress-strain loop is shifted to the right side (ratcheting). Figure 5a demonstrates that the inelastic material behaviour is related both to hardening and softening and that there is a strong dependence on the temperature. Concerning the creep behaviour as shown in Fig. 5b, the three creep stages are observable, but there is a strong dependence on the loading direction, i.e., whether tensile or compressive loads are applied.

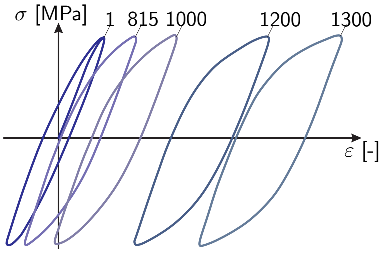
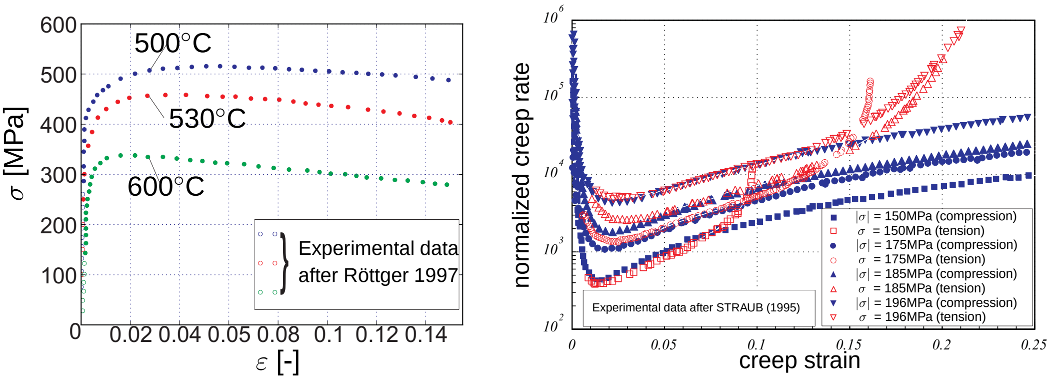


Figure 4. Cyclic creep for stress-controlled symmetric loading cycles, after [37].



(a) Influence of the temperature on the stress-strain curve, after [35].

(b) Different behaviour under tension and compression, after [36].

Figure 5. Inelastic material behaviour.

The basic idea for modelling is the assumption that the material under consideration behaves like a binary mixture. As illustrated in Fig. 6a, the total stress σ is composed of the stress σ_{h} for the hard phase and the stress σ_{s} for the soft phase:

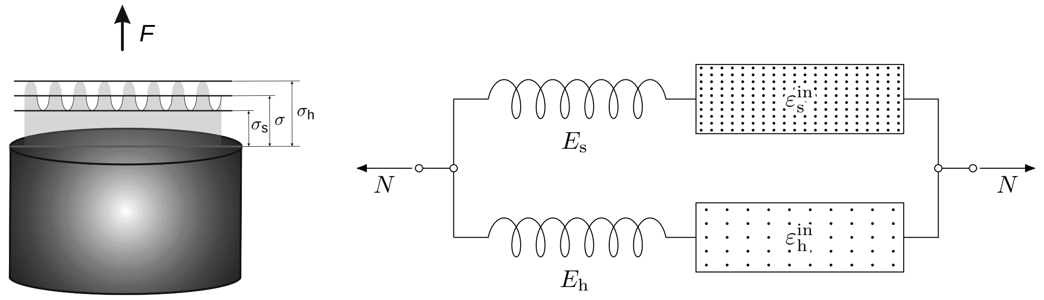
$$\sigma = \eta_{\mathrm{s}}\sigma_{\mathrm{s}} + \eta_{\mathrm{h}}\sigma_{\mathrm{h}}, \tag{28}$$

where η_{s} and η_{h} are the volume fractions of the inelastic soft and hard phase, respectively. Furthermore, we assume that the iso-strain assumption is fulfilled. In this case, we have a rheological behaviour with a parallel connection, cf. Fig. 6b:

$$\dot{\varepsilon} = \dot{\varepsilon}_{\mathrm{s}} = \dot{\varepsilon}_{\mathrm{h}}. \tag{29}$$

The further development starts from the following two assumptions:

- kinematic hardening follows for different inelastic properties, i.e., if the volume fraction is kept constant, the proposed binary mixture model results in a backstress model similar to the approach in Ref. [38], and
- if the volume fraction of the inelastic hard constituent decreases towards a saturation value, the creep rate increases after the minimum creep rate is reached. This allows for the consideration of softening effects.



