Introduction

In this exposition we aim to develop some basic tools on pseudo–differential operators as they are needed in applications to partial differential equations on non–smooth spaces, e.g., manifolds with (polyhedral) singularities. The exposition has been written in connection with a series of lectures by the author for an "Advanced Course on Pseudodifferential Operators and their Applications", held in Tbilisi, Georgia, April 18–28, 1998, at the I. Vekua Institute of Applied Mathematics.

The calculus of pseudo–differential operators (and, more generally, the field of microlocal analysis) has a long tradition building upon the basic works of Kohn and Nirenberg [29], Hörmander [26] and of many other authors. A large number of monographs has been written on this topic; see, for instance, Treves [90], Kumano–go [33], Eskin [12], Rempel and Schulze [42], Hörmander [27]. Over time, the calculus itself has often been refined and adapted to new applications. The classes of permissible symbols have been greatly extended and the functional analytic background of the associated spaces of distributions has been enormously enriched.

The present material should be of help to young mathematicians who are interested in applications to configurations with singularities. Many problems and models arising in applied sciences and mathematical physics (in mechanics, crack theory, elasticity, hydrodynamics, scattering theory, to name just a few areas) typically refer to non–smooth configurations, and this requires a rather "non–standard" view of the traditional techniques. In fact, the standard calculus does not answer the simplest questions on the asymptotics of solutions to mixed or transmission problems when the interfaces have corner or cuspidal points. Such problems may even pose a challenge for an experienced mathematician with knowledge of the classical theory.

In the past few years numerous papers and monographs have appeared that are devoted to the specific, new structures that enter into play when the models contain geometric singularities, cf. Schulze [58], [61], for example. These developments have also been stimulated by the hope of (and certain progress toward) getting generalisations of the Atiyah–Singer index theorem for (pseudo–differential operator algebras on) manifolds with singularities and by new applications in geometry and topology. The role of the present notes in this context is to single out a number of beautiful new aspects among these developments and to give an introduction. Consequently, well-known facts about the Fourier transform, Sobolev spaces, standard symbol spaces and the like will not be proved here; they may be found in any textbook on pseudo–differential and Fourier integral operators.

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Chapter 1

Basic elements of the pseudo–differential calculus

1.1 Symbols and oscillatory integrals

1.1.1 Fourier transform and Sobolev spaces

The Fourier transform in $\mathbb{R}^n$ is defined by the formula

$$(Fu)(\xi) = \int\limits_{\mathbb{R}^n} e^{-ix\xi} u(x)\, dx, \tag{1.1.1}$$

$x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$, $x\xi = \sum_{j=1}^n x_j \xi_j$, $i = \sqrt{-1}$. We also write $\hat{u}(\xi) = (Fu)(\xi)$. It is well–known that F induces isomorphisms

$$F : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n), \qquad F : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n),$$

with $\mathcal{S}(\mathbb{R}^n)$ being the Schwartz space in $\mathbb{R}^n$ and $L^2(\mathbb{R}^n)$ the space of square integrable functions with $(u, v) = \int_{\mathbb{R}^n} u(x)\overline{v(x)}\, dx$ as the scalar product. The inverse of F is given by the formula

$$(F^{-1}g)(x) = \int\limits_{\mathbb{R}^n} e^{ix\xi} g(\xi)\, đ\xi, \qquad đ\xi = (2\pi)^{-n}\, d\xi.$$

The Fourier transform satisfies relations

$$(Fu, f) = (2\pi)^n (u, F^{-1}f), \qquad \langle Fu, f\rangle = \langle u, Ff\rangle, \tag{1.1.2}$$

where $\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)g(x)\,dx$. Then F extends to the space $\mathcal{S}'(\mathbb{R}^n)$ of temperate distributions in $\mathbb{R}^n$ as an isomorphism

$$F : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n).$$

We do not prove here such standard facts and refer, for instance to Treves [89] or Hörmander [27], vol. 1. Set $\langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}$ and denote by $[\xi]$ any (fixed) strictly positive function in $C^\infty(\mathbb{R}^n)$ with $[\xi] = |\xi|$ for $|\xi| >$ const for some constant > 0. Then we have $c_1[\xi] \le \langle\xi\rangle \le c_2[\xi]$ for suitable $c_1, c_2 > 0$.

The Sobolev space $H^s(\mathbb{R}^n)$ in $\mathbb{R}^n$ of smoothness $s \in \mathbb{R}$ is defined as the subspace of all $u \in \mathcal{S}'(\mathbb{R}^n)$ for which

$$\|u\|_{H^s(\mathbb{R}^n)} = \left\{ \int \langle\xi\rangle^{2s} |\hat{u}(\xi)|^2\, đ\xi \right\}^{\frac{1}{2}} \tag{1.1.3}$$

is finite. It can equivalently be obtained as the completion of $\mathcal{S}(\mathbb{R}^n)$ (or $C_0^\infty(\mathbb{R}^n)$) with respect to the norm (1.1.3). By virtue of relation (1.1.2) there is a canonical isomorphism between $H^0(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$.

Let us also define the weighted Sobolev spaces

$$H^{s;\delta}(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \ \langle x\rangle^\delta u \in H^s(\mathbb{R}^n)\}$$

for arbitrary reals s, δ, endowed with the norm $\|u\|_{H^{s;\delta}(\mathbb{R}^n)} = \|\langle x\rangle^\delta u\|_{H^s(\mathbb{R}^n)}$.

Theorem 1.1.1 *There are canonical continuous embeddings*

$$H^{s';\delta'}(\mathbb{R}^n) \hookrightarrow H^{s;\delta}(\mathbb{R}^n)$$

for $s' \ge s$, $\delta' \ge \delta$, that are compact for $s' > s$, $\delta' > \delta$.

We have

$$\mathcal{S}(\mathbb{R}^n) = \varprojlim_{k\in\mathbb{N}} H^{k;k}(\mathbb{R}^n) \tag{1.1.4}$$

($\varprojlim$ denotes the projective limit; $\mathbb{N} = \{0, 1, 2, \dots\}$).

Definition 1.1.2 *Let E be a Banach space and $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ be a strongly continuous group of isomorphisms of E (i.e., $\kappa_\lambda e \in C(\mathbb{R}_+, E)$ for each $e \in E$; $\kappa_\lambda\kappa_\mu = \kappa_{\lambda\mu}$ for all $\lambda, \mu \in \mathbb{R}_+$). Then the space $\mathcal{W}^s(\mathbb{R}^n, E)$, $s \in \mathbb{R}$, is defined as the completion of $\mathcal{S}(\mathbb{R}^n, E)$ $(= \mathcal{S}(\mathbb{R}^n) \otimes_\pi E)$ with respect to the norm*

$$\|u\|_{\mathcal{W}^s(\mathbb{R}^n,E)} = \left\{ \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} \|\kappa^{-1}(\xi)\hat{u}(\xi)\|_E^2\, đ\xi \right\}^{\frac{1}{2}}, \tag{1.1.5}$$

where $\kappa(\xi) := \kappa_{[\xi]}$. For $\kappa_\lambda = \mathrm{id}_E$ for all $\lambda \in \mathbb{R}_+$ we write $H^s(\mathbb{R}^n, E)$ ($\otimes_\pi$ denotes the completed projective tensor product between the respective spaces).

It is then obvious that the operator $T = F^{-1}\kappa^{-1}(\xi)F$ induces an isomorphism

$$T : \mathcal{W}^s(\mathbb{R}^n, E) \to H^s(\mathbb{R}^n, E) \tag{1.1.6}$$

for all $s \in \mathbb{R}$. The basic properties of the spaces $\mathcal{W}^s(\mathbb{R}^n, E)$ are investigated in Schulze [58], [66].

Example 1.1.3 *Let $E = H^{s;\delta}(\mathbb{R}^n)$ and $(\kappa_\lambda u)(x) = \lambda^{\frac{n}{2}} u(\lambda x)$, $\lambda \in \mathbb{R}_+$. Then $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ is a strongly continuous group of isomorphisms on the space $H^{s;\delta}(\mathbb{R}^n)$. In particular, for $\delta = 0$, i.e., $E = H^s(\mathbb{R}^n)$, we get a canonical isomorphism*

$$\mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^n)) = H^s(\mathbb{R}^{q+n}).$$

This relation can be regarded as an anisotropic reformulation of Sobolev spaces in $\mathbb{R}^{q+n}$ with respect to a corresponding splitting of variables, cf. [66], Section 3.1.3.

Remark 1.1.4 *We have*

$$\mathcal{W}^s(\mathbb{R}^n, E) \subset \mathcal{S}'(\mathbb{R}^n, E) = \mathcal{L}(\mathcal{S}(\mathbb{R}^n), E).$$

The space $\mathcal{W}^s(\mathbb{R}^n, E)$, $s \in \mathbb{R}$, depends on the specific choice of $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, it is fixed and omitted in our notation. However $\mathcal{W}^\infty(\mathbb{R}^n, E) = \bigcap_{s\in\mathbb{R}} \mathcal{W}^s(\mathbb{R}^n, E)$ is independent of $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$.

We also define spaces with weight $\delta \in \mathbb{R}$ at infinity

$$\mathcal{W}^{s;\delta}(\mathbb{R}^n, E) = \{u \in \mathcal{S}'(\mathbb{R}^n, E) : \ \langle x\rangle^\delta u \in \mathcal{W}^s(\mathbb{R}^n, E)\},$$

with the norm $\|u\|_{\mathcal{W}^{s;\delta}(\mathbb{R}^n,E)} = \|\langle x\rangle^\delta u\|_{\mathcal{W}^s(\mathbb{R}^n,E)}$.

Theorem 1.1.5 *Let E' and E be Hilbert spaces endowed with $\{\kappa'_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively. Suppose $E' \hookrightarrow E$ is a continuous embedding and $\kappa'_\lambda = \kappa_\lambda|_{E'}$ for all $\lambda \in \mathbb{R}_+$. Then we have continuous embeddings*

$$\mathcal{W}^{s';\delta'}(\mathbb{R}^n, E') \hookrightarrow \mathcal{W}^{s;\delta}(\mathbb{R}^n, E) \tag{1.1.7}$$

for all $s' \geq s$, $\delta' \geq \delta$. If $E' \hookrightarrow E$ is compact and $s' > s$, $\delta' > \delta$, then (1.1.7) *is compact.*

It is often useful to glue different spaces together to new ones. Let us fix the following notation. Let E be a Fréchet space with an operation of multiplication from the left by elements a in an algebra A. Set

$$[a]E = \text{completion of } \{ae : \ e \in E\} \text{ in } E. \tag{1.1.8}$$

In an analogous sense we form $E[b]$ or $[a]E[b]$ for $a, b \in A$. Moreover, for Fréchet spaces E_0, E_1 that are vector subspaces of a topological Hausdorff space H we write

$$E_0 + E_1 = \{e_0 + e_1 : \ e_0 \in E_0, e_1 \in E_1\} \tag{1.1.9}$$

equipped with the Fréchet topology from the bijection

$$E_0 + E_1 \cong (E_0 \oplus E_1)/\Delta \qquad \text{where } \Delta = \{(e, -e) : \ e \in E_0 \cap E_1\}.$$

We call (1.1.9) the non–direct sum of the spaces E_0, E_1. In a similar manner we can introduce finite non–direct sums of arbitrary length.

Definition 1.1.6 *Let X be a closed compact C^∞ manifold, $n = \dim X$, $\{U_1, \dots, U_N\}$ an open covering by coordinate neighbourhoods, $\{\varphi_1, \dots, \varphi_N\}$ a subordinate partition of unity and $\chi_j : U_j \to \mathbb{R}^n$, $j = 1, \dots, N$ a system of charts. Then $H^s(X)$ for $s \in \mathbb{R}$ is defined as the completion of $C^\infty(X)$ with respect to the norm*

$$\|u\|_{H^s(X)} = \left\{ \sum_{j=1}^{N} \|(\varphi_j u) \circ \chi_j^{-1}\|^2_{H^s(\mathbb{R}^n)} \right\}^{\frac{1}{2}}$$

(the definition is correct, i.e., we get equivalent norms by changing the involved charts and the partition of unity).

Then we have $H^s(X) = \sum_{j=1}^{N} [\varphi_j] H^s(X)$.

Concerning the (completed) projective tensor product $\otimes_\pi$ between Fréchet spaces E and F there is the following result:

Theorem 1.1.7 *Every element $h \in E \otimes_\pi F$ has an expansion*

$$h = \sum_{j=0}^{\infty} \lambda_j e_j \otimes f_j$$

with sequences $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$, $\sum_{j=0}^{\infty} |\lambda_j| < \infty$, and $\{e_j\}_{j\in\mathbb{N}} \subset E$, $\{f_j\}_{j\in\mathbb{N}} \subset F$, tending to zero in the respective spaces for $j \to \infty$.

1.1.2 Symbol spaces

Symbols (or amplitude functions) are functions $a(x, \xi)$ dependent on the variables $x \in \Omega$, $\Omega \subseteq \mathbb{R}^n$ open, and the covariables $\xi \in \mathbb{R}^n$. The idea of pseudo–differential operators is to pass to operators $a(x, D)$, where $\xi = (\xi_1, \ldots, \xi_n)$ is replaced by the corresponding differentiations $D = (D_{x_1}, \ldots, D_{x_n})$, $D_{x_j} = -i\partial_{x_j}$, $\partial_{x_j} = \partial/\partial x_j$. If $a(x, \xi)$ is a polynomial with respect to ξ, it is clear what this means (when the variables are identified with operators of multiplication from the left). In general (the "non–polynomial" case) we have to explain two basic aspects, namely

(i) suitable spaces of admitted symbols,

(ii) rules that determine a correspondence $\mathrm{Op} : a(x, \xi) \to a(x, D)$.

For the symbol spaces there are many choices. In the general definition the variables x may vary in an open set U in $\mathbb{R}^m$. If we then form operators, we suppose $U = \Omega \times \Omega$ for open $\Omega \subseteq \mathbb{R}^n$ and denote the variables (x, x').

Definition 1.1.8 *The symbol space $S^\mu(U \times \mathbb{R}^n)$ for $\mu \in \mathbb{R}$, $U \subseteq \mathbb{R}^m$ open, is defined as the set of all $a(x, \xi) \in C^\infty(U \times \mathbb{R}^n)$ which satisfy the estimates*

$$|D_x^\alpha D_\xi^\beta a(x, \xi)| \le c\langle\xi\rangle^{\mu - |\beta|} \tag{1.1.10}$$

for all $x \in K$ for arbitrary $K \subset\subset U$, $\xi \in \mathbb{R}^n$ and all multi–indices $\alpha \in \mathbb{N}^m$, $\beta \in \mathbb{N}^n$, with constants $c = c(\alpha, \beta, K) > 0$.

The best constants in these symbol estimates form a semi–norm system in which $S^\mu(U \times \mathbb{R}^n)$ is a Fréchet space. We then have

$$S^{-\infty}(U \times \mathbb{R}^n) := \bigcap_{\mu \in \mathbb{R}} S^\mu(U \times \mathbb{R}^n) = C^\infty(U, \mathcal{S}(\mathbb{R}^n)).$$

A function $\chi(\xi) \in C^\infty(\mathbb{R}^n)$ is called an excision function if $\chi(\xi) = 0$ for $|\xi| < c_0$, $\chi(\xi) = 1$ for $|\xi| > c_1$, for certain $0 < c_0 < c_1 < \infty$.

Let $S^{(\mu)}(U \times (\mathbb{R}^n \setminus \{0\}))$ be the subspace of all $f(x, \xi) \in C^\infty(U \times (\mathbb{R}^n \setminus \{0\}))$ with

$$f(x, \lambda\xi) = \lambda^\mu f(x, \xi)$$

for all $\lambda \in \mathbb{R}_+$ and $(x, \xi) \in U \times (\mathbb{R}^n \setminus \{0\})$. If $f(x, \xi)$ is such a function, we have $\chi(\xi) f(x, \xi) \in S^\mu(U \times \mathbb{R}^n)$ for every excision function $\chi(\xi)$.

Definition 1.1.9 *An element $a(x,\xi) \in S^{\mu}(U \times \mathbb{R}^n)$, $\mu \in \mathbb{R}$, is called classical if there exist functions $a_{(\mu-j)}(x,\xi) \in S^{(\mu-j)}(U \times (\mathbb{R}^n \setminus \{0\}))$ for all $j \in \mathbb{N}$ such that $a(x,\xi) - \sum_{j=0}^{N} \chi(\xi) a_{(\mu-j)}(x,\xi) \in S^{\mu-(N+1)}(U \times \mathbb{R}^n)$ for all $N \in \mathbb{N}$. The subspace of all classical symbols $\in S^{\mu}(U \times \mathbb{R}^n)$ is denoted by $S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^n)$.*

The so–called homogeneous components $a_{(\mu-j)}(x,\xi)$ of order $\mu-j$, $j \in \mathbb{N}$, are uniquely determined by $a(x,\xi)$. Thus there is a system of linear operators $S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^n) \to S^{(\mu-j)}(U \times (\mathbb{R}^n \setminus \{0\}))$, $j \in \mathbb{N}$, defined by $a(x,\xi) \to a_{(\mu-j)}$. Moreover, we have linear operators $S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^n) \to S^{\mu-(N+1)}(U \times \mathbb{R}^n)$ for all $N = -1, 0, 1, 2, \dots$, defined as the embedding for $N = -1$ and as the difference in Definition 1.1.9 for $N \in \mathbb{N}$, for any fixed χ. Then $S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^n)$ is a Fréchet space in the projective limit topology.

If a relation holds both for symbols in S^{μ} and S^{μ}_{cl} we shortly write $S^{\mu}_{(\mathrm{cl})}$. We denote by $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^n)$ the subspace of all elements in $S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^n)$ that are independent of x. These are closed subspaces, and we have

$$S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^n) = C^{\infty}(U, S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^n)) = C^{\infty}(U) \otimes_{\pi} S^{\mu}_{(\mathrm{cl})}(\mathbb{R}^n). \tag{1.1.11}$$

The following relations are obvious:

$$S^{\tilde{\mu}}_{(\mathrm{cl})}(U \times \mathbb{R}^n) \subset S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^n)$$

for $\mu \geq \tilde{\mu}$ (and $\mu - \tilde{\mu} \in \mathbb{N}$ in the classical case),

$$D_x^{\alpha} D_{\xi}^{\beta} S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^n) \subset S^{\mu-|\beta|}_{(\mathrm{cl})}(U \times \mathbb{R}^n)$$

for all $\alpha \in \mathbb{N}^m$, $\beta \in \mathbb{N}^n$, and

$$S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^n) S^{\nu}_{(\mathrm{cl})}(U \times \mathbb{R}^n) \subset S^{\mu+\nu}_{(\mathrm{cl})}(U \times \mathbb{R}^n)$$

for all $\mu, \nu \in \mathbb{R}$. The left hand side of the latter formula means the space of all products between elements of the involved spaces.

If we are given a sequence of symbols $a_j(x,\xi) \in S^{\mu_j}_{(\mathrm{cl})}(U \times \mathbb{R}^n)$, $j \in \mathbb{N}$, with $\mu_j \to -\infty$ as $j \to \infty$ ($\mu_j = \mu - j$ for a $\mu \in \mathbb{R}$ in the classical case), then a symbol $a(x,\xi) \in S^{\mu}_{(\mathrm{cl})}(U \times \mathbb{R}^n)$ for $\mu = \max\{\mu_j : j \in \mathbb{N}\}$ is called an asymptotic sum of the $a_j(x,\xi)$ if for every $M \in \mathbb{N}$ there is an $N(M) \in \mathbb{N}$ with

$$a(x,\xi) - \sum_{j=0}^{N} a_j(x,\xi) \in S^{\mu-M}_{(\mathrm{cl})}(U \times \mathbb{R}^n)$$

for all $N \geq N(M)$. Every such $a(x,\xi)$ is uniquely determined by the sequence, modulo $S^{-\infty}(U \times \mathbb{R}^n)$. We write $a(x,\xi) \sim \sum_{j=0}^{\infty} a_j(x,\xi)$.

Theorem 1.1.10 *Let $a_j(x,\xi) \in S^{\mu_j}(U \times \mathbb{R}^n)$, $j \in \mathbb{N}$, be an arbitrary sequence. Then under the above assumptions on the orders there exists an asymptotic sum $a(x,\xi) \in S^{\mu}(U \times \mathbb{R}^n)$. For $a_j(x,\xi) \in S^{\mu-j}_{\mathrm{cl}}(U \times \mathbb{R}^n)$, $j \in \mathbb{N}$, we get automatically $a(x,\xi) \in S^{\mu}_{\mathrm{cl}}(U \times \mathbb{R}^n)$.*

Choosing an excision function $\chi(\xi)$ an asymptotic sum can be obtained by $a(x,\xi) = \sum_{j=0}^{\infty} \chi(\xi/c_j) a_j(x,\xi)$ as a convergent sum in the space $S^{\mu}(U \times \mathbb{R}^n)$ when $\{c_j\}_{j\in\mathbb{N}}$ is a sequence of constants > 0, tending to ∞ sufficiently fast as $j \to \infty$.

Concerning the choice of an operator to a symbol $a(x,x',\xi) \in S^{\mu}(\Omega \times \Omega \times \mathbb{R}^n)$, $\Omega \subseteq \mathbb{R}^n$ open, we set

$$\operatorname{Op}(a)u(x) = \int\limits_{\mathbb{R}^n} e^{ix\xi} \left\{ \int e^{-ix'\xi} a(x,x',\xi) u(x')\, dx' \right\} đ\xi. \tag{1.1.12}$$

We first assume $u \in C_0^{\infty}(\Omega)$. It is then easy to see that the double integral makes sense and induces a continuous operator

$$\operatorname{Op}(a) : C_0^{\infty}(\Omega) \to C^{\infty}(\Omega).$$

Especially for $a(x,\xi) \in S^{\mu}(\Omega \times \mathbb{R}^n)$ we can write

$$\operatorname{Op}(a)u(x) = \int\limits_{\mathbb{R}^n} e^{ix\xi} a(x,\xi) \hat{u}(\xi)\, đ\xi \tag{1.1.13}$$

which is convergent because of $\hat{u}(\xi) \in \mathcal{S}(\mathbb{R}^n)$. The correspondence $a(x,\xi) \to \operatorname{Op}(a)$ is called an operator convention.

For the case of a symbol $a(\xi) \in S^{\mu}(\mathbb{R}^n)$ with "constant coefficients" we immediately obtain the following result:

Proposition 1.1.11 *The map $a(\xi) \to \operatorname{Op}(a)$ induces a continuous operator*

$$\operatorname{Op} : S^{\mu}(\mathbb{R}^n) \to \mathcal{L}(H^s(\mathbb{R}^n), H^{s-\mu}(\mathbb{R}^n))$$

for each $s \in \mathbb{R}$, and we have

$$\| \operatorname{Op}(a) \|_{\mathcal{L}(H^s(\mathbb{R}^n), H^{s-\mu}(\mathbb{R}^n))} \leq \sup_{\xi \in \mathbb{R}^n} |a(\xi)| \langle \xi \rangle^{-\mu}.$$

This is an obvious consequence of (1.1.3) and (1.1.13).

Let us now pass to operator–valued variants of the scalar symbols of Definitions 1.1.8, 1.1.9. To motivate the general definition we first consider a scalar symbol $p(y,\xi,\eta) \in S^{\mu}(\Omega_y \times \mathbb{R}^{n+q}_{\xi,\eta})$ for an open set $\Omega \subseteq \mathbb{R}^q$. This can be viewed as a (y,η)–dependent family of symbols in $S^{\mu}(\mathbb{R}^n_{\xi})$. Applying Op with respect to the x–variables in $\mathbb{R}^n$, written Op_x, we get a (y,η)–dependent operator function

$$a(y,\eta) := \mathrm{Op}_x(p)(y,\eta) : H^s(\mathbb{R}^n) \to H^{s-\mu}(\mathbb{R}^n),$$

$s \in \mathbb{R}$, cf. Proposition 1.1.11.

We shall show below that $a(y,\eta)$ is an operator–valued symbol in the following sense. Let E and $\tilde{E}$ be Hilbert spaces, endowed with strongly continuous groups of isomorphisms $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively.

Definition 1.1.12 *The symbol space $S^{\mu}(U \times \mathbb{R}^q; E, \tilde{E})$ for $\mu \in \mathbb{R}$, $U \subseteq \mathbb{R}^p$ open, is defined as the set of all $a(y,\eta) \in C^{\infty}(U \times \mathbb{R}^q, \mathcal{L}(E,\tilde{E}))$ which satisfy the estimates*

$$\|\tilde{\kappa}^{-1}(\eta)\{D_y^{\alpha} D_\eta^{\beta} a(y,\eta)\}\kappa(\eta)\|_{\mathcal{L}(E,\tilde{E})} \leq c\langle\eta\rangle^{\mu-|\beta|} \tag{1.1.14}$$

for all $y \in K$ for arbitrary $K \subset\subset U$ and $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^q$, with constants $c = c(\alpha,\beta,K) > 0$.

The best constants in these symbol estimates form a semi–norm system in which $S^{\mu}(U \times \mathbb{R}^q; E, \tilde{E})$ is a Fréchet space.

The groups $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$ in Definition 1.1.12 are fixed. Of course, $S^{\mu}(U \times \mathbb{R}^q; E, \tilde{E})$ depends on this choice. However

$$S^{-\infty}(U \times \mathbb{R}^q; E, \tilde{E}) = \bigcap_{\mu\in\mathbb{R}} S^{\mu}(U \times \mathbb{R}^q; E, \tilde{E})$$

is independent of the groups and coincides with $C^{\infty}(U, \mathcal{S}(\mathbb{R}^q, \mathcal{L}(E,\tilde{E})))$.

Remark 1.1.13 *The trivial group actions $\kappa_\lambda = \mathrm{id}_E$ or $\tilde{\kappa}_\lambda = \mathrm{id}_{\tilde{E}}$ for all $\lambda \in \mathbb{R}_+$ are admitted. For the case $E = \mathbb{C}^N$ ($\tilde{E} = \mathbb{C}^{\tilde{N}}$) we always assume $\kappa_\lambda = \mathrm{id}_E$ ($\tilde{\kappa}_\lambda = \mathrm{id}_{\tilde{E}}$).*

Let $S^{(\mu)}(U \times (\mathbb{R}^q \setminus \{0\}); E, \tilde{E})$ be the subspace of all $f(y,\eta) \in C^{\infty}(U \times (\mathbb{R}^q \setminus \{0\}), \mathcal{L}(E,\tilde{E}))$ with

$$f(y,\lambda\eta) = \lambda^{\mu}\tilde{\kappa}_\lambda f(y,\eta)\kappa_\lambda^{-1} \tag{1.1.15}$$

for all $\lambda \in \mathbb{R}_+$ and $(y,\eta) \in U \times (\mathbb{R}^q \setminus \{0\})$. For such functions we have $\chi(\eta)f(y,\eta) \in S^\mu(U \times \mathbb{R}^q; E, \tilde{E})$, when $\chi(\eta)$ is any excision function.

Relation (1.1.15) is also called twisted homogeneity of operator–valued functions.

Definition 1.1.14 *An element $a(y,\eta) \in S^\mu(U \times \mathbb{R}^q; E, \tilde{E})$, $\mu \in \mathbb{R}$, is called classical if there exist functions $a_{(\mu-j)}(y,\eta) \in S^{(\mu-j)}(U \times (\mathbb{R}^q \setminus \{0\}); E, \tilde{E})$ for all $j \in \mathbb{N}$ such that $a(y,\eta) - \sum_{j=0}^N \chi(\eta)a_{(\mu-j)}(y,\eta) \in S^{\mu-(N+1)}(U \times \mathbb{R}^q, E, \tilde{E})$ for all $N \in \mathbb{N}$. The subspace of all classical symbols in $S^\mu(U \times \mathbb{R}^q; E, \tilde{E})$ is denoted by $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \tilde{E})$.*

Similarly to the case of scalar symbols above a classical symbol $a(y,\eta)$ has unique homogeneous components $a_{(\mu-j)}(y,\eta) =: \sigma_\wedge^{\mu-j}(a)(y,\eta)$, $j \in \mathbb{N}$. The space $S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \tilde{E})$ can be endowed with a natural Fréchet topology, cf. [66].

The subspaces of symbols with constant coefficients $S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; E, \tilde{E})$ are closed in $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \tilde{E})$, and we have analogues of relation (1.1.11).

Remark 1.1.15 *Let $a_{(\mu)}(y,\eta) \in S^{(\mu)}(U \times (\mathbb{R}^q \setminus \{0\}); E, \tilde{E})$ and $\chi(\eta)$ be an excision function. Then we have $(\chi a_{(\mu)})(y,\eta) \in S^\mu_{\mathrm{cl}}(U \times \mathbb{R}^q; E, \tilde{E})$ and $\sigma^\mu_\wedge(\chi a_{(\mu)}) = a_{(\mu)}$.*

Asymptotic sums of symbols are defined for the operator–valued case in a similar manner as for scalar symbols above, and we have an immediate analogue of Theorem 1.1.10 for operator–valued symbols. The evident explicit formulation is left to the reader.

Definitions 1.1.12, 1.1.14 have generalisations to the case when $\tilde{E}$ is a Fréchet space, written as a projective limit $\tilde{E} = \varprojlim_{k\in\mathbb{N}} \tilde{E}^k$ of Hilbert spaces with continuous embeddings $\tilde{E}^{k+1} \hookrightarrow \tilde{E}^k$ for all k, with groups $\{\tilde{\kappa}_{\lambda,k}\}_{\lambda\in\mathbb{R}_+}$, acting on $\tilde{E}^k$, with the property $\tilde{\kappa}_{\lambda,k+1}u = \tilde{\kappa}_{\lambda,k}u$ for all $\lambda \in \mathbb{R}_+$, $u \in \tilde{E}^{k+1}$. We shortly write $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+} = \{\tilde{\kappa}_{\lambda,0}\}_{\lambda\in\mathbb{R}_+}$ also for the induced action on $\tilde{E}^k$. We then have the symbol spaces $S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \tilde{E}^{k+1})$, $k \in \mathbb{N}$, with continuous embeddings to the corresponding spaces with $\tilde{E}^k$, and we define

$$S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \tilde{E}) = \varprojlim_{k\in\mathbb{N}} S^\mu_{(\mathrm{cl})}(U \times \mathbb{R}^q; E, \tilde{E}^k).$$

Proposition 1.1.16 *Assume $p(y,\xi,\eta) \in S^\mu(\Omega_y \times \mathbb{R}^{n+q}_{\xi,\eta})$, $\mu \in \mathbb{R}$, $\Omega \subseteq \mathbb{R}^q$ open, and define $a(y,\eta) = \mathrm{Op}_x(p)(y,\eta)$. Set $E = H^s(\mathbb{R}^n)$, $\tilde{E} = H^{s-\mu}(\mathbb{R}^n)$,*

$s \in \mathbb{R}$, with $(\kappa_\lambda u)(x) = \lambda^{\frac{n}{2}} u(\lambda x)$ for $\lambda \in \mathbb{R}_+$, u in E or $\tilde{E}$. Then we have $a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; E, \tilde{E})$.

Proof. We have to show the symbol estimates (1.1.14). Let us consider for simplicity the case that the symbol p is independent of y. The general case is easy and left to the reader. First let $\beta = 0$. A trivial calculation shows that

$$\kappa_\lambda^{-1} \operatorname{Op}_x(p)(\eta)\kappa_\lambda = \operatorname{Op}_x(p_\lambda)(\eta)$$

for $p_\lambda(\xi,\eta) = p(\lambda\xi,\eta)$. Inserting $\lambda = [\eta]$ and using Proposition 1.1.11 we obtain for $\|\cdot\| = \|\cdot\|_{\mathcal{L}(H^s(\mathbb{R}^n), H^{s-\mu}(\mathbb{R}^n))}$

$$\|\kappa^{-1}(\eta) \operatorname{Op}_x(p)(\eta)\kappa(\eta)\| = \|\operatorname{Op}_x(p_{[\eta]})(\eta)\| \le \sup_{\xi\in\mathbb{R}^n} |p([\eta]\xi,\eta)|\langle\xi\rangle^{-\mu}. \quad (1.1.16)$$

By virtue of $|p(\xi,\eta)| \le c\langle\xi,\eta\rangle^\mu$ for a constant $c > 0$ we get

$$\sup_{\xi\in\mathbb{R}^n} |p([\eta]\xi,\eta)|\langle\xi\rangle^{-\mu} \le c \sup_{\xi\in\mathbb{R}^n} \langle[\eta]\xi,\eta\rangle^\mu\langle\xi\rangle^{-\mu} \le \tilde{c}\langle\eta\rangle^\mu$$

for another constant $\tilde{c} > 0$. This is just the symbol estimate for $\beta = 0$. For arbitrary β we can proceed in an analogous manner, using the fact that $D_\eta^\beta p(\xi,\eta)$ is of order $\mu - |\beta|$ in (ξ,η). $\square$

Remark 1.1.17 *Under the assumptions of Proposition* 1.1.16 *we have even a more precise result, namely*

$$\|\kappa^{-1}(\eta)\{D_y^\alpha D_\eta^\beta a(y,\eta)\}\kappa(\eta)\|_{\mathcal{L}(H^s(\mathbb{R}^n), H^{s-\mu+|\beta|}(\mathbb{R}^n))} \le c\langle\eta\rangle^{\mu-|\beta|} \quad (1.1.17)$$

for all $s \in \mathbb{R}$, $\alpha,\beta \in \mathbb{N}^q$, $y \in K$ for arbitrary $K \subset\subset \Omega$, with constants $c = c(\alpha,\beta,K) > 0$.

This is an easy consequence of Proposition 1.1.16 and of $D_y^\alpha D_\eta^\beta p(y,\xi,\eta) \in S^{\mu-|\beta|}(\Omega \times \mathbb{R}^{n+q}_{\xi,\eta})$. The symbol estimates (1.1.17) are also satisfied for symbols $p(x,y,\xi,\eta) \in S^\mu(\mathbb{R}^n_x \times \Omega_y \times \mathbb{R}^{n+q}_{\xi,\eta})$ that are independent of x for $|x| > R$ for some $R > 0$, cf. [66].

1.1.3 Elements of the standard calculus

Given a symbol $a(x,x',\xi) \in S^\mu(\Omega\times\Omega\times\mathbb{R}^n)$, $\Omega \subseteq \mathbb{R}^n$ open, the associated operator $\mathrm{Op}(a)$, cf. formula (1.1.1), is also interpreted as a so–called oscillatory integral, formally written as a double integral

$$\mathrm{Op}(a)u(x) = \iint e^{i(x-x')\xi} a(x,x',\xi)u(x')\, dx' đ\xi, \tag{1.1.18}$$

first for $u \in C_0^\infty(\Omega)$. This is not convergent in the usual sense unless $\mu < -n$. However if $\omega(\xi) \in C_0^\infty(\mathbb{R}^n)$ is a function with $\omega(\xi) = 1$ in a neighbourhood of $\xi = 0$, then the limit

$$\lim_{\varepsilon\to 0} \iint e^{i(x-x')\xi} a(x,x',\xi)\omega_\varepsilon(\xi)u(x')\, dx' đ\xi \tag{1.1.19}$$

for $\omega_\varepsilon(\xi) = \omega(\varepsilon\xi)$ exists for arbitrary $a(x,x',\xi) \in S^\mu(\Omega\times\Omega\times\mathbb{R}^n)$, $\mu \in \mathbb{R}$, and it is independent of the choice of ω. A proof of this result may be found in any textbook on pseudo–differential operators. The meaning of (1.1.18) for general $a(x,x',\xi)$ is just (1.1.19), and it coincides with (1.1.1) above, especially with the usual double integral for $\mu < -n$.