(a) Stress concept in binary mixture model.

(b) Rheological model with two branches in parallel (soft phase at the top and hard phase at the bottom), each branch consist of an elastic and an inelastic element, after [33].

Figure 6. Binary mixture model.

Then, the behaviour of both constituents can be described by the following two equations:

$$\text{soft}: \qquad \dot{\varepsilon}_{\mathrm{s}} = \frac{\dot{\sigma}}{E} + \operatorname{sgn}(\sigma_{\mathrm{s}})\, f_\sigma(|\sigma_{\mathrm{s}}|) f_T(T), \tag{30}$$

$$\text{hard}: \qquad \dot{\varepsilon}_{\mathrm{h}} = \frac{\dot{\sigma}}{E} + \frac{\sigma_{\mathrm{h}} - \sigma}{\sigma_{\mathrm{h}_*} - \sigma}|\dot{\varepsilon}|, \tag{31}$$

where E is the macroscopic Young's modulus, which is assumed to be identical for both constituents, i.e., $E = E_{\mathrm{h}} = E_{\mathrm{s}}$. The variables f_σ and f_T denote response functions with respect to the stress and the temperature T, respectively, and σ_{h_*} is the saturation stress in the inelastic hard constituent.

The aim is to express the unknowns either by

- known variables,
- material parameters to identify, or
- internal state variables.

In the following, let us assume constant temperature and constant stress, as in standard creep tests. Thus, Eq. (30) is simplified:

$$\dot{\varepsilon}^{\mathrm{cr}} = \dot{\varepsilon}_{\mathrm{s}} = \operatorname{sgn}(\sigma_{\mathrm{s}})\, f_\sigma(|\sigma_{\mathrm{s}}|) f_T(T). \tag{32}$$

In addition, due to mass conservation, the sum of the volume fractions equals 1, i.e.:

$$\eta_{\mathrm{s}} + \eta_{\mathrm{h}} = 1. \tag{33}$$

Equation (28) is modified considering Eq. (33):

$$\sigma = (1 - \eta_{\mathrm{h}})\sigma_{\mathrm{s}} + \eta_{\mathrm{h}}\sigma_{\mathrm{h}}. \tag{34}$$

Now, we introduce a new variable, the softening variable $\mathit{\Gamma}$, which is based on the assumption that the volume fraction of the hard constituent decreases with time towards the saturation value η_{h_0}:

$$\mathit{\Gamma} = c_{\mathrm{h}}\frac{\eta_{\mathrm{h}}}{\eta_{\mathrm{s}}} \qquad \forall\ 1 \geq \mathit{\Gamma} \geq \mathit{\Gamma}_* = \frac{\eta_{\mathrm{h}_*}}{\eta_{\mathrm{s}_*}}, \tag{35}$$

where $c_{\mathrm{h}} = \frac{\eta_{\mathrm{s}_0}}{\eta_{\mathrm{h}_0}}$ is a material parameter which should be identified. Note that the asterisk marks variables with respect to the saturation state. Based on the introduced softening variable $\mathit{\Gamma}$, the unknown volume fraction η_{h} can be expressed as

$$\eta_{\mathrm{h}} = \frac{\mathit{\Gamma}}{\mathit{\Gamma} + c_{\mathrm{h}}}. \tag{36}$$

By replacing the volume fraction in Eq. (34) with Eq. (36), one obtains:

$$\sigma = \left(1 - \frac{\mathit{\Gamma}}{\mathit{\Gamma} + c_{\mathrm{h}}}\right)\sigma_{\mathrm{s}} + \frac{\mathit{\Gamma}}{\mathit{\Gamma} + c_{\mathrm{h}}}\sigma_{\mathrm{h}}. \tag{37}$$

Further algebraic manipulations result in:

$$\sigma c_{\mathrm{h}} = \sigma_{\mathrm{s}} c_{\mathrm{h}} + \mathit{\Gamma}(\sigma_{\mathrm{h}} - \sigma). \tag{38}$$

Making use of the overstress concept, the variable β, referred to as “backstress” or “kinematic stress”, is introduced:

$$\beta = \frac{1}{c_{\mathrm{h}}} \left(\sigma_{\mathrm{h}} - \sigma\right). \tag{39}$$