$\mathrm{Op}(a)$ is called the pseudo–differential operator in Ω associated with the symbol (or amplitude function) $a(x,x',\xi)$. We set

$$L^\mu(\Omega) = \{\mathrm{Op}(a) :\ a \in S^\mu(\Omega\times\Omega\times\mathbb{R}^n)\}, \tag{1.1.20}$$
$$L^\mu_{\mathrm{cl}}(\Omega) = \{\mathrm{Op}(a) :\ a \in S^\mu_{\mathrm{cl}}(\Omega\times\Omega\times\mathbb{R}^n)\}. \tag{1.1.21}$$

(1.1.21) is a subspace of (1.1.20), consisting of the so–called classical pseudo–differential operators. In future, if a notation or result is valid both for general and classical symbols (or operators), we write subscript (cl). In other words, $L^\mu_{(\mathrm{cl})} = \{\mathrm{Op}(a) :\ a \in S^\mu_{(\mathrm{cl})}\}$.

We now turn to a list of further definitions and standard results on pseudo–differential operators that we need below. For proofs (as far as they are not given here) and more details we refer, for instance, to [27], [33], [90].

First it is easy to check that every $A \in L^\mu(\Omega)$ induces a continuous operator $A : C_0^\infty(\Omega) \to C^\infty(\Omega)$. Let $K_A(x,x')$ be its distributional kernel, i.e., the element $K_A \in \mathcal{D}'(\Omega\times\Omega)$ with $\langle A\varphi,\psi\rangle = \langle K_A, \varphi\otimes\psi\rangle$ for arbitrary $\varphi,\psi \in C_0^\infty(\Omega)$, with $\langle\cdot,\cdot\rangle$ being the pairing between distributions and test functions or $\langle f,g\rangle = \int f(x)g(x)\,dx$. Notice that for $\mu + k < -n$, $k \in \mathbb{N}$, we have

$$K_A(x,x') = \int e^{i(x-x')\xi} a(x,x',\xi)\, đ\xi \in C^k(\Omega);$$

then $Au(x) = \int K_A(x,x')u(x')\,dx'$. Another property is the pseudo–locality: For every $\chi(x,x') \in C^\infty(\Omega\times\Omega)$ which vanishes in a neighbourhood of $\operatorname{diag}\Omega\times\Omega = \{(x,x) : x \in \Omega\}$ we have $\chi K_A \in C^\infty(\Omega\times\Omega)$. In other words, if $\omega(x,x') \in C^\infty(\Omega\times\Omega)$ is a function which equals 1 in a neighbourhood of $\operatorname{diag}\Omega\times\Omega$, we have $\operatorname{Op}(a) - \operatorname{Op}(\omega a) \in L^{-\infty}(\Omega)$. Here $L^{-\infty}(\Omega)$ denotes the space of all integral operators in Ω with kernel in $C^\infty(\Omega\times\Omega)$; this coincides with $\bigcap_{\mu\in\mathbb{R}} L^\mu(\Omega)$.

Let

$$C \subset \Omega\times\Omega, \qquad \operatorname{diag}\Omega\times\Omega \subset \operatorname{int} C, \tag{1.1.22}$$

and suppose that C is relatively closed, with

$$C \cap \pi_i^{-1}(M) \text{ compact, } i = 1,2, \text{ for every compact } M \subset \Omega, \tag{1.1.23}$$

where $\pi_i : \Omega\times\Omega \to \Omega$ is the projection to the ith component, i.e., $\pi_1(x,x') = x$, $\pi_2(x,x') = x'$. We call an operator $A \in L^\mu(\Omega)$ properly supported if the support of its distributional kernel in $\Omega\times\Omega$ is contained in a set C with these properties.

Remark 1.1.18 *A properly supported pseudo–differential operator A induces continuous operators*

$$A : C_0^\infty(\Omega) \to C_0^\infty(\Omega), \qquad A : C^\infty(\Omega) \to C^\infty(\Omega).$$

Definition 1.1.19 *An element $a_L(x,\xi) \in S^\mu_{(\mathrm{cl})}(\Omega_x\times\mathbb{R}^n)$ ($a_R(x',\xi) \in S^\mu_{(\mathrm{cl})}(\Omega_{x'}\times\mathbb{R}^n)$) is called a left (right) symbol of an operator $A \in L^\mu_{(\mathrm{cl})}(\Omega)$ if $A = \operatorname{Op}(a_L) \mod L^{-\infty}(\Omega)$ ($A = \operatorname{Op}(a_R) \mod L^{-\infty}(\Omega)$).*

Theorem 1.1.20 *Let $A \in L^\mu_{(\mathrm{cl})}(\Omega)$ be properly supported and set*

$$\sigma(A)(x,\xi) = e^{-ix\xi} A_{x'}(e^{ix'\xi}), \qquad \xi \in \mathbb{R}^n, \tag{1.1.24}$$

where $A_{x'}$ means the action of A with respect to the variable x' (that is possible by Remark 1.1.18). Then $\sigma(A)(x,\xi) \in S^\mu_{(\mathrm{cl})}(\Omega\times\mathbb{R}^n)$, and we have $A = \operatorname{Op}(\sigma(A))$ (i.e., $\sigma(A)$ is a left symbol of A).

Theorem 1.1.21 *Every operator $A \in L^\mu_{(\mathrm{cl})}(\Omega)$ has a left and a right symbol $a_L(x,\xi) \in S^\mu_{(\mathrm{cl})}(\Omega_x\times\mathbb{R}^n)$ and $a_R(x',\xi) \in S^\mu_{(\mathrm{cl})}(\Omega_{x'}\times\mathbb{R}^n)$, respectively. If A*

is written as $\mathrm{Op}(a)$ *for an* $a(x,x',\xi) \in S^{\mu}_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^n)$*, then there are asymptotic expansions,*

$$a_L(x,\xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} D_\xi^\alpha \partial_{x'}^\alpha a(x,x',\xi)|_{x'=x},$$
$$a_R(x',\xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} (-D_\xi)^\alpha \partial_x^\alpha a(x,x',\xi)|_{x=x'}.$$

The left and right symbols are $\bmod S^{-\infty}(\Omega \times \mathbb{R}^n)$ *uniquely determined by* A*.*

Theorem 1.1.22 *The formal adjoint* A^* *of an* $A \in L^{\mu}_{(\mathrm{cl})}(\Omega)$*, defined by* $(Au,v)_{L^2(\Omega)} = (u, A^* v)_{L^2(\Omega)}$ *for all* $u,v \in C_0^\infty(\Omega)$*, belongs to* $L^{\mu}_{(\mathrm{cl})}(\Omega)$*. If* $a(x,\xi)$ *is a left symbol of* A *then every left symbol* $a^*(x,\xi)$ *of* A^* *has the asymptotic expansion*

$$a^*(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} D_\xi^\alpha \partial_x^\alpha \overline{a(x,\xi)}.$$

Theorem 1.1.23 *Let* $A \in L^{\mu}_{(\mathrm{cl})}(\Omega)$*,* $B \in L^{\mu}_{(\mathrm{cl})}(\Omega)$*, and suppose that* A *or* B *is properly supported. Then we have* $AB \in L^{\mu+\nu}_{(\mathrm{cl})}(\Omega)$*. If* $a(x,\xi)$ *and* $b(x,\xi)$ *are left symbols of* A *and* B*, respectively, then every left symbol of* AB *has the asymptotic expansion*

$$a\#b(x,\xi) \sim \sum_\alpha \frac{1}{\alpha!} (D_\xi^\alpha a(x,\xi)) \partial_x^\alpha b(x,\xi)$$

In the sequel we denote by $a\#b$ any fixed choice of a symbol with the latter asymptotic expansion and call it the Leibniz product between a and b.

Remark 1.1.24 *Let* $A \in L^{\mu}_{(\mathrm{cl})}(\Omega)$ *and choose a properly supported* $A_0 \in L^{\mu}_{(\mathrm{cl})}(\Omega)$ *with* $A - A_0 \in L^{-\infty}(\Omega)$*. Then the map* $A \to \sigma(A_0)$ *with* σ *being defined by* (1.1.24) *induces an isomorphism*

$$L^{\mu}_{(\mathrm{cl})}(\Omega)/L^{-\infty}(\Omega) \to S^{\mu}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)/S^{-\infty}(\Omega \times \mathbb{R}^n). \tag{1.1.25}$$

Let $\chi : \Omega \to \tilde{\Omega}$ be a diffeomorphism, $\Omega, \tilde{\Omega} \subseteq \mathbb{R}^n$ open. Then we have the function pull–backs $\chi^* : C_0^\infty(\tilde{\Omega}) \to C_0^\infty(\Omega)$, $C^\infty(\tilde{\Omega}) \to C^\infty(\Omega)$. To an operator $A \in L^{\mu}_{(\mathrm{cl})}(\Omega)$, $A : C_0^\infty(\Omega) \to C^\infty(\Omega)$, we can define the operator push–forward $\chi_* A = (\chi^*)^{-1} A \chi^* : C_0^\infty(\tilde{\Omega}) \to C^\infty(\tilde{\Omega})$.

Denote by $d\chi(x)$ the Jacobian matrix of χ at the point x.

Theorem 1.1.25 *The operator push–forward χ_* under a diffeomorphism $\chi : \Omega \to \tilde{\Omega}$ induces isomorphisms*

$$\chi_* : L^\mu_{(\mathrm{cl})}(\Omega) \to L^\mu_{(\mathrm{cl})}(\tilde{\Omega})$$

for all $\mu \in \mathbb{R}$. This gives rise (via (1.1.25)) to isomorphisms

$$S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)/S^{-\infty}(\Omega \times \mathbb{R}^n) \to S^\mu_{(\mathrm{cl})}(\tilde{\Omega} \times \mathbb{R}^n)/S^{-\infty}(\tilde{\Omega} \times \mathbb{R}^n)$$

for all $\mu \in \mathbb{R}$. If $a_L(x, \xi) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ is a left symbol of $A \in L^\mu_{(\mathrm{cl})}(\Omega)$ and $\tilde{a}_L(y, \eta) \in S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n)$ a left symbol of $\tilde{A} = \chi_* A$, then we have

$$\tilde{a}_L(\chi(x), {}^T d\chi(x)^{-1}\xi) = a_L(x, \xi) \mod S^{-1}_{(\mathrm{cl})}(\Omega \times \mathbb{R}^n). \tag{1.1.26}$$

Remark 1.1.26 *As a consequence of (1.1.26) we see that the homogeneous principal symbol $a_{(\mu)}(x, \xi)$ of order μ of an operator $A \in L^\mu_{\mathrm{cl}}(\Omega)$ is invariant in the sense $\tilde{a}_{(\mu)}(\chi(x), {}^T d\chi(x)^{-1}\xi) = a_{(\mu)}(x, \xi)$ when $\tilde{a}_{(\mu)}(y, \eta)$ denotes the homogeneous principal symbol of $\chi_* A \in L^\mu_{\mathrm{cl}}(\tilde{\Omega})$.*

Remark 1.1.27 *Relation (1.1.26) is the consequence of the following more explicit description of $\tilde{a}_L(y, \eta)$. Define $\delta(x, z) = \chi(z) - \chi(x) - d\chi(x)(z - x)$ and set $\Phi_\alpha(x, \eta) = D^\alpha_z e^{i\delta(x,z)\eta}|_{z=x}$, $\alpha \in \mathbb{N}^n$, which is a polynomial in η of degree $\le |\alpha|/2$, where $\Phi_0(x, \eta) = 1$. Then we have an asymptotic expansion*

$$\tilde{a}_L(y, \eta)|_{y=\chi(x)} \sim \sum_\alpha \frac{1}{\alpha!}(\partial^\alpha_\xi a_L)(x, {}^T d\chi(x)\eta)\Phi_\alpha(x, \eta).$$

If X is a C^∞ manifold and $\chi : U \to \Omega$ a chart on X, we set $L^\mu_{(\mathrm{cl})}(U) = \{A = (\chi^{-1})_* B : \ B \in L^\mu_{(\mathrm{cl})}(\Omega)\}$. Let $\mathcal{U} = \{U_k\}_{k\in\mathbb{N}}$ be a locally finite open covering of X by coordinate neighbourhoods U_k, $\{\varphi_k\}_{k\in\mathbb{N}}$ a subordinate partition of unity, $\{\psi_k\}_{k\in\mathbb{N}}$ another system of functions $\psi_k \in C^\infty_0(U_k)$ with $\varphi_k\psi_k = \varphi_k$ for all k. Set

$$L^\mu_{(\mathrm{cl})}(X) = \Big\{\sum_{k\in\mathbb{N}} \varphi_k A_k \psi_k + C : \ A_k \in L^\mu_{(\mathrm{cl})}(U_k), C \in L^{-\infty}(X)\Big\},$$

where $L^{-\infty}$ is the space of all operators with kernel in $C^\infty(X \times X)$ (a Riemannian metric on X is fixed which gives an identification between kernels and operators). The spaces $L^{\mu}_{(\mathrm{cl})}(X)$, $\mu \in \mathbb{R}$, are Fréchet in a natural way. Every $A \in L^{\mu}(X)$ defines a continuous operator

$$A : C_0^{\infty}(X) \to C^{\infty}(X),$$

and it has a distributional kernel $K_A \in \mathcal{D}'(X \times X)$. Similarly as above we define properly supported operators by requiring $(\pi_i^{-1}M) \cap \operatorname{supp} K_A$ compact for every compact $M \subset X$, where $\pi_i : X \times X \to X$ projects to the ith component, $i = 1, 2$.

Then every $A \in L^{\mu}(X)$ has a decomposition $A = A_0 + C$ for properly supported $A_0 \in L^{\mu}(X)$ and $C \in L^{-\infty}(X)$. Properly supported elements $A \in L^{\mu}(X)$ induce continuous operators $C_0^{\infty}(X) \to C_0^{\infty}(X)$ and $C^{\infty}(X) \to C^{\infty}(X)$.

Theorem 1.1.28 *Let $A_j \in L^{\mu_j}(X)$, $\mu_j \in \mathbb{R}$, $j \in \mathbb{N}$, be an arbitrary sequence, $\mu_j \to -\infty$ as $j \to \infty$. Then there exists an $A \in L^{\mu}(X)$ for $\mu = \max\{\mu_j : j \in \mathbb{N}\}$ such that for every $M \in \mathbb{N}$ there exists an $N(M) \in \mathbb{N}$ with $A - \sum_{j=0}^{N} A_j \in L^{\mu-M}(X)$ for all $N \geq N(M)$, and A is unique modulo $L^{-\infty}(X)$.*

We write $A \sim \sum_{j=0}^{\infty} A_j$ and call any such A an asymptotic sum of the A_j.

Denote by T^*X the cotangent bundle of X (with the transition maps $(\chi, {}^T d\chi^{-1}) : \Omega \times \mathbb{R}^n \to \tilde{\Omega} \times \mathbb{R}^n$ for a coordinate diffeomorphism $\chi : \Omega \to \tilde{\Omega}$); then every $A \in L^{\mu}_{\mathrm{cl}}(X)$ has a homogeneous principal symbol $\sigma^{\mu}_{\psi}(A)(x, \xi)$ of order μ which is a function in $C^{\infty}(T^*X \setminus 0)$ ($\setminus 0$ indicates minus the zero section that corresponds locally to $\xi = 0$).

Let us denote by $S^{(\mu)}(T^*X \setminus 0)$ the subspace of all $p(x, \xi) \in C^{\infty}(T^*X \setminus 0)$ with $p(x, \lambda\xi) = \lambda^{\mu} p(x, \xi)$ for all $\lambda \in \mathbb{R}_+$, $(x, \xi) \in T^*X \setminus 0$. The space $S^{(\mu)}(T^*X \setminus 0)$ is Fréchet in the topology induced by $C^{\infty}(T^*X \setminus 0)$. Then $A \to \sigma^{\mu}_{\psi}(A)$ is a linear continuous operator $\sigma : L^{\mu}_{\mathrm{cl}}(X) \to S^{(\mu)}(T^*X \setminus 0)$.

Theorem 1.1.29 *The sequence*

$$0 \to L^{\mu-1}_{\mathrm{cl}}(X) \xrightarrow{\iota} L^{\mu}_{\mathrm{cl}}(X) \xrightarrow{\sigma} S^{(\mu)}(T^*X \setminus 0) \to 0$$

(with the canonical embedding $\iota : L^{\mu-1}_{\mathrm{cl}}(X) \to L^{\mu}_{\mathrm{cl}}(X)$) is exact and there exists a linear continuous operator $\mathrm{op} : S^{(\mu)}(T^*X \setminus 0) \to L^{\mu}_{\mathrm{cl}}(X)$ *such that* $\sigma \circ \mathrm{op} = \mathrm{id}$.

Every choice of a map op that is right inverse to σ is called an operator convention . Given an element $p_{(\mu)}(x,\xi) \in S^{(\mu)}(T^*X \setminus 0)$ with the system $p_{(\mu),k}(x,\xi)$ of local representatives with respect to charts $\chi_k : U_k \to \Omega_k \subseteq \mathbb{R}^n$ on X, $k \in \mathbb{N}$, we can set

$$\operatorname{op}(p_{(\mu)}) = \sum_{k\in\mathbb{N}} \varphi_k\{(\chi_k^{-1})_* \operatorname{Op}(\vartheta p_{(\mu),k})\}\psi_k \tag{1.1.27}$$

where $\vartheta(\xi)$ is an excision function in $\mathbb{R}^n$. The charts and the functions φ_k, ψ_k are arbitrary but fixed.

Theorem 1.1.30 *Every $A \in L^\mu(X)$ induces continuous operators $A : H^s_{\mathrm{comp}}(X) \to H^{s-\mu}_{\mathrm{loc}}(X)$ for all $s \in \mathbb{R}$. If X is compact, we have*

$$A : H^s(X) \to H^{s-\mu}(X) \tag{1.1.28}$$

for all $s \in \mathbb{R}$.