Based on Eq. (38) and by introducing the backstress, one obtains finally:

$$\sigma_{\mathrm{s}} = \sigma - \beta \mathit{\Gamma}. \tag{40}$$

For the internal variables β and $\mathit{\Gamma}$, it holds:

- the backstress β describes the stress accumulation in the inelastic hard phase. Since it is an internal variable, an evolution equation is required,
- one can show by proper algebraic manipulations that from the iso-strain condition and by the chosen rate formulation for the inelastic hard phase, a formulation of the Frederick-Armstrong type follows:

$$\dot{\beta} = \frac{E}{c_{\mathrm{h}}} \dot{\varepsilon}^{\mathrm{cr}} \left(1 - \frac{\beta}{\beta_*}\right), \tag{41}$$

- the softening variable $\mathit{\Gamma}$ controls the volume change of the inelastic hard phase and is motivated by microstructural changes,
- the following evolution equation for $\mathit{\Gamma}$ is adopted from literature:

$$\dot{\mathit{\Gamma}} = A_{\mathrm{s}}(\mathit{\Gamma}_* - \mathit{\Gamma})\dot{\varepsilon}^{\mathrm{cr}} \tag{42}$$

  with the material parameter A_{s}.

For further details, the interested reader is referred to Ref. [32]. Let us note that for constant stress and temperature, the evolution equations both for the backstress and the softening variable can be integrated and written in terms of the creep strain $\varepsilon^{\mathrm{cr}}$:

$$\beta = \beta_* \left[1 - \exp\left(-\frac{E}{c_{\mathrm{h}} \beta_*} \varepsilon^{\mathrm{cr}}\right)\right], \tag{43}$$

$$\mathit{\Gamma} = \mathit{\Gamma}_* - (\mathit{\Gamma}_* - 1) \exp\left(-A_{\mathrm{s}} \varepsilon^{\mathrm{cr}}\right). \tag{44}$$

For the required response functions, the following statements are valid:

- the stress and temperature response functions f_σ and f_T of the inelastic soft phase need to be defined,
- a temperature response function for the Young’s modulus must be formulated, and
- the saturation variables β_* and $\mathit{\Gamma}_*$ are stress-dependent and therefore additional response functions need to be found.

The following response functions have been identified:

- the viscosity function for the stress dependence of the creep rate:

$$f_\sigma(\sigma) = a_0 \sinh(B\sigma), \tag{45}$$

- the Arrhenius function for the temperature dependence:

$$f_T\left(T\right) = \exp\left(-\frac{\alpha}{T}\right), \tag{46}$$

- the following linear function for the saturation backstress β_*:

$$\beta_*\left(\sigma\right) = H_*\sigma, \tag{47}$$

- and the following stress response function for the saturation softening variable $\mathit{\Gamma}_*$:

$$\mathit{\Gamma}_*(x) = \frac{a_{\mathit{\Gamma}}}{1 + b_{\mathit{\Gamma}} \exp\left(-\frac{x}{c_{\mathit{\Gamma}}}\right)} \tag{48}$$

  with the material parameters a_0, B, α, H_*, $a_{\mathit{\Gamma}}$, $b_{\mathit{\Gamma}}$, and $c_{\mathit{\Gamma}}$.

The last step is the inclusion of damage by introducing the scalar damage variable ω following Rabotnov's approach, cf. [31]. The following evolution equation is applied to describe damage according to [31]:

$$\dot{\omega} = r\left(\omega\right) h\left(\sigma\right) \frac{|\dot{\varepsilon}^{\mathrm{cr}}|}{\varepsilon_*\left(\sigma\right)} \tag{49}$$

with the response functions:

$$r\left(\omega\right) = l\omega^{\frac{l-1}{l}}, \tag{50}$$

$$\varepsilon_*\left(\sigma\right) = \varepsilon_{\mathrm{br}} + \frac{a_\varepsilon}{1 + b_\varepsilon \exp\left(-\frac{|\sigma|}{c_\varepsilon}\right)}, \tag{51}$$

$$h\left(\sigma\right) = \frac{\sigma + |\sigma|}{2\sigma}, \tag{52}$$

whereby the variables l, $\varepsilon_{\mathrm{br}}$, a_ε, b_ε, and c_ε are material parameters to be determined by calibrating the constitutive model against experimental data. A detailed description of this calibration procedure is given in Ref. [33]. The following comments concerning the calibration process can be given:

- The Frederick-Armstrong-type approach for the backstress follows naturally from the proposed model.
- The evolution of microstructure is taken into account.
- The stress accumulation in only one inelastic hard phase does not seem to be sufficient.
- With the improved backstress formulation, all creep curves are described with higher accuracy.