The fixed Riemannian metric on X gives rise to the space $L^2(X)$ (identified with $H^0(X)$) with a scalar product $(\cdot,\cdot)_{L^2(X)}$ when X is closed and compact. In general we have the local spaces $L^2_{\mathrm{loc}}(X)$ with a corresponding sesquilinear pairing $(\cdot,\cdot)_{L^2_{\mathrm{loc}}(X)}$. Define the formal adjoint A^* by $(Au,v)_{L^2_{\mathrm{loc}}(X)} = (u,A^*v)_{L^2_{\mathrm{loc}}(X)}$ for all $u,v \in C_0^\infty(X)$.

Theorem 1.1.31 *$A \in L^\mu_{(\mathrm{cl})}(X)$ implies $A^* \in L^\mu_{(\mathrm{cl})}(X)$. If A is classical, then $\sigma^\mu_\psi(A^*) = \overline{\sigma^\mu_\psi(A)}$.*

Theorem 1.1.32 *Let $A \in L^\mu_{(\mathrm{cl})}(X)$, $B \in L^\nu_{(\mathrm{cl})}(X)$, and suppose that A or B is properly supported. Then we have $AB \in L^{\mu+\nu}_{(\mathrm{cl})}(X)$. If A and B are classical then $\sigma^{\mu+\nu}_\psi(AB) = \sigma^\mu_\psi(A)\sigma^\nu_\psi(B)$.*

Definition 1.1.33 *An operator $A \in L^\mu_{\mathrm{cl}}(X)$ is called elliptic of order μ if its homogeneous principal symbol $\sigma^\mu_\psi(A)$ of order μ does not vanish on $T^*X \setminus 0$.*

Theorem 1.1.34 *Let X be a closed compact C^∞ manifold and let $A \in L^\mu_{\mathrm{cl}}(X)$ be elliptic. Then the operator* (1.1.28) *is a Fredholm operator for every $s \in \mathbb{R}$. Conversely, if* (1.1.28) *is Fredholm for an $s = s_0 \in \mathbb{R}$ then A is elliptic and hence* (1.1.28) *is Fredholm for all $s \in \mathbb{R}$.*

Theorem 1.1.35 *Let X be a closed compact C^∞ manifold and let $A \in L^\mu_{\mathrm{cl}}(X)$ be elliptic. Then there is a parametrix $P \in L^{-\mu}_{\mathrm{cl}}(X)$ in the sense*

$$PA - 1, AP - 1 \in L^{-\infty}(X). \tag{1.1.29}$$

In order to construct P we first set $\tilde{P} = \mathrm{op}(\sigma^\mu_\psi(A)^{-1})$ in the sense of Theorem 1.1.29 above, cf. also formula (1.1.27). Then, by Theorem 1.1.29 and Theorem 1.1.32 we see that $C = \tilde{P}A - 1 \in L^{-1}_{\mathrm{cl}}(X)$. By virtue of Theorem 1.1.28 we can form an operator $D \in L^{-1}_{\mathrm{cl}}(X)$ as $D \sim \sum_{j=0}^\infty (-1)^j C^j$. Then $(1+D)(1+C) = 1 \mod L^{-\infty}(X)$ and hence $(1+D)\tilde{P}A = 1 \mod L^{-\infty}(X)$. Thus $P = (1+D)\tilde{P}$ is an element of $L^{-\mu}_{\mathrm{cl}}(X)$ that satisfies the first relation in (1.1.29). In an analoguous manner we find a $Q \in L^{-\mu}_{\mathrm{cl}}(X)$ with $AQ - 1 \in L^{-\infty}(X)$. An easy algebraic calculation then shows that $P = Q \mod L^{-\infty}(X)$ and hence P also satisfies the second relation in (1.1.29).

Corollary 1.1.36 *Let $A \in L^\mu_{\mathrm{cl}}(X)$ be elliptic. Then $\ker_s A = \{u \in H^s(X) : Au = 0\}$ is a finite–dimensional subspace $N_+ \subset C^\infty(X)$, independent of s. Moreover, if we set $\mathrm{im}_s A = \{f \in H^{s-\mu}(X) : \ f = Au$ for an $u \in H^s(X)\}$, there is a finite–dimensional subspace $N_- \subset C^\infty(X)$ such that $\mathrm{im}_s A \cap N_- = \{0\}$ and $\mathrm{im}_s A + N_- = H^{s-\mu}(X)$ for all $s \in \mathbb{R}$. In other words we may write $\ker_s = \ker$, $\mathrm{im}_s = \mathrm{im}$, and the index*

$$\mathrm{ind}\, A = \dim\ker A - \dim\operatorname{coker} A$$

is independent of s ($\operatorname{coker} A = H^{s-\mu}(X)/\,\mathrm{im}\, A$)

1.1.4 Parameter–dependent operators

Let us set $\Lambda = \mathbb{R}^l$ and consider symbols $a(x, x', \xi, \lambda) \in S^\mu_{\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^{n+l})$ for an open set $\Omega \subseteq \mathbb{R}^n$, where $(\xi, \lambda) \in \mathbb{R}^{n+l}$ plays the role of the covariable. Then, for every fixed $\lambda_0 \in \Lambda$ we have $a(x, x', \xi, \lambda_0) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^n)$, and we can form the spaces of parameter–dependent pseudo–differential operators

$$L^\mu_{(\mathrm{cl})}(\Omega; \Lambda) = \{\mathrm{Op}_x(a)(\lambda) : \ a(x, x', \xi, \lambda) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^{n+l}_{\xi,\lambda})\}.$$

The basic constructions with pseudo–differential operators of the preceding section carry over to the parameter–dependent case, and we do not repeat all details when they are completely analogous. Note that

$$L^{-\infty}(\Omega; \Lambda) = \mathcal{S}(\Lambda, L^{-\infty}(\Omega))$$

($L^{-\infty}(\Omega)$ is equipped with the Fréchet topology from the bijection $L^{-\infty}(\Omega) \cong C^\infty(\Omega \times \Omega)$).

Let us consider a relatively closed subset

$$C \subset \Omega \times \Omega \qquad \text{with } \operatorname{diag} \Omega \times \Omega \subset \operatorname{int} C \tag{1.1.30}$$

and suppose

$$C \cap \pi_i^{-1}(M) \text{ compact, } i = 1, 2, \text{ for every compact } M \subset \Omega, \tag{1.1.31}$$

where $\pi_i : \Omega \times \Omega \to \Omega$ is the projection to the i th component. We call an operator family $A(\lambda) \in L^\mu_{(\mathrm{cl})}(\Omega; \Lambda)$ properly supported if the support of its distributional kernel in $\Omega \times \Omega$ is contained in a set C with (1.1.30), (1.1.31), where C is independent of $\lambda \in \Lambda$. Analogously to the case without parameters we can always write $A(\lambda) = A_0(\lambda) + G(\lambda)$ for arbitrary $A(\lambda) \in L^\mu_{(\mathrm{cl})}(\Omega; \Lambda)$ with properly supported $A_0(\lambda) \in L^\mu_{(\mathrm{cl})}(\Omega; \Lambda)$ and a $G(\lambda) \in L^{-\infty}(\Omega; \Lambda)$. It suffices to set $A_0(\lambda) = \mathrm{Op}_x(\omega a)(\lambda)$ when $A(\lambda) = \mathrm{Op}_x(a)(\lambda)$ for an amplitude function $a(x, x', \xi, \lambda) \in S^\mu_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^{n+l})$, where $\omega(x, x') \in C^\infty(\Omega \times \Omega)$ is an arbitrary properly supported function that equals 1 in an open neighbourhood of $\operatorname{diag} \Omega \times \Omega$. Let $L^\mu_{(\mathrm{cl})}(\Omega; \Lambda)_C$ be the subspace of all $A(\lambda) \in L^\mu_{(\mathrm{cl})}(\Omega; \Lambda)$ whose distributional kernels are supported by C, for all $\lambda \in \Lambda$.

Theorem 1.1.37 *The space of functions*

$$\{\sigma(A)(x, \xi, \lambda) = e^{-ix\xi} A(\lambda) e^{ix\xi} : \ A(\lambda) \in L^\mu_{(\mathrm{cl})}(\Omega; \Lambda)_C\}$$

is a closed subspace $S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^{n+l})_C$ *of* $S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^{n+l})$*, and*

$$\mathrm{Op}_x : S^\mu_{(\mathrm{cl})}(\Omega \times \mathbb{R}^{n+l}) \to L^\mu_{(\mathrm{cl})}(\Omega; \Lambda)_C$$

is an isomorphism.

If X is a C^∞ manifold and $\chi : U \to \Omega$ a chart on X, we set $L^\mu_{(\mathrm{cl})}(U; \Lambda) = \{A(\lambda) = (\chi^{-1})_* B(\lambda) : \ B(\lambda) \in L^\mu_{(\mathrm{cl})}(\Omega; \Lambda)\}$. Let $\mathcal{U} = \{U_k\}_{k \in \mathbb{N}}$ be a locally finite open covering of X by coordinate neighbourhoods U_k, $\{\varphi_k\}_{k \in \mathbb{N}}$ a subordinate partion of unity, $\{\psi_k\}_{k \in \mathbb{N}}$ another system of functions $\psi_k \in C_0^\infty(U_k)$ with $\varphi_k \psi_k = \varphi_k$ for all k. Set

$$L^\mu_{(\mathrm{cl})}(X; \Lambda) = \Big\{ \sum_{k \in \mathbb{N}} \varphi_k A_k(\lambda) \psi_k + C(\lambda) : \ A_k(\lambda) \in L^\mu_{(\mathrm{cl})}(U_k; \Lambda), \\ C(\lambda) \in L^{-\infty}(X; \Lambda) \Big\} \tag{1.1.32}$$

If $A(\lambda) \in L^{\mu}_{\mathrm{cl}}(X;\Lambda)$ we denote by $\sigma^{\mu}_{\psi,\mathrm{p}}(A)$ its parameter–dependent homogeneous principal symbol of order μ (in $(\xi,\lambda) \neq 0$). This is an element of $C^{\infty}(T^*X \times \Lambda \setminus 0)$, where 0 stands for $(\xi,\lambda) = 0$.

Remark 1.1.38 *The space $L^{\mu}_{(\mathrm{cl})}(X;\Lambda)$ is Fréchet in a natural way. The construction of its topology is similar as above in the case without parameters. We then have*

$$L^{\mu}_{(\mathrm{cl})}(X;\Lambda) \subset C^{\infty}(\Lambda, L^{\mu}_{(\mathrm{cl})}(X))$$

and $D^{\alpha}_{\lambda}A(\lambda) \in L^{\mu-|\alpha|}_{(\mathrm{cl})}(X;\Lambda)$ for each multi–index $\alpha \in \mathbb{N}^l$.

We now assume that X is compact. Furthermore, for simplicity we consider classical operators.

Definition 1.1.39 *An $A(\lambda) \in L^{\mu}_{\mathrm{cl}}(X;\Lambda)$ is called parameter–dependent elliptic of order μ, if the parameter–dependent homogeneous principal symbol $\sigma^{\mu}_{\psi,\mathrm{p}}(A)(x,\xi,\lambda)$ of order μ does not vanish on $T^*X \times \Lambda \setminus 0$.*

Example 1.1.40 *Let X be compact, $\{U_1,\dots,U_N\}$ an open covering of X by coordinate neighbourhoods and $\{\varphi_1,\dots,\varphi_N\}$ as above. Set*

$$R^{\mu}(\lambda) = \sum_{k=1}^{N} \varphi_k A_k(\lambda)\psi_k,$$

cf. notation (1.1.32), with operators $A_k(\lambda)$ whose local symbols are $(c + |\xi|^2 + |\lambda|^2)^{\mu/2}$ for a constant $c > 0$. Then $R^{\mu}(\lambda) \in L^{\mu}_{\mathrm{cl}}(X;\Lambda)$ is parameter–dependent elliptic (of order μ).

Theorem 1.1.41 *Let $A(\lambda) \in L^{\mu}_{\mathrm{cl}}(X;\Lambda)$ be parameter–dependent elliptic of order μ. Then there exists an element $B(\lambda) \in L^{-\mu}_{\mathrm{cl}}(X;\Lambda)$ such that*

$$A(\lambda)B(\lambda) - 1, B(\lambda)A(\lambda) - 1 \in L^{-\infty}(X;\Lambda). \tag{1.1.33}$$

Moreover,

$$A(\lambda) : H^s(X) \to H^{s-\mu}(X) \tag{1.1.34}$$

is a family of Fredholm operators of index zero, and there is a constant $c > 0$ such that (1.1.34) is an isomorphism for each $\lambda \in \Lambda$, $|\lambda| \geq c$ and every $s \in \mathbb{R}$.

The family $B(\lambda) \in L^{-\mu}_{\mathrm{cl}}(X;\Lambda)$ is called a parameter–dependent parametrix of $A(\lambda) \in L^{\mu}_{\mathrm{cl}}(X;\Lambda)$. It can be constructed in an analogous manner as for the case without parameters. Since parameter–dependent ellipticity of $A(\lambda)$ entails usual ellipticity of $A(\lambda_0)$ for each λ_0, the Fredholm property of (1.1.34) for every $\lambda \in \Lambda$, $s \in \mathbb{R}$ is clear. From (1.1.33) it follows that the operators (1.1.34) are isomorphisms for $|\lambda|$ sufficiently large. Thus, in particular, $\operatorname{ind} A(\lambda) = 0$ is a consequence of the homotopy invariance of the index.

Let us point out that when $A(\lambda) \in L^{\mu}_{\mathrm{cl}}(X;\Lambda)$ is paramter–dependent elliptic, then we have $\sigma^{\mu}_{\psi,\mathrm{p}}(A)(x,\xi,\lambda) \neq 0$ for all $(x,\xi) \in T^*X$ (including $\xi = 0$) for $\lambda \neq 0$.

1.2 Calculus with operator–valued symbols

1.2.1 Pseudo–differential operators

Let E and $\tilde{E}$ be Hilbert spaces with group actions $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$, respectively. Consider operator–valued symbols $a(y,y',\eta) \in S^{\mu}_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^q; E, \tilde{E})$, $\mu \in \mathbb{R}$, $\Omega \subseteq \mathbb{R}^q$ open, and associated pseudo–differential operators

$$\operatorname{Op}(a)u(y) = \iint e^{i(y-y')\eta} a(y,y',\eta)u(y')\, dy' đ\eta \tag{1.2.1}$$

which induce continuous operators $\operatorname{Op}(a) : C_0^\infty(\Omega, E) \to C^\infty(\Omega, \tilde{E})$.

The scalar theory of pseudo–differential operators, cf. Section 1.1.3 above, has a natural generalisation to the operator–valued case. Conversely, the results of the operator–valued theory specialise to the scalar situation when we set $E = \tilde{E} = \mathbb{C}$ and $\kappa_\lambda = \tilde{\kappa}_\lambda = \mathrm{id}_{\mathbb{C}}$ for all $\lambda \in \mathbb{R}_+$.

The expression (1.2.1) is interpreted in the sense of oscillatory integrals, similarly to (1.1.19). We set

$$L^{\mu}_{(\mathrm{cl})}(\Omega; E, \tilde{E}) = \{\operatorname{Op}(a) :\ a \in S^{\mu}_{(\mathrm{cl})}(\Omega \times \Omega \times \mathbb{R}^q; E, \tilde{E})\};$$

the elements of $L^{\mu}_{(\mathrm{cl})}(\Omega; E, \tilde{E})$ are called (classical) pseudo–differential operators with operator–valued symbols (in the abstract edge approach).

Consider the space $\mathcal{L}(E, \tilde{E})$ of continuous operators $E \to \tilde{E}$ in the norm topology. Every $A \in L^{\mu}(\Omega; E, \tilde{E})$ has an $\mathcal{L}(E, \tilde{E})$–valued distributional kernel $K_A(y,y') \in \mathcal{D}'(\Omega \times \Omega, \mathcal{L}(E, \tilde{E})$ $(= \mathcal{L}(C_0^\infty(\Omega \times \Omega), \mathcal{L}(E, \tilde{E})))$.

In (1.2.1) we assumed $u \in C_0^\infty(\Omega, E)$. We can also insert $u \in C_0^\infty(\Omega)$. In that case we get a continuous operator

$$\mathrm{Op}(a) : C_0^\infty(\Omega) \to C^\infty(\Omega, \mathcal{L}(E, \tilde{E})).$$

There is then an associated $\mathcal{L}(E, \tilde{E})$–valued distributional kernel $K_A \in \mathcal{D}'(\Omega \times\Omega, \mathcal{L}(E, \tilde{E}))$.

Similarly to the scalar case we can introduce the notion of a properly supported operator $A \in L^\mu_{(\mathrm{cl})}(\Omega; E, \tilde{E})$. Every element $A \in L^\mu_{(\mathrm{cl})}(\Omega; E, \tilde{E})$ admits a decomposition $A = A_0 + C$, where A_0 is properly supported and $C \in L^{-\infty}(\Omega; E, \tilde{E})$.

A properly supported operator A induces continuous maps

$$\begin{aligned} A &: C_0^\infty(\Omega, E) \to C_0^\infty(\Omega, \tilde{E}), & C^\infty(\Omega, E) &\to C^\infty(\Omega, \tilde{E}), \\ A &: C_0^\infty(\Omega) \to C_0^\infty(\Omega, \mathcal{L}(E, \tilde{E})), & C^\infty(\Omega) &\to C^\infty(\Omega, \mathcal{L}(E, \tilde{E})). \end{aligned}$$

Let us define $\mathcal{W}^s_{\mathrm{comp}}(\Omega, E)$, $s \in \mathbb{R}$, as the subspace of all $u \in \mathcal{E}'(\Omega, E)$ whose extension by zero to $\mathbb{R}^q$ belongs to $\mathcal{W}^s(\mathbb{R}^q, E)$ and have compact support in Ω. The space $\mathcal{W}^s_{\mathrm{comp}}(\Omega, E)$ is an inductive limit of Hilbert spaces. Moreover, we denote by $\mathcal{W}^s_{\mathrm{loc}}(\Omega, E)$ the subspace of all $u \in \mathcal{D}'(\Omega, E)$ with $\varphi u \in \mathcal{W}^s_{\mathrm{comp}}(\Omega, E)$ for every $\varphi \in C_0^\infty(\Omega)$. The space $\mathcal{W}^s_{\mathrm{loc}}(\Omega, E)$ is Fréchet.

Theorem 1.2.1 *Every $A \in L^\mu(\Omega; E, \tilde{E})$ induces continuous operators*

$$A : \mathcal{W}^s_{\mathrm{comp}}(\Omega, E) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{E})$$

for all $s \in \mathbb{R}$. If A is properly supported, $A : \mathcal{W}^s_{\mathrm{comp}}(\Omega, E) \to \mathcal{W}^{s-\mu}_{\mathrm{comp}}(\Omega, \tilde{E})$, $\mathcal{W}^s_{\mathrm{loc}}(\Omega, E) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}}(\Omega, \tilde{E})$ are also continuous, $s \in \mathbb{R}$.

The proof is an easy consequence of a corresponding result in $\mathbb{R}^q$ for $A = \mathrm{Op}(a)$ when $a(y, y', \eta)$ has a compact support with respect to $(y, y') \in \mathbb{R}^q \times \mathbb{R}^q$, cf. [66]. In particular, for $a(\eta) \in S^\mu(\mathbb{R}^q; E, \tilde{E})$ (symbols with constant coefficients) we have the continuity globally in $\mathbb{R}^q$:

Theorem 1.2.2 *Let $a(\eta) \in S^\mu(\mathbb{R}^q; E, \tilde{E})$. Then* $\mathrm{Op}(a)$ *induces continuous operators*

$$\mathrm{Op}(a) : \mathcal{W}^{s;\delta}(\mathbb{R}^q, E) \to \mathcal{W}^{s-\mu;\delta}(\mathbb{R}^q, \tilde{E})$$

for all $s, \delta \in \mathbb{R}$.

The proof for $\delta = 0$ is very easy: We have

$$\begin{aligned}
\| \operatorname{Op}(a)u\|^2_{\mathcal{W}^{s-\mu}(\mathbb{R}^q,\tilde{E})} &= \int \langle\eta\rangle^{2(s-\mu)} \|\tilde{\kappa}^{-1}(\eta)a(\eta)(Fu)(\eta)\|^2_{\tilde{E}}\, d\eta \\
&\le \int \langle\eta\rangle^{-2\mu} \|\tilde{\kappa}^{-1}(\eta)a(\eta)\kappa(\eta)\|^2_{\mathcal{L}(E,\tilde{E})} \\
&\quad \times \langle\eta\rangle^{2s} \|\kappa^{-1}(\eta)(Fu)(\eta)\|^2_E\, d\eta \\
&\le \{\sup_{\eta\in\mathbb{R}^q} \langle\eta\rangle^{-\mu} \|\tilde{\kappa}^{-1}(\eta)a(\eta)\kappa(\eta)\|_{\mathcal{L}(E,\tilde{E})}\}^2 \|u\|^2_{\mathcal{W}^s(\mathbb{R}^q,E)}.
\end{aligned}$$

This gives us also an estimate for the operator norm, namely

$$\| \operatorname{Op}(a)\|_{\mathcal{L}(\mathcal{W}^s(\mathbb{R}^q,E),\mathcal{W}^{s-\mu}(\mathbb{R}^q,\tilde{E}))} \le \sup_{\eta\in\mathbb{R}^q} \langle\eta\rangle^{-\mu} \|\tilde{\kappa}^{-1}(\eta)a(\eta)\kappa(\eta)\|_{\mathcal{L}(E,\tilde{E})}$$

which is finite by Definition 1.1.12. A proof for general δ may be found in Dorschfeldt, Grieme and Schulze [9] or Seiler [84]. An easy tensor product argument and the operator–valued analogue of relation (1.1.11), cf. Theorem 1.1.7, allows us to reduce the proof of Theorem 1.2.1 to Theorem 1.2.2, using the following result.

Proposition 1.2.3 *The operator of multiplication $\mathcal{M}_\varphi$ by a function $\varphi \in \mathcal{S}(\mathbb{R}^q)$ induces a continuous operator $\mathcal{M}_\varphi : \mathcal{W}^s(\mathbb{R}^q, E) \to \mathcal{W}^s(\mathbb{R}^q, E)$, and $\varphi \to \mathcal{M}_\varphi$ is continuous as an operator $\mathcal{S}(\mathbb{R}^q) \to \mathcal{L}(\mathcal{W}^s(\mathbb{R}^q, E))$ for each $s \in \mathbb{R}$.*

This is a special case of a corresponding continuity result for global (in $\mathbb{R}^q$) pseudo–differential operators with operator–valued symbols, see, e.g., [84]. The proof in [84] is in fact much more general; we do not want to formulate it in full generality here.

Remark 1.2.4 *We have $\mathcal{M}_\varphi \in S^0(\mathbb{R}^q; E, E)$, and $\mathcal{M}_\varphi$ is independent of the covariable η. We shall obtain below many other examples of operator–valued symbols $p \in S^\mu_{(\mathrm{cl})}(\mathbb{R}^q; E, \tilde{E})$ that are independent of η, and it may happen that $\mu \neq 0$, cf. Example* 1.2.10 *below. In this case we have*

$$\operatorname{Op}(p)u = \iint e^{i(y-y')\eta} pu(y')\, dy' đ\eta = pu$$

for $u \in \mathcal{W}^s(\mathbb{R}^q, E)$, where p on the right hand side acts as an operator on the values of u for every fixed $y \in \mathbb{R}^q$. In other words

$$p : \mathcal{W}^s(\mathbb{R}^q, E) \to \mathcal{W}^{s-\mu}(\mathbb{R}^q, \tilde{E}) \qquad (1.2.2)$$

is continuous for each $s \in \mathbb{R}$, cf. Theorem 1.2.2.

Let E, $\tilde{E}$ be Hilbert spaces, $p : E \to \tilde{E}$ an arbitrary isomorphism and $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ given on E. Define $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$ on $\tilde{E}$ by $\tilde{\kappa}_\lambda = p\kappa_\lambda p^{-1}$, $\lambda \in \mathbb{R}_+$. Then we have $p \in S^0_{\mathrm{cl}}(\mathbb{R}^q; E, \tilde{E})$ and (1.2.2) *(for $\mu = 0$) is an isomorphism.*

As noted above the scalar pseudo–differential calculus in the sense of Section 1.1.3 extends in a natural way to the operator–valued case. Therefore we do not formulate all results in the corresponding modification. Details may be found in the books [61], [66]. Let us only consider a few typical points. First every $A \in L^\mu(\Omega; E, \tilde{E})$ has left and right symbols $a_L(y, \eta)$, $a_R(y', \eta)$ in $S^\mu(\Omega \times \mathbb{R}^q; E, \tilde{E})$, unique $\mathrm{mod}\, S^{-\infty}(\Omega \times \mathbb{R}^q; E, \tilde{E})$, and A equals $\mathrm{Op}(a_L)$ and $\mathrm{Op}(a_R)$ mod $L^{-\infty}(\Omega; E, \tilde{E})$. Especially for classical operators we have $a_L, a_R \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; E, \tilde{E})$. The homogeneous component $a_{L,(\mu)}(y, \eta)$ of $a_L(y, \eta)$ of order μ (cf. Definition 1.1.14) is called the homogeneous principal symbol of $A \in L^\mu_{\mathrm{cl}}(\Omega; E, \tilde{E})$, and we write

$$\sigma^\mu_\wedge(A)(y, \eta) = a_{L,(\mu)}(y, \eta).$$

Similarly to the scalar theory, if $A = \mathrm{Op}(a)$ is an operator with arbitrary $a(y, y', \eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \Omega \times \mathbb{R}^q; E, \tilde{E})$, we have

$$\sigma^\mu_\wedge(A)(y, \eta) = a_{(\mu)}(y, y', \eta)|_{y'=y}.$$

Theorem 1.2.5 *Let $A \in L^\mu_{(\mathrm{cl})}(\Omega; E_0, \tilde{E})$, $B \in L^\nu_{(\mathrm{cl})}(\Omega; E, \tilde{E}_0)$ and suppose that A or B is properly supported. Then we have $AB \in L^{\mu+\nu}_{(\mathrm{cl})}(\Omega; E, \tilde{E})$. If $a(y, \eta)$ and $b(y, \eta)$ are left symbols of A and B, respectively, then every left symbol of AB has the asymptotic expansion*

$$a\#b(y, \eta) \sim \sum_\alpha \frac{1}{\alpha!}(D^\alpha_\eta a(y, \eta))\partial^\alpha_y b(y, \eta).$$

In particular, for classical A and B we have

$$\sigma^{\mu+\nu}_\wedge(AB)(y, \eta) = \sigma^\mu_\wedge(A)(y, \eta)\sigma^\nu_\wedge(B)(y, \eta).$$

Remark 1.2.6 *Let $a(\eta) \in S^\mu(\mathbb{R}^q; E_0, \tilde{E})$, $b(\eta) \in S^\nu(\mathbb{R}^q; E_0, E)$. Then we have* $\mathrm{Op}(a)\,\mathrm{Op}(b) = \mathrm{Op}(ab)$.

1.2.2 Examples

We now consider some examples of operator–valued symbols as they occur in the pseudo–differential calculus on spaces with operator–valued symbols.

Let E be a Hilbert space with group action $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$. Then a symbol

$$b(y,y',\eta)\in S^\mu(\Omega\times\Omega\times\mathbb{R}^q;E,\mathbb{C}^N),\qquad N\in\mathbb{N},$$

(with $\mathbb{C}^N$ being equipped with the identity as group action, cf. Remark 1.1.13) is called an "abstract" trace symbol of order μ. Moreover, a

$$c(y,y',\eta)\in S^\mu(\Omega\times\Omega\times\mathbb{R}^q;\mathbb{C}^M,E),\qquad M\in\mathbb{N},\tag{1.2.3}$$

is called an "abstract" potential symbol of order μ. A similar notation makes sense when we replace E in (1.2.3) by a Fréchet subspace F that is a projective limit of Hilbert spaces F^k, $k\in\mathbb{N}$, where $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ acts on every F^k by restriction. The pointwise adjoint $c^*(y,y',\eta)$ (via $(c^*u,v)_{\mathbb{C}^M}=(u,cv)_E$ for all $u\in E$, $v\in\mathbb{C}^M$, with $(\cdot,\cdot)_{\mathbb{C}^M}$ and $(\cdot,\cdot)_E$ being the corresponding (sesquilinear) scalar products) gives us a potential symbol and vice versa.

Observe that when $b(y,\eta)\in S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;E,\mathbb{C}^N)$ and $c(y,\eta)\in S^\mu_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;\mathbb{C}^M,\tilde{E})$ are classical trace and potential symbols, with the groups $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}}$ and $\{\tilde{\kappa}_\lambda\}_{\lambda\in\mathbb{R}_+}$ on E and $\tilde{E}$, respectively, for the homogeneous principal symbols of order μ we have

$$\sigma^\mu_\wedge(b)(y,\lambda\eta)=\lambda^\mu\sigma^\mu_\wedge(b)(\eta)\kappa_\lambda^{-1},\qquad \sigma^\mu_\wedge(c)(y,\lambda\eta)=\lambda^\mu\tilde{\kappa}_\lambda\sigma^\mu_\wedge(c)(y,\eta)\tag{1.2.4}$$

for all $\lambda\in\mathbb{R}_+$, $(y,\eta)\in\Omega\times(\mathbb{R}^q\setminus\{0\})$.

The following example is useful in the pseudo–differential analysis of boundary value problems with the transmission property.

Example 1.2.7 *Let us fix a function $\beta(t)\in\mathcal{S}(\overline{\mathbb{R}}_+)$ $(=\mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+})$ and form the operator family $b(\eta):L^2(\mathbb{R}_+)\to\mathbb{C}$ by the expression*

$$b(\eta)u=[\eta]^{\mu-\frac{1}{2}}\int_0^\infty\beta(r[\eta])u(r)\,dr.$$

By virtue of the $b\in C^\infty(\mathbb{R}^q,\mathcal{L}(L^2(\mathbb{R}_+),\mathbb{C}))$ and the homogeneity $b(\lambda\eta)=\lambda^\mu b(\eta)\kappa_\lambda^{-1}$, $\lambda\geq 1$, $|\eta|\geq\mathrm{const}$, we get $b(\eta)\in S^\mu_{\mathrm{cl}}(\mathbb{R}^q;L^2(\mathbb{R}_+),\mathbb{C})$, cf. Remark 1.1.15. *By duality we get a symbol*

$$g(\eta)v=[\eta]^{\mu-\frac{1}{2}}\overline{\beta}(r[\eta])c,\qquad c\in\mathbb{C},$$

that belongs to $S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q;\mathbb{C},L^2(\mathbb{R}_+))$*; the homogeneity means* $g(\lambda\eta)=\lambda^{\mu}\kappa_\lambda g(\eta)$ *for* $\lambda\geq 1$, $|\eta|\geq \mathrm{const}$.

For the homogeneous principal symbols we have

$$\sigma^{\mu}_{\wedge}(b)(\eta)u=|\eta|^{\mu-\frac{1}{2}}\int\limits_0^{\infty}\beta(r|\eta|)u(r)\,dr,\qquad \eta\neq 0,$$

$$\sigma^{\mu}_{\wedge}(g)(\eta)c=|\eta|^{\mu-\frac{1}{2}}\beta(r|\eta|)c,\qquad \eta\neq 0.$$

Example 1.2.8 *Let* $H^s(\mathbb{R}^n)$, $s\in\mathbb{R}$, *be endowed with the group of isomorphisms* $(\kappa_\lambda u)(x)=\lambda^{\frac{n}{2}}u(\lambda x)$, $\lambda\in\mathbb{R}_+$. *Then the operator of multiplication* $\mathcal{M}_\varphi$ *by a function* $\varphi\in\mathcal{S}(\mathbb{R}^n)$ *belongs to* $S^0(\mathbb{R}^q;H^s(\mathbb{R}^n),H^s(\mathbb{R}^n))$ *(it has constant coefficients and is even independent of* $\eta\in\mathbb{R}^q$*). The map* $\varphi\to\mathcal{M}_\varphi$ *is continuous in the sense* $\mathcal{S}(\mathbb{R}^n)\to S^0(\mathbb{R}^q;H^s(\mathbb{R}^n),H^s(\mathbb{R}^n))$ *for each* $s\in\mathbb{R}$, *cf. [56], [66].*

If necessary, when different dimensions play a role, we also write

$$(\kappa^{(n)}_{\lambda}u)(x)=\lambda^{\frac{n}{2}}u(\lambda x),\qquad \lambda>0. \tag{1.2.5}$$

Example 1.2.9 *Consider a differential operator*

$$A(x,y,D_x,D_y)=\sum_{|\alpha|+|\beta|\leq\mu}c_{\alpha\beta}(x,y)D^{\alpha}_{x}D^{\beta}_{y}$$

in $\mathbb{R}^n\times\Omega\ni(x,y)$, $\Omega\subseteq\mathbb{R}^q$ *open, with coefficients* $c_{\alpha\beta}\in C^{\infty}(\mathbb{R}^n\times\Omega)$ *that are independent of* x *for* $|x|>R$ *for a constant* $R>0$. *Then*

$$a(y,\eta)=\sum_{|\alpha|+|\beta|\leq\mu}c_{\alpha\beta}(x,y)D^{\alpha}_{x}\eta^{\beta}:H^s(\mathbb{R}^n)\to H^{s-\mu}(\mathbb{R}^n)$$

is an element in $S^{\mu}(\Omega\times\mathbb{R}^q;H^s(\mathbb{R}^n),H^{s-\mu}(\mathbb{R}^n))$ *for all* $s\in\mathbb{R}$. *In the case that the coefficients* $c_{\alpha\beta}$ *are independent of* x *we have*

$$a(y,\eta)\in S^{\mu}_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;H^s(\mathbb{R}^n),H^{s-\mu}(\mathbb{R}^n))$$

for all $s\in\mathbb{R}$ *and* $\sigma^{\mu}_{\wedge}(y,\eta)=\sum_{|\alpha|+|\beta|=\mu}c_{\alpha\beta}(y)D^{\alpha}_{x}\eta^{\beta}$.

The following two examples are discussed in [66] in more detail, cf. also the application in Section 1.2.3 below.

Example 1.2.10 *Let $k \in \mathbb{N}$, $k \geq 1$, and consider the operator* $\mathrm{r} : \mathcal{S}(\mathbb{R}^k) \to \mathbb{C}$, *defined by* $\mathrm{r}u = u(0)$. *Then* r *extends by continuity to a map*

$$\mathrm{r} : H^s(\mathbb{R}^k) \to \mathbb{C} \tag{1.2.6}$$

for every $s \in \mathbb{R}$, $s - \frac{k}{2} > 0$, and (1.2.6) *is a symbol (with constant coefficients, even independent of η) in $S^{\frac{k}{2}}_{\mathrm{cl}}(\mathbb{R}^q; H^s(\mathbb{R}^k), \mathbb{C})$, and we have $\sigma^{\frac{k}{2}}_{\wedge}(\mathrm{r}) = \mathrm{r}$. If z is the variable in $\mathbb{R}^k$, we have more generally $\mathrm{r}D^\alpha_z \in S^{\frac{k}{2}+|\alpha|}_{\mathrm{cl}}(\mathbb{R}^q; H^s(\mathbb{R}^k), \mathbb{C})$ for $s \in \mathbb{R}$, $s - \frac{k}{2} - |\alpha| > 0$, $\alpha \in \mathbb{N}^k$ and $\sigma^{\frac{k}{2}+|\alpha|}_{\wedge}(\mathrm{r}D^\alpha_z) = \mathrm{r}D^\alpha_z$.*

In order to see (1.2.6) we first assume $u \in \mathcal{S}(\mathbb{R}^k)$ and write $u(z) = \int e^{iz\zeta}\hat{u}(\zeta)\, đ\zeta$. Then, for $-2s < -k$ we have

$$\begin{aligned} |u(0)| &= \left| \int \hat{u}(\zeta)\, đ\zeta \right| \leq \int \langle\zeta\rangle^{-s}\langle\zeta\rangle^{s}|\hat{u}(\zeta)|\, đ\zeta \\ &\leq (2\pi)^{-k}\left\{\int \langle\zeta\rangle^{-2s}\, d\zeta\right\}^{\frac{1}{2}}\left\{\int \langle\zeta\rangle^{2s}|\hat{u}(\zeta)|^2\, d\zeta\right\}^{\frac{1}{2}} \\ &\leq c\|u\|_{H^s(\mathbb{R}^k)}. \end{aligned} \tag{1.2.7}$$

This gives us the extension (1.2.6). The asserted symbol property of r is a consequence of $\mathrm{r}\kappa^{(k)}_\lambda u = \mathrm{r}\lambda^{\frac{k}{2}}u(\lambda z) = \lambda^{\frac{k}{2}}\mathrm{r}u$, $\lambda \in \mathbb{R}_+$, cf. formula (1.2.5), (1.2.7) and Remark 1.1.15. For D^α_z it suffices to observe $(D^\alpha_z\kappa^{(k)}_\lambda u)(z) = \lambda^{|\alpha|}\kappa^{(k)}_\lambda D^\alpha_z u(z)$ for all $\lambda \in \mathbb{R}_+$ and hence D^α_z is homogeneous (in the operator–valued sense) of order $|\alpha|$; also here there is no η–dependence, but $D^\alpha_z \in S^{|\alpha|}(\mathbb{R}^q; H^s(\mathbb{R}^k), H^{s-|\alpha|}(\mathbb{R}^k))$ for all $s \in \mathbb{R}$.