In the following, let us briefly indicate how to extend the one-dimensional model to three dimensions. First, let us assume small strains and isotropy for the elastic behaviour. Second, if necessary, scalar variables are replaced by the corresponding second-order tensors, such as the backstress tensor $\boldsymbol{\beta}$. Third, some variables, particularly the stresses, are substituted by their equivalent properties. Here, we choose the von Mises hypothesis. This procedure results in the following three-dimensional evolution equations:

- an evolution equation for the creep strain:

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{cr}} = \frac{3}{2} f_\sigma \left(\frac{\bar{\sigma}_{\mathrm{vM}}}{1-\omega} \right) f_T\left(T\right) \frac{\bar{\boldsymbol{s}}}{\bar{\sigma}_{\mathrm{vM}}} - \frac{1}{2G} \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathit{\Gamma} \boldsymbol{\beta} \right) \tag{53}$$

  with the effective stress deviator $\bar{\boldsymbol{s}} = \boldsymbol{s} - \mathit{\Gamma}\boldsymbol{\beta}$ and the effective von Mises stress $\bar{\sigma}_{\mathrm{vM}} = \sqrt{\frac{3}{2}\mathrm{tr}(\bar{\boldsymbol{s}})^2}$,
- an evolution equation for the backstress tensor:

$$\dot{\boldsymbol{\beta}} = A_{\mathrm{h}} \left(\frac{2}{3} \dot{\boldsymbol{\varepsilon}}^{\mathrm{cr}} - \dot{\varepsilon}_{\mathrm{vM}} \frac{\boldsymbol{\beta}}{\beta_*\left(\sigma_{\mathrm{vM}}\right)} \right), \tag{54}$$

  whereby the variables $\dot{\varepsilon}_{\mathrm{vM}}$ and σ_{vM} represent the equivalent strain rate and stress of von Mises-type,
- an evolution equation for the softening variable:

$$\dot{\mathit{\Gamma}} = A_{\mathrm{s}} \left[\mathit{\Gamma}_*(\sigma_{\mathrm{vM}}) - \mathit{\Gamma} \right] \dot{\varepsilon}^{\mathrm{cr}}_{\mathrm{vM}}, \tag{55}$$

- and an evolution equation for the damage variable:

$$\dot{\omega} = r(\omega) h(\boldsymbol{\sigma}) \frac{\dot{\varepsilon}^{\mathrm{cr}}_{\mathrm{vM}}}{\varepsilon_*(\sigma_{\mathrm{vM}})}, \quad h(\boldsymbol{\sigma}) = \frac{\sigma_I + |\sigma_I|}{2\sigma_{\mathrm{vM}}}, \tag{56}$$

  whereby σ_I represents the largest principal stress.

Further details are given in Ref. [31], where also some verification results are presented. The following conclusions were drawn:

- Within its range of calibration, the binary mixture model reproduces the material behaviour with high accuracy.
- Physical processes on a microstructural level are accounted for. Nevertheless, the material parameters are determined based on the results of macroscopic creep tests.
- Complex creep curves including hardening, softening, and damage are accurately represented by the proposed constitutive model.
- In addition, the model is able to predict the material behaviour if loading conditions differ from the range of calibration.
- Implementations into the finite element method are possible, cf. Ref. [39].
- In some cases, softening is overestimated by the model.
- For lower temperatures, the model overestimates hardening, which should be improved in the future.
- The transition from linear elastic to non-linear stress-strain behaviour is currently too strict and should be refined in forthcoming papers.

Finally, such an approach is applicable to other materials as well, e.g., to model the creep behaviour of thermoplastics, as done in Ref. [40].

7. Summary

Rheological models are a powerful tool for describing complex material behavior. The advantages of the inductive and the deductive description of the individual response to external loads are brought together. After a brief historical overview of the rheological model method, the method is introduced in a variant developed by Palmov. Application examples (two-dimensional continua, creep of heat-resistant alloys) conclude the article.

More applications can be established: binary mixtures, plastics, etc. New challenges are related to the integration of the symbolic tensor calculus and the basics of continuum mechanics, cf. [28].

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