More generally we have $\mathrm{r}D^\alpha_z \in S^{\frac{k}{2}+|\alpha|}_{\mathrm{cl}}(\mathbb{R}^q; H^s(\mathbb{R}^{q+k}), H^{s-\frac{k}{2}-|\alpha|}(\mathbb{R}^q))$ for $s \in \mathbb{R}$, $s - \frac{k}{2} - |\alpha| > 0$, $\alpha \in \mathbb{N}^k$, and $\sigma^{\frac{k}{2}+|\alpha|}_{\wedge}(\mathrm{r}D^\alpha_z) = \mathrm{r}D^\alpha_z$.

Example 1.2.11 *Let $q, k \in \mathbb{N}$, $q \geq 1$, $k \geq 1$, and consider the operator* $\mathrm{r} : \mathcal{S}(\mathbb{R}^{q+k}) \to \mathcal{S}(\mathbb{R}^q)$, *defined by* $(\mathrm{r}u)(y) = u(y, 0)$, $u(y, z) \in \mathcal{S}(\mathbb{R}^{q+k})$. *Then* r *extends by continuity to a map*

$$\mathrm{r} : H^s(\mathbb{R}^{q+k}) \to H^{s-\frac{k}{2}}(\mathbb{R}^q)$$

for every $s \in \mathbb{R}$ with $s - \frac{k}{2} > 0$, and we have $\mathrm{r} \in S^{\frac{k}{2}}_{\mathrm{cl}}(\mathbb{R}^q; H^s(\mathbb{R}^{q+k}), H^{s-\frac{k}{2}}(\mathbb{R}^q))$ (r is a symbol with constant coefficients that is independent of η; the groups acting in the Sobolev spaces are as above), and we have $\sigma^{\frac{k}{2}}_{\wedge}(\mathrm{r}) = \mathrm{r}$.

Proposition 1.2.12 *Let $\Omega \subseteq \mathbb{R}^q$ be an open set, X a closed compact C^∞ manifold and*

$$p(y,\eta) = r^{-\mu} \sum_{k+|\alpha|\le\mu} c_{k\alpha}(r,y)(rD_r)^k(r\eta)^\alpha$$

be an operator function, $\mu \in \mathbb{N}$, with coefficients $c_{k\alpha}(r,y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(k+|\alpha|)}(X))$, $k+|\alpha| \le \mu$, that are independent of r for $r > R$ for a constant $R > 0$. Then we have

$$p(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$$

for all $s \in \mathbb{R}$. If the coefficients $c_{k\alpha}$ are independent of r, then $p(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$ for all $s \in \mathbb{R}$, and we have in this case

$$\sigma^\mu_\wedge(p)(y,\eta) = r^{-\mu} \sum_{k+|\alpha|\le\mu} c_{k\alpha}(y)(rD_r)^k(r\eta)^\alpha.$$

1.2.3 Remarks on classical Sobolev spaces

In this section we want to interprete some standard properties of the classical Sobolev spaces in the Euclidean space in terms of pseudo–differential operators with operator–valued symbols (with constant coefficients).

Let us have a look at the symbols of Example 1.2.10 in the preceding section. Choose a cut–off function $\omega \in C_0^\infty(\mathbb{R}^k)$, i.e., a function $\omega(z)$ that equals 1 in an open neighbourhood of the origin. Then

$$p(\eta) : c \to \frac{1}{\alpha!}[\eta]^\mu z^\alpha \omega(z[\eta])c \qquad \text{for } c \in \mathbb{C},$$

$\mu \in \mathbb{R}$, $\alpha \in \mathbb{N}^k$, is a potential symbol in the sense of the notation above, namely

$$p(\eta) \in S_{\mathrm{cl}}^{\mu-|\alpha|-\frac{k}{2}}(\mathbb{R}^q; \mathbb{C}, H^s(\mathbb{R}^k)) \tag{1.2.8}$$

for arbitrary $s \in \mathbb{R}$. In fact, we have $p(\eta)c \in H^s(\mathbb{R}^k)$ for all $s \in \mathbb{R}$, $\eta \in \mathbb{R}^q$, $c \in \mathbb{C}$, where $p(\eta) \in C^\infty(\mathbb{R}^q, \mathcal{L}(\mathbb{C}, H^s(\mathbb{R}^k)))$, and

$$\begin{aligned} p(\lambda\eta)c &= \frac{1}{\alpha!}\lambda^\mu z^\alpha \omega(\lambda z[\eta])c \\ &= \frac{1}{\alpha!}\lambda^{\mu-|\alpha|-\frac{k}{2}} \kappa_\lambda^{(k)}\{z^\alpha \omega(z[\eta])\}c \end{aligned}$$

for all $\lambda \geq 1$, $|\eta| \geq$ const for a constant > 0. Thus, by virtue of Theorem 1.2.2 we get continuous operators

$$\mathrm{Op}(p) : \mathcal{W}^t(\mathbb{R}^q, \mathbb{C}) = H^t(\mathbb{R}^q) \to \mathcal{W}^{t-\mu+|\alpha|+\frac{k}{2}}(\mathbb{R}^q, H^s(\mathbb{R}^k))$$

for all $s, t \in \mathbb{R}$.

Moreover, denoting (as above) by r the operator of restriction to $z = 0$, i.e., $\mathrm{r}u = u(0)$, the operator function $d(\eta) : u(z) \to \langle\eta\rangle^{-\mu}\mathrm{r}D_z^\alpha u$ represents a classical trace symbol

$$d(\eta) \in S_{\mathrm{cl}}^{-\mu+|\alpha|+\frac{k}{2}}(\mathbb{R}^q; H^s(\mathbb{R}^k), \mathbb{C}) \tag{1.2.9}$$

for $s > |\alpha| + \frac{k}{2}$. This is a consequence of $d(\eta) \in C^\infty(\mathbb{R}^q, \mathcal{L}(H^s(\mathbb{R}^k), \mathbb{C}))$ and of the homogeneities

$$\mathrm{r} = \lambda^{\frac{k}{2}}\mathrm{r}(\kappa_\lambda^{(k)})^{-1}, \qquad D_z^\alpha = \lambda^{|\alpha|}\kappa_\lambda^{(k)}D_z^\alpha(\kappa_\lambda^{(k)})^{-1},$$

i.e., $\mathrm{r}D_z^\alpha = \lambda^{|\alpha|+\frac{k}{2}}\mathrm{r}D_z^\alpha(\kappa_\lambda^{(k)})^{-1}$ for all $\lambda > 0$, i.e.,

$$\begin{aligned} d(\lambda\eta) &= \lambda^{-\mu+|\alpha|+\frac{k}{2}}[\eta]^{-\mu}\mathrm{r}D_z^\alpha(\kappa_\lambda^{(k)})^{-1} \\ &= \lambda^{-\mu+|\alpha|+\frac{k}{2}}d(\eta)(\kappa_\lambda^{(k)})^{-1} \end{aligned}$$

for all $\lambda > 0$, $|\eta| > $ const for a constant > 0. It follows that $\mathrm{Op}(d)$ induces continuous operators

$$\mathrm{Op}(d) : \mathcal{W}^{t-\mu+|\alpha|+\frac{k}{2}}(\mathbb{R}^q, H^s(\mathbb{R}^k)) \to \mathcal{W}^t(\mathbb{R}^q, \mathbb{C}) = H^t(\mathbb{R}^q) \tag{1.2.10}$$

for all $s, t \in \mathbb{R}$, $s > |\alpha| + \frac{k}{2}$. We have $d(\eta) \circ p(\eta) = \mathrm{id}_\mathbb{C}$ for all $\eta \in \mathbb{R}^q$ which implies $\mathrm{Op}(d) \circ \mathrm{Op}(p) = \mathrm{id}_{H^t(\mathbb{R}^q)}$ for all $t \in \mathbb{R}$. It follows that $\mathrm{Op}(p) \circ \mathrm{Op}(d) : \mathcal{W}^{t-\mu+|\alpha|+\frac{k}{2}}(\mathbb{R}^q, H^s(\mathbb{R}^k)) \to \mathcal{W}^{t-\mu+|\alpha|+\frac{k}{2}}(\mathbb{R}^q, H^s(\mathbb{R}^k))$ is a continuous projection to $\mathrm{im}\,\mathrm{Op}(p)$.

Remark 1.2.13 (1.2.10) *can be applied to* $\mu = 0$, $t = s - |\alpha| - \frac{k}{2}$, $k \geq 1$. *Let us denote* (1.2.9) *and* (1.2.8) *in this case by* d_α *and* p_α, *respectively,* $\alpha \in \mathbb{N}^k$ *fixed. Then* (1.2.10) *takes the form*

$$\mathrm{Op}(d_\alpha) : \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}^k)) = H^s(\mathbb{R}^{q+k}) \to H^{s-|\alpha|-\frac{k}{2}}(\mathbb{R}^q)$$

for $s > |\alpha| + \frac{k}{2}$. *The operator* $\mathrm{Op}(d_\alpha)$ *is nothing else than the extension of* $f(y,z) \to (D_z^\alpha f)(y,0)$, $\mathcal{S}(\mathbb{R}^{q+k}) \to \mathcal{S}(\mathbb{R}^q)$ *by continuity to a continuous map*

$$\mathrm{r}D_z^\alpha : H^s(\mathbb{R}^{q+k}) \to H^{s-|\alpha|-\frac{k}{2}}(\mathbb{R}^q), \qquad s > \alpha + \frac{k}{2}. \tag{1.2.11}$$

By virtue of $\mathrm{Op}(d_\alpha)\,\mathrm{Op}(p_\alpha) = \mathrm{id}_{H^{s-|\alpha|-\frac{k}{2}}(\mathbb{R}^q)}$ *the operator* (1.2.11) *is surjective and*

$$\begin{aligned}&\{\mathrm{Op}(p_\alpha)v : \ v \in H^{s-|\alpha|-\frac{k}{2}}(\mathbb{R}^q)\}\\ &\qquad\qquad = \left\{F^{-1}_{\eta\to y}\left[\frac{1}{\alpha!}z^\alpha\omega(z[\eta])\hat{v}(\eta)\right] : \ v \in H^{s-|\alpha|-\frac{k}{2}}(\mathbb{R}^q)\right\}\end{aligned}$$

is a complement of $\ker \mathrm{Op}(d_\alpha)$ *(the operator* $1 - \mathrm{Op}(p_\alpha)\,\mathrm{Op}(d_\alpha)$ *is a continuous projection to that complement).*

Notice that (1.2.11) *also satisfies the homogeneity* $rD_z^\alpha = \lambda^{|\alpha|+\frac{k}{2}}\kappa_\lambda^{(q)} rD_z^\alpha (\kappa_\lambda^{(q+k)})^{-1}$ *for all* $\lambda > 0$, *i.e., we have* $rD_z^\alpha \in S_{\mathrm{cl}}^{|\alpha|+\frac{k}{2}}(\mathbb{R}^q; H^s(\mathbb{R}^{q+k}), H^{s-|\alpha|-\frac{k}{2}}(\mathbb{R}^q))$ *for all* $s > \alpha + \frac{k}{2}$.

Consider the vector of symbols

$$\boldsymbol{d}_m(\eta) = \{d_\alpha(\eta)\}_{|\alpha|\le m}, \qquad \boldsymbol{p}_m(\eta) = \{p_\alpha(\eta)\}_{|\alpha|\le m}$$

for some $m \in \mathbb{N}$. Then

$$\mathrm{Op}(\boldsymbol{d}_m) : H^s(\mathbb{R}^{q+k}) \to \bigoplus_{\substack{\alpha\in\mathbb{N}^k\\ |\alpha|\le m}} H^{s-|\alpha|-\frac{k}{2}}(\mathbb{R}^q) \tag{1.2.12}$$

for arbitrary $s > m + \frac{k}{2}$ corresponds to the vector of restriction operators $u(y,z) \to \{(D_z^\alpha u)(y,0)\}_{|\alpha|\le m}$ which is surjective. The space

$$\left\{\sum_{\substack{\alpha\in\mathbb{N}^k\\ |\alpha|\le m}} F^{-1}_{\eta\to y}\left[\frac{1}{\alpha!}z^\alpha\omega(z[\eta])\hat{v}_\alpha(\eta)\right] : v_\alpha \in H^{s-|\alpha|-\frac{k}{2}}(\mathbb{R}^q), |\alpha| \le m\right\} \tag{1.2.13}$$

is a complement of

$$\ker \mathrm{Op}(\boldsymbol{d}_m) = \{u \in H^s(\mathbb{R}^{q+k}) : (D_z^\alpha u)(y,0) = 0, |\alpha| \le m\}. \tag{1.2.14}$$

Denote $H_0^s(\mathbb{R}^{q+k}\setminus\mathbb{R}^k)$ the closure of $C_0^\infty(\mathbb{R}^{q+k}\setminus\mathbb{R}^k)$ in the space $H^s(\mathbb{R}^{q+k})$. Then, for $m(s,k)=\max\{j\in\mathbb{N}:\ s>j+\frac{k}{2}\}$ we have

$$H_0^s(\mathbb{R}^{q+k}\setminus\mathbb{R}^k)\subseteq\ker(\boldsymbol{d}_{m(s,k)}) \tag{1.2.15}$$

since $C_0^\infty(\mathbb{R}^{q+k}\setminus\mathbb{R}^k)\subseteq\ker(\boldsymbol{d}_{m(s,k)})$ and because of the continuity of (1.2.12).

It is often useful to study a generalisation of the Sobolev spaces in $\mathbb{R}^{q+k}\ni(y,z)$, namely the subspace $H^{s,t}(\mathbb{R}^{q+k})$ of all $u\in\mathcal{S}'(\mathbb{R}^{q+k})$ with

$$\left\{\int\langle\eta,\zeta\rangle^{2s}\langle\eta\rangle^{2t}|(F_{(y,z)\to(\eta,\zeta)}u)(\eta,\zeta)|^2\,d\eta d\zeta\right\}^{\frac{1}{2}}<\infty,$$

$s,t\in\mathbb{R}$, cf. [25]. We have $H^{s,0}(\mathbb{R}^{q+k})=H^s(\mathbb{R}^{q+k})=\mathcal{W}^s(\mathbb{R}^q,H^s(\mathbb{R}^k))$. The expression

$$\left\{\int\langle\eta,\zeta\rangle^{2s}[\eta]^{2t}|(F_{(y,z)\to(\eta,\zeta)}u)(\eta,\zeta)|^2\,d\eta d\zeta\right\}^{\frac{1}{2}}<\infty,$$

is an equivalent norm on $H^{s,t}(\mathbb{R}^{q+k})$. The operator $m_t(\eta)$ of multiplication by $[\eta]^t$ on $H^s(\mathbb{R}^k)$ has the homomogeneity $m_t(\lambda\eta)=\lambda^t\kappa_\lambda^{(k)}m_t(\eta)(\kappa_\lambda^{(k)})^{-1}$ for all $\lambda\geq 1$, $|\eta|\geq$ const for a constant >0 and it is C^∞ in $\eta\in\mathbb{R}^q$. Thus $m_t(\eta)\in S^t_{\mathrm{cl}}(\mathbb{R}^q;H^s(\mathbb{R}^k),H^s(\mathbb{R}^k))$. Moreover, $m_t(\eta):H^s(\mathbb{R}^k)\to H^s(\mathbb{R}^k)$ is an isomorphism for all $\eta\in\mathbb{R}^q$. It follows that

$$\mathrm{Op}(m_t^{-1}):\mathcal{W}^s(\mathbb{R}^q,H^s(\mathbb{R}^k))\to\mathcal{W}^{s+t}(\mathbb{R}^q,H^s(\mathbb{R}^k))$$

is an isomorphism. By virtue of

$$H^{s,t}(\mathbb{R}^{q+k})=\mathcal{W}^{s+t}(\mathbb{R}^q,H^s(\mathbb{R}^k))$$

(which is an easy generalisation of Example 1.1.3) we get an isomorphism

$$\mathrm{Op}(m_t^{-1}):H^s(\mathbb{R}^{q+k})\to H^{s,t}(\mathbb{R}^{q+k}).$$

Remark 1.2.14 *In boundary value problems it is important to have Sobolev spaces in the half–space $\mathbb{R}_+\times\mathbb{R}^q=\mathbb{R}_+^{1+q}$. They are defined*

$$H^s(\mathbb{R}_+^{1+q})=\mathrm{r}_q^+H^s(\mathbb{R}^{1+q})$$

where $\mathrm{r}_q^+:\mathcal{D}'(\mathbb{R}^{1+q})=\mathcal{D}'(\mathbb{R}_+^{1+q})$ is the restriction. Especially for $q=0$ the restriction operator $\mathrm{r}_0^+:H^s(\mathbb{R})\to H^s(\mathbb{R}_+)$ satisfies the relation $\mathrm{r}_0^+=\kappa_\lambda\mathrm{r}_0^+\kappa_\lambda^{-1}$

for all $\lambda \in \mathbb{R}_+$, *with* $(\kappa_\lambda u)(t) = \lambda^{\frac{1}{2}} u(\lambda t)$ *on* $H^s(\mathbb{R})$ *and* $H^s(\mathbb{R}_+)$, *respectively. This gives us*

$$\mathrm{r}_0^+ \in S_{\mathrm{cl}}^0(\mathbb{R}^q; H^s(\mathbb{R}), H^s(\mathbb{R}_+)).$$

Then the above operator $\mathrm{r}_q^+ : H^s(\mathbb{R}^{1+q}) \to H^s(\mathbb{R}_+^{1+q})$ *corresponds to*

$$\mathrm{r}_q^+ = \mathrm{Op}(r_0^+) : \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R})) \to \mathcal{W}^s(\mathbb{R}^q, H^s(\mathbb{R}_+)),$$

cf. Remark 1.2.4, *i.e.*, $\mathcal{W}^s(\mathbb{R}^s, H^s(\mathbb{R}_+)) = H^s(\mathbb{R}_+^{1+q})$.

Let us make some further remarks on Sobolev spaces or more general scales of (separable) complex Hilbert spaces $\{H^s\}_{s\in\mathbb{R}}$. Let us take the scalar product $(\cdot,\cdot)_{H^0} =: (\cdot,\cdot)$ in the space H^0 as reference scalar product. The scale of Sobolev spaces $H^s(\mathbb{R}^n)$ (and other scales such as $\mathcal{K}^{s,s}(X^\wedge)$, cf. Section 2.1.1 below, or $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,s}(X^\wedge))$) have the following properties:

(i) There are continuous embeddings $H^{s'} \hookrightarrow H^s$ for all $s' \geq s$.

(ii) The space $H^\infty = \bigcap_{s\in\mathbb{R}} H^s$ is dense in H^s for each s.

(iii) The scalar product $(\cdot,\cdot) : E^\infty \times E^\infty \to \mathbb{C}$ extends to a non–degenerate sesquilinear pairing $H^s \times H^{-s} \to \mathbb{C}$ for all $s \in \mathbb{R}$, where $|(u,v)| \leq \|u\|_s \|v\|_{-s}$ for all $u \in H^s$, $v \in H^{-s}$ ($\|\cdot\|_r$ is the norm in H^r) and

$$\|u\|_s = \sup_{\substack{v\in H^{-s}\\ v\neq 0}} \frac{|(u,v)|}{\|v\|_{-s}}, \qquad \|v\|_{-s} = \sup_{\substack{u\in H^{s}\\ u\neq 0}} \frac{|(u,v)|}{\|u\|_{s}}.$$

If $A : H^s \to H^0$ is a continuous operator, we can form its formal adjoint A^* by $(Au, v) = (u, A^*v)$ for all $u, v \in H^\infty$. This has an extension to a continuous operator $A^* : H^0 \to H^{-s}$.

In fact, let us set $f(u) = (Au, v)$ for fixed $v \in H^\infty$. This is a linear continuous functional on the space $H^s \ni u$. It can be written $f(u) = (u, v^*)$ for a $v^* \in H^{-s}$. Set $v^* = A^*v$. Then

$$|(u, A^*v)| = |(Au, v)| \leq \|Au\|_0 \|v\|_0.$$

Since

$$\begin{aligned}\|A^*v\|_{-s} &= \sup_{\substack{u\in H^{s}\\ u\neq 0}} \frac{|(u, A^*v)|}{\|u\|_s} = \sup_{\substack{u\in H^{s}\\ u\neq 0}} \frac{|(Au, v)|}{\|u\|_s} \\ &\leq \sup_{\substack{u\in H^{s}\\ u\neq 0}} \frac{\|Au\|_0\|v\|_0}{\|u\|_s} \leq \sup_{\substack{u\in H^{s}\\ u\neq 0}} \frac{\|A\|\|u\|_s\|v\|_0}{\|u\|_s} = \|A\|\|v\|_0,\end{aligned}$$

we get the asserted continuity of A^*.

Let us form the composition

$$L = A^*A : H^s \to H^{-s}. \tag{1.2.16}$$

We then have the following property:

Lemma 1.2.15 *Assume* $A : H^s \to H^0$ *is injective. Then also* (1.2.16) *is an injective operator. Moreover,* $\{Lu : \ u \in H^s\}$ *is dense in the space* H^{-s}*. Especially if* L *is a Fredholm operator, we have* $\operatorname{im} L = H^{-s}$*, i.e.,* (1.2.16) *is an isomorphism.*

Proof. In order to check the injectivity of L we observe that $Lu = 0$, $u \in H^s$, is equivalent to $(Lu, v) = 0$ for all $v \in H^s$. But $(Lu, v) = (A^*Au, v) = (Au, Av) = 0$ for $u = v$ yields $\|Au\|^2 = 0$, i.e., $u = 0$, since A is injective. Secondly, let us show that $(Lu, v) = 0$ for some $v \in H^s$ and all $u \in H^s$ implies $v = 0$; then $\operatorname{im} L$ is dense in H^{-s}. We have $(Lu, v) = (A^*Au, v) = 0$; this is equivalent to $(Au, Av) = 0$. Setting $u = v$ we get $(Av, Av) = 0$ and thus, as before, $v = 0$. Finally, if L is a Fredholm operator, its image is closed. In that case, (1.2.16) is an isomorphism. □

Chapter 2

Conormal asymptotics and meromorphic operator functions

2.1 Operators of Fuchs type and the Mellin transform

2.1.1 Mellin transform and associated Sobolev spaces

Partial differential equations on a configuration with (geometric) singularities require new technicalities, compared with the smooth situation. In the simplest case of isolated singular points of conical type the analysis is organised with respect to a splitting of variables, namely in (r, x) with x varying along the base X of the (local) cone and $r \in \mathbb{R}_+$, the axial variable, the distance to the corresponding conical point. For boundary value problems in domains with conical singularities the base X is a compact C^∞ manifold, with C^∞ boundary; in the case of "surfaces" with conical singularities X is a closed compact C^∞ manifold. In the present exposition we content ourselves with the second case; for boundary value problems we refer to the papers of Schrohe and Schulze [47], [48] and to the book [68] of the author. A basic observation is that the typical differential operators (of order $\mu \in \mathbb{N}$) are of

Fuchs type

$$A = r^{\delta} \sum_{j=0}^{\mu} a_j(r) \left(-r\frac{\partial}{\partial r}\right)^j r^{-\gamma} \tag{2.1.1}$$

for suitable reals γ, δ, with operator–valued $a_j(r) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}(X))$. In many cases it is natural to assume $\delta = \gamma - \mu$, especially for $\gamma = 0$

$$A = r^{-\mu} \sum_{j=0}^{\mu} a_j(r) \left(-r\frac{\partial}{\partial r}\right)^j . \tag{2.1.2}$$

It is convenient to interprete operators of Fuchs type in terms of the Mellin transform

$$(Mu)(z) = \int_0^\infty r^{z-1} u(r)\, dr. \tag{2.1.3}$$

In order to formulate the basic properties we first assume that $u(r)$ is a scalar function; below we tacitly extend it to several classes of vector–valued functions. The variable z varies in a subset of $\mathbb{C}$, usually on a so–called weight line

$$\Gamma_\beta = \{z \in \mathbb{C} : \ \mathrm{Re}\, z = \beta\}.$$

The standard function and distribution spaces on $\mathbb{R}$ can be transformed to Γ_β by the bijection $\Gamma_\beta \to \mathbb{R}$, $z \to \mathrm{Im}\, z$; in particular, we get the spaces

$$\mathcal{S}(\Gamma_\beta), \mathcal{S}'(\Gamma_\beta), H^s(\Gamma_\beta), L^2(\Gamma_\beta), \ldots .$$

Incidentally we also use the notation $\Gamma_B = \bigcup_{\beta \in B} \Gamma_\beta$ for any subset $B \subseteq \mathbb{R}$. Let us substitute the diffeomorphism $\chi : \mathbb{R}_+ \to \mathbb{R}$, $\chi(r) = -\log r = t$, into the Mellin transform. Then, for $u \in C_0^\infty(\mathbb{R}_+)$ we obtain

$$(Mu)\left(\frac{1}{2} - \gamma + i\tau\right) = \int_{-\infty}^{\infty} e^{-i\tau t} (S_\gamma u)(t)\, dt \tag{2.1.4}$$

for $(S_\gamma u)(t) = e^{-(\frac{1}{2}-\gamma)t} u(e^{-t})$, $\gamma \in \mathbb{R}$ fixed. Notice that $S_\gamma : C_0^\infty(\mathbb{R}_+) \to C^\infty(\mathbb{R})$ extends to an isomorphism $S_\gamma : r^\gamma L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$. Hence $FS_\gamma : r^\gamma L^2(\mathbb{R}_+) \to L^2(\mathbb{R})$ is also an isomorphism. Thus, if we set

$$(M_\gamma u)(z) = Mu(z)|_{\Gamma_{\frac{1}{2}-\gamma}}, \qquad \gamma \in \mathbb{R},$$

M_γ induces an isomorphism

$$M_\gamma : r^\gamma L^2(\mathbb{R}_+) \to L^2(\Gamma_{\frac{1}{2}-\gamma}) \tag{2.1.5}$$

with the inverse

$$(M_\gamma^{-1} g)(r) = \frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{1}{2}-\gamma}} r^{-z} g(z)\, dz, \tag{2.1.6}$$

as a consequence of (2.1.4). We call M_γ the weighted Mellin transform, with weight $\gamma \in \mathbb{R}$. For $\gamma = 0$ we also write M instead of M_0. We see from formula (2.1.4) that the properties of the Mellin transform can be derived from those of the Fourier transform. Nevertheless it is not always adequate to replace the Mellin transform by the Fourier transform in new coordinates. The reason is that in many standard singular configurations, e.g., mixed or transmission problems, the Fourier transform on the r–axis, restricted to $\mathbb{R}_+$, occurs together with the Mellin transform on the r–half–axis at the same time. There are typical interactions between Fourier and Mellin descriptions (Mellin operator conventions), but the logarithmic substitution only transforms these effects from 0 to infinity.

Remark 2.1.1 *The Mellin transform* (2.1.3)*, first applied to $u \in C_0^\infty(\mathbb{R}_+)$, induces a continuous operator $M : C_0^\infty(\mathbb{R}_+) \to \mathcal{A}(\mathbb{C})$, and $u \to Mu|_{\frac{1}{2}-\gamma}$ is continuous as a map $M_\gamma : C_0^\infty(\mathbb{R}_+) \to \mathcal{S}(\Gamma_{\frac{1}{2}-\gamma})$ for every $\gamma \in \mathbb{R}$. The extension by continuity gives us the isomorphism* (2.1.5).

Let us set

$$(\delta_\lambda u)(r) = u(\lambda r), \qquad \lambda \in \mathbb{R}_+. \tag{2.1.7}$$

Then, if a function on $\mathbb{R}_+$ satisfies the relation $(\delta_\lambda u)(r) = \lambda^c u(r)$ for a $c \in \mathbb{C}$ we call $u(r)$ homogeneous of order c. Especially we have $\delta_\lambda r^{-z} = \lambda^{-z} r^{-z}$, $z \in \mathbb{C}$, so r^{-z} is of homogeneity order $-z$. Writing formula (2.1.6) in the form $u(r) = \frac{1}{2\pi i}\int_{\Gamma_{\frac{1}{2}-\gamma}} r^{-z} g(z)\, dz$ we can interprete this as a superposition of homogeneous functions with respect to r, in other words, every $u(r) \in r^\gamma L^2(\mathbb{R}_+)$ can be decomposed as a superposition of homogeneous functions, with the density $(2\pi i)^{-1} g(z)$.

The description of operators by means of the Mellin transform is analogous to that of pseudo–differential operators in terms of the Fourier transform. We have $M^{-1}zMu = -r\frac{\partial}{\partial r}u$, say, for $u \in C_0^\infty(\mathbb{R}_+)$, and then a Fuchs type differential operator $A = \sum_{j=0}^{\mu} a_j(r)\left(-r\frac{\partial}{\partial r}\right)^j$ gets the form

$$\mathrm{op}_M(a) = M^{-1}a(r,z)M \qquad \text{for } a(r,z) = \sum_{j=0}^{\mu} a_j(r)z^j. \tag{2.1.8}$$

Here $a(r,z)$ is involved as a function of $(r,z) \in \mathbb{R}_+ \times \Gamma_{\frac{1}{2}}$. Note that the function $a(r,z)$ can be recovered in terms of the action by

$$a(r,z) = r^{-z}\{\mathrm{op}_M(a)\}r^z.$$

More generally, for $a(r,z)$ taken for $(r,z) \in \mathbb{R}_+ \times \Gamma_{\frac{1}{2}-\gamma}$, we can form the weighted Mellin pseudo–differential operator

$$\mathrm{op}_M^\gamma(a) = M_\gamma^{-1}a(r,z)M_\gamma.$$

If necessary we indicate the transformation of variables as subscripts: $M_{\gamma,r\to z}$ and $M^{-1}_{\gamma,z\to r}$.

Remark 2.1.2 *The Mellin symbol $a(r,z)$ can be recovered by the operator $A = \sum_{j=0}^{\mu} a_j(r)(-r\frac{\partial}{\partial r})^j$, namely $a(r,z) = r^z A r^{-z}$.*

Remark 2.1.3 *If $a = a(z)$ is independent of r, we have*

$$\mathrm{op}_M^\gamma(a) = \delta_\lambda^{-1}\,\mathrm{op}_M^\gamma(a)\delta_\lambda$$

for all $\lambda \in \mathbb{R}_+$.

Examples of operators of Fuchs type can be obtained by substituting polar coordinates into a differential operator

$$\tilde{A} = \sum_{|\alpha|\le\mu} c_\alpha(\tilde{x})D_{\tilde{x}}^\alpha$$

in $\mathbb{R}^{n+1}_{\tilde{x}}$ with coefficients $c_\alpha \in C^\infty(\mathbb{R}^{n+1})$. Denote by $\chi : \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{R}_+ \times S^n$ the diffeomorphism $\chi(\tilde{x}) = (r, \tilde{x}/|\tilde{x}|)$, $r = |\tilde{x}|$; then we get $A = \tilde{A}|_{\mathbb{R}_{n+1}\setminus\{0\}} \circ \chi^{-1} = r^{-\mu}\sum_{j=0}^{\mu} a_j(r)(-r\frac{\partial}{\partial r})^j$ with coefficients $a_j(r) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}(S^n))$. Especially for $\tilde{A} = \sum_{j=1}^{n+1}\tilde{x}_j\frac{\partial}{\partial\tilde{x}_j}$ we obtain $A = r\frac{\partial}{\partial r}$.

Let $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times \mathbb{R}^n)$ for $s, \gamma \in \mathbb{R}$ be the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)} = \left\{ \frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{n+1}{2}-\gamma}} \int\limits_{\mathbb{R}^n} (1+|z|^2+|\xi|^2)^s |(M_{r\to z}F_{x\to\xi}u)(z,\xi)|^2 \, dz đ\xi \right\}^{\frac{1}{2}}.$$

If X is a closed compact C^∞ manifold we set $X^\wedge = \mathbb{R}_+ \times X$, called the infinite (open stretched) cone with base X, and construct the space $\mathcal{H}^{s,\gamma}(X^\wedge)$ as follows. We choose a finite atlas $\chi_j : U_j \to \mathbb{R}^n$ on X, $j = 1, \ldots, N$, such that $\{U_1, \ldots, U_N\}$ is an open covering of X, and a subordinate partition of unity $\{\varphi_1, \ldots, \varphi_N\}$. Then $\mathcal{H}^{s,\gamma}(X^\wedge)$ is the completion of $C_0^\infty(\mathbb{R}_+ \times X)$ with respect to the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(X^\wedge)} = \left\{ \sum_{j=1}^N \|(\varphi_j u) \circ \chi_j^{-1}\|^2_{\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times\mathbb{R}^n)} \right\}^{\frac{1}{2}}.$$

Note that we have $\mathcal{H}^{s,\gamma}(X^\wedge) \subset H^s_{\mathrm{loc}}(X^\wedge)$ for every $s, \gamma \in \mathbb{R}$. Set $\mathcal{H}^{\infty,\gamma}(X^\wedge) = \bigcap_{s\in\mathbb{R}} \mathcal{H}^{s,\gamma}(X^\wedge)$. Then it can be proved that

$$\mathcal{H}^{\infty,\gamma}(X^\wedge) = r^{-\frac{n}{2}} \mathcal{H}^{\infty,\gamma}(\mathbb{R}_+) \otimes_\pi C^\infty(X),$$

cf. [48], Lemma 1.4.5.

The spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ can also be obtained by another more concise construction. Choose an element $R^s(\tau) \in L^s_{\mathrm{cl}}(X; \mathbb{R}_\tau)$ that is parameter–dependent elliptic of order $s \in \mathbb{R}$, cf. Example 1.1.40. Applying Theorem 1.1.41 we can find $R^s(\tau)$ in such a way that $R^s(\tau) : H^t(X) \to H^{t-s}(X)$ is an isomorphism for arbitrary $t \in \mathbb{R}$, $\tau \in \mathbb{R}$. It suffices to replace $|\lambda|^2$ in Example 1.1.40 by $|\tau|^2 + c$ for a sufficiently large $c > 0$. It is then easy to verify the following equivalent characterisation:

Proposition 2.1.4 *$\mathcal{H}^{s,\gamma}(X^\wedge)$ is the completion $C_0^\infty(X^\wedge)$ with respect to the norm*

$$\left\{ \frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{n+1}{2}-\gamma}} \|R^s(\operatorname{Im} z)(M_{r\to z}u)(z)\|^2_{L^2(X)} \, dz \right\}^{\frac{1}{2}}.$$

In an analogous manner we can define a Sobolev space $H^s(\mathbb{R}\times X)$ on the infinite cylinder by replacing the Mellin transform by the one–dimensional Fourier transform $F = F_{t\to\tau}$. In other words, $H^s(\mathbb{R}\times X)$ is the completion of $C_0^\infty(\mathbb{R}\times X)$ with respect to the norm

$$\left(\int \|R^s(\tau)(F_{t\to\tau}v)(\tau)\|^2_{L^2(\mathbb{R})}\,d\tau\right)^{\frac{1}{2}}.$$

Using (2.1.4) we thus obtain isomorphisms

$$S_{\gamma-\frac{n}{2}} : \mathcal{H}^{s,\gamma}(X^\wedge) \to H^s(\mathbb{R}\times X)$$

for arbitrary $s,\gamma\in\mathbb{R}$. It is well–known that $v\in H^s(\mathbb{R}\times X)$ implies $\varphi v\in H^s(\mathbb{R}\times X)$ for each $\varphi\in C_{\mathrm{b}}^\infty(\mathbb{R})$ (subscript b indicates the subspace of all $\varphi\in C^\infty(\mathbb{R})$ with $\sup_{t\in\mathbb{R}}|D_t^k\varphi(t)|<\infty$ for all $k\in\mathbb{N}$).

By virtue of $S_{\gamma-\frac{n}{2}}(\psi u)(t) = \psi(e^{-t})(S_{\gamma-\frac{n}{2}}u)(t)$ it follows that $u\in\mathcal{H}^{s,\gamma}(X^\wedge)$ implies $\psi u\in\mathcal{H}^{s,\gamma}(X^\wedge)$ for arbitrary $\psi(r)$ with $\psi(e^{-t})\in C_{\mathrm{b}}^\infty(\mathbb{R})$, that is, $\psi\in C^\infty(\mathbb{R}_+)$ with $\sup_{r\in\mathbb{R}_+}|(r\frac{\partial}{\partial r})^k\psi(r)|<\infty$ for all $k\in\mathbb{N}$.

Remark 2.1.5 *The operator $\mathcal{M}_\varphi$ of multiplication by $\varphi\in C_0^\infty(\overline{\mathbb{R}}_+)$ preserves $\mathcal{H}^{s,\gamma}(X^\wedge)$ and the map $\varphi\to\mathcal{M}_\varphi$ represents a continuous operator $C_0^\infty(\overline{\mathbb{R}}_+)\to\mathcal{L}(\mathcal{H}^{s,\gamma}(X^\wedge))$ for each $s,\gamma\in\mathbb{R}$.*

This is a special case of the relations above.

Remark 2.1.6 *The transformation $I^{(n)} : u(r,x)\to r^{-n-1}u(r^{-1},x)$ induces isomorphisms $I^{(n)} : \mathcal{H}^{s,\gamma}(X^\wedge)\to\mathcal{H}^{s,-\gamma}(X^\wedge)$ for all $s,\gamma\in\mathbb{R}$, and we have $(MI^{(n)}u)(z) = (Mu)(n+1-z)$. Moreover, if $h(z)=\sum_{j=0}^\mu a_jz^j$ with constants a_j, we have $I^{(n)}\operatorname{op}_M(h)(I^{(n)})^{-1}u = \operatorname{op}_M(f)u$ for $f(z)=h(n+1-z)$.*

For the unit sphere $X = S^n$ in $\mathbb{R}^{n+1}$ we can form $\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times S^n)$ as a subspace of $H^s_{\mathrm{loc}}(\mathbb{R}^{n+1}\setminus\{0\})$ when we identify $\mathbb{R}^{n+1}\setminus\{0\}$ with $\mathbb{R}_+\times S^n$ via polar coordinates. Since the specific influence of the Mellin transform only concerns a neighbourhood of the origin, it is interesting also to look at spaces of the kind

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+\times S^n) = [\omega]\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times S^n) + [1-\omega]H^s(\mathbb{R}^{n+1}) \tag{2.1.9}$$

for a cut–off function $\omega(r)$ (i.e., $\omega\in C^\infty(\overline{\mathbb{R}}_+)$, $\omega(r)=1$ near $r=0$, $\omega(r)=0$ for $r>$ const for a constant >0) or more generally

$$\mathcal{K}^{s,\gamma;\varrho}(\mathbb{R}_+\times S^n) = [\omega]\mathcal{H}^{s,\gamma}(\mathbb{R}_+\times S^n) + [1-\omega]H^{s;\varrho}(\mathbb{R}^{n+1}).$$

Especially for $n = 0$ we have the spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$, $H^{s;\varrho}(\mathbb{R}_+) = H^{s;\varrho}(\mathbb{R})|_{\mathbb{R}_+}$, and we can form

$$\mathcal{K}^{s,\gamma;\varrho}(\mathbb{R}_+) = [\omega]\mathcal{H}^{s,\gamma}(\mathbb{R}_+) + [1-\omega]H^{s;\varrho}(\mathbb{R}_+), \tag{2.1.10}$$

and we set $\mathcal{K}^{s,\gamma}(\mathbb{R}_+) := \mathcal{K}^{s,\gamma;0}(\mathbb{R}_+)$.

Similar spaces make sense when we replace S^n by an arbitrary closed compact C^∞ manifold X. Let us choose a finite atlas $\vartheta_j : \mathbb{R}_+ \times U_j \to \mathbb{R}^{n+1} \setminus \{0\}$, $j = 1, \dots, N$, where $\{U_1, \dots, U_N\}$ is an open covering of X and ϑ_j is homogeneous in the sense $\vartheta_j(\lambda r, x) = \lambda \vartheta_j(r, x)$ for all $(r, x) \in \mathbb{R}_+ \times U_j$. Thus $\vartheta_j(\mathbb{R}_+ \times U_j)$ is an open conical subset of $\mathbb{R}^{n+1} \setminus \{0\}$. We then define $H^{s;\varrho}(X^\wedge)$ as the completion of $C_0^\infty(\mathbb{R}_+ \times X)$ with respect to the norm

$$\|u\|_{H^{s;\varrho}(X^\wedge)} = \Big\{ \sum_{j=1}^{N} \|(\varphi_j u) \circ \vartheta_j^{-1}\|^2_{H^{s;\varrho}(\mathbb{R}^{n+1})} \Big\}^{\frac{1}{2}}$$

(with the above partition of unity $\{\varphi_j\}_{j=1,\dots,N}$) and set

$$\mathcal{K}^{s,\gamma;\varrho}(X^\wedge) = [\omega]\mathcal{H}^{s,\gamma}(X^\wedge) + [1-\omega]H^{s;\varrho}(X^\wedge), \tag{2.1.11}$$

$\mathcal{K}^{s,\gamma}(X^\wedge) := \mathcal{K}^{s,\gamma;0}(X^\wedge)$. The definition is correct in the sense that we get equivalent norms by changing the data (charts, partition of unity, cut–off function ω). Notice that

$$r^{-\delta}\mathcal{K}^{s,\gamma;\varrho}(X^\wedge) = \mathcal{K}^{s,\gamma-\delta;\varrho+\delta}(X^\wedge) \tag{2.1.12}$$

for all $s, \gamma, \delta, \varrho \in \mathbb{R}$.

Remark 2.1.7 *Let $u(r,x) \in \mathcal{K}^{s,\gamma;\varrho}(X^\wedge)$ and set $(\kappa_\lambda u)(r,x) = \lambda^{\frac{n+1}{2}} u(\lambda r, x)$ for $\lambda \in \mathbb{R}_+$, $n = \dim X$. Then $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ is a strongly continuous group of isomorphisms on $\mathcal{K}^{s,\gamma;\varrho}(X^\wedge)$ for arbitrary $s, \gamma, \varrho \in \mathbb{R}$.*

Proposition 2.1.8 *Let A be a differential operator of Fuchs type, written in the form* (2.1.1). *Assume $a_j(r) \in C_{\mathrm{b}}^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}(X))$, $j = 0, \dots, \mu$. Then A induces continuous operators*

$$A : \mathcal{K}^{s,\gamma+\frac{n}{2};\varrho}(X^\wedge) \to \mathcal{K}^{s-\mu,\delta+\frac{n}{2};\varrho-\gamma+\delta}(X^\wedge)$$

for all $s \in \mathbb{R}$.

An explicit proof is given in connection with Proposition 3.1.1 below (in a slightly simplified situation).

2.1.2 Solutions with asymptotics

Consider an equation of Fuchs type on $\mathbb{R}_+$

$$r^{-\mu}\sum_{j=0}^{\mu} a_j\left(-r\frac{\partial}{\partial r}\right)^j u = f \tag{2.1.13}$$

with constant coefficients a_j, $j = 0, \dots, \mu$, and assume $f(r) \in \mathcal{S}(\overline{\mathbb{R}}_+) = \mathcal{S}(\mathbb{R})|_{\overline{\mathbb{R}}_+}$.

Proposition 2.1.9 *Let $a(z) = \sum_{j=0}^{\mu} a_j z^j \neq 0$ for all $z \in \Gamma_{\frac{1}{2}}$. Then equation* (2.1.13) *has a unique solution $u \in L^2(\mathbb{R}_+)$.*

Proof. Equation (2.1.13) can be written in the form $\mathrm{op}_M(a)u = r^\mu f$. By assumption we have $a(z) \neq 0$ on $\Gamma_{\frac{1}{2}}$. Thus we can apply $\mathrm{op}_M(a^{-1})$ to $r^\mu f \in L^2(\mathbb{R}_+)$. In view of $\mathrm{op}_M(a^{-1})\,\mathrm{op}_M(a) = \mathrm{id}$ we get the solution $u = \mathrm{op}_M(a^{-1})(r^\mu f)$ as an element of $L^2(\mathbb{R}_+)$, cf. (2.1.5). □

Proposition 2.1.9 plays the role of an example. In the case of operators with constant coefficients it illustrates the idea to obtain solutions via pseudo–differential operators. The example gives us in fact more precise information, and it can be generalised in various ways. First observe that we have $a(z)|_{\Gamma_\beta} \in S^\mu_{\mathrm{cl}}(\Gamma_\beta)$ for all $\beta \in \mathbb{R}$. We did not assume that the leading coefficient a_μ does not vanish. If we do so, we get $a^{-1}(z)|_{\Gamma_\beta} \in S^{-\mu}_{\mathrm{cl}}(\Gamma_\beta)$ for all $\beta \in \mathbb{R}$ for which Γ_β does not contain a zero of $a(z)$. For all those $\beta = \frac{1}{2} - \gamma$ the operator

$$\mathrm{op}^\gamma_M(a) : \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{H}^{s-\mu,\gamma}(\mathbb{R}_+)$$

is an isomorphism for every $s \in \mathbb{R}$, with the inverse $\mathrm{op}^\gamma_M(a^{-1})$. Then equation (2.1.13) represents an isomorphism $\mathcal{H}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{R}_+)$ for every $s \in \mathbb{R}$, and we get the following result:

Proposition 2.1.10 *Let $a(z) = \sum_{j=0}^{\mu} a_j z^j \neq 0$ for all $z \in \Gamma_{\frac{1}{2}-\gamma}$ and $a_\mu \neq 0$. Then equation* (2.1.13) *has a unique solution $u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ for every $f \in \mathcal{H}^{s-\mu,\gamma-\mu}(\mathbb{R}_+)$, $s \in \mathbb{R}$ arbitrary.*

In the latter proposition we did not use the fact that $a(z)$ is a polynomial. We may admit an arbitrary symbol $a(z) \in S^\mu_{\mathrm{cl}}(\Gamma_{\frac{1}{2}-\gamma})$ that does not vanish on $\Gamma_{\frac{1}{2}-\gamma}$ and which is elliptic in the sense that its homogeneous principal part is non–zero on non–vanishing covectors. However in the case of a polynomial $a(z)$ we can formulate associated operators for all weights $\gamma \in \mathbb{R}$, except for a discrete subset, and we know that $a^{-1}(z)$ is meromorphic. This is an important extra information which leads to results on asymptotics of solutions.

In this connection it is useful to introduce certain spaces of meromorphic functions which play the role of symbols for pseudo–differential operators with respect to the Mellin transform.

Definition 2.1.11 *Let* **As** *be the set of all sequences of pairs*

$$R = \{(r_j, n_j)\}_{j \in \mathbb{Z}},$$

where $r_j \in \mathbb{C}$, $n_j \in \mathbb{N}$, $|\operatorname{Re} r_j| \to \infty$ *as* $|j| \to \infty$. *The elements of* **As** *are called discrete asymptotic types for (scalar) Mellin symbols. We denote by* M_R^μ *the space of all* $f(z) \in \mathcal{A}(\mathbb{C} \setminus \pi_{\mathbb{C}} R)$ *(where* $\pi_{\mathbb{C}} R = \{r_j\}_{j \in \mathbb{N}}$*) such that for any* $\pi_{\mathbb{C}} R$*–excision function* $\chi(z)$ *(that is,* $\chi(z) \in C^\infty(\mathbb{C})$, $\chi(z) = 0$ *for* $\operatorname{dist}(z, \pi_{\mathbb{C}} R) < \varepsilon_0$, $\chi(z) = 1$ *for* $\operatorname{dist}(z, \pi_{\mathbb{C}} R) > \varepsilon_1$ *for certain* $0 < \varepsilon_0 < \varepsilon_1$*) we have* $\chi(z) f(z)|_{\Gamma_\beta} \in S_{\mathrm{cl}}^\mu(\Gamma_\beta)$ *for every* $\beta \in \mathbb{R}$*, uniformly in* $c \le \beta \le c'$ *for arbitrary* $c \le c'$*; moreover* $f(z)$ *is meromorphic with poles at the points* $z = r_j$ *of multiplicities* $n_j + 1$*, for all* $j \in \mathbb{Z}$*.*

For $\pi_{\mathbb{C}} R = \emptyset$ we denote the corresponding space by M_O^μ.

M_R^μ is a Fréchet space in a natural way. Set $M_R^{-\infty} = \bigcap_{\mu \in \mathbb{R}} M_R^\mu$ in the topology of the projective limit.

Example 2.1.12 *The function* e^{z^2} *belongs to* $M_O^{-\infty}$*. Moreover, if* $\Gamma(z)$ *is Euler's* Γ*–function, for every* $\varrho \in \mathbb{R}$ *we have*

$$f_\varrho(z) = \frac{\Gamma(1-z)}{\Gamma(1-z+\varrho)} \in M_R^{-\varrho},$$

where $R = \{(1+k, 0)\}_{k \in \mathbb{N}}$*. Furthermore,* $g^\pm(z) = (1 - e^{\mp 2\pi i z})^{-1} \in M_P^0$, $g(z) = e^{-i\pi z} g^-(z) \in M_P^{-\infty}$ *for* $P = \{(j, 0)\}_{j \in \mathbb{Z}}$, $g^+(z) g^-(z) \in M_Q^{-\infty}$ *for* $Q = \{(k, 1)\}_{k \in \mathbb{Z}}$*.*

Theorem 2.1.13 *For every* $R \in$ **As** *and every* $\mu \in \mathbb{R}$ *we have* $M_R^\mu = M_O^\mu + M_R^{-\infty}$ *in the Fréchet topology of the non–direct sum.*

We now introduce subspaces of $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ with so–called discrete asymptotics. First let $\Theta = (\vartheta, 0]$ for $-\infty \le \vartheta < 0$ and set

$$\mathcal{K}^{s,\gamma}_{\Theta}(\mathbb{R}_+) = \bigcap_{\varepsilon>0} \mathcal{K}^{s,\gamma-\vartheta-\varepsilon}(\mathbb{R}_+),$$

in the Fréchet topology of the projective limit. The elements of $\mathcal{K}^{s,\gamma}_{\Theta}(\mathbb{R}_+)$ are flat of order Θ, relative to the weight γ. The singular functions of the asymptotics will be of the form $\omega(r)r^{-p}\log^k r$ for a cut–off function $\omega(r)$ and $p \in \mathbb{C}$, $\operatorname{Re} p < \frac{1}{2} - \gamma$, $k \in \mathbb{N}$. Those functions belong to $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$.

Definition 2.1.14 *Let $\gamma \in \mathbb{R}$ and*

$$P = \{(p_j, m_j)\}_{j=0,\dots,N}, \tag{2.1.14}$$

$N = N(P)$, be a sequence of pairs, $p_j \in \mathbb{C}$, $m_j \in \mathbb{N}$, and let $-\infty < \vartheta < 0$ such that

$$\frac{1}{2} - \gamma + \vartheta < \operatorname{Re} p_j < \frac{1}{2} - \gamma \tag{2.1.15}$$

for all $j = 0, \dots, N$. We then denote by $\mathcal{K}^{s,\gamma}_P(\mathbb{R}_+)$ the subspace of all $u \in \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ such that there are coefficients $c_{jk} = c_{jk}(u)$ with

$$u(r) - \omega(r)\sum_{j=0}^{N}\sum_{k=0}^{m_j} c_{jk} r^{-p_j}\log^k r \in \mathcal{K}^{s,\gamma}_{\Theta}(\mathbb{R}_+).$$

Let $\boldsymbol{g} = (\gamma, \Theta)$ with $\gamma \in \mathbb{R}$ and $\Theta = (-\vartheta, 0]$, $-\infty < \vartheta < 0$, being a weight interval. Denote by $\operatorname{As}(\boldsymbol{g})$ the set of all P of the form (2.1.14) such that (2.1.15) holds for all j. The elements of $\operatorname{As}(\boldsymbol{g})$ are called discrete asymptotic types to the weight data $\boldsymbol{g}$. For $\Theta = (-\infty, 0]$ we denote by $\operatorname{As}(\boldsymbol{g})$ the set of all sequences

$$P = \{(p_j, m_j)\}_{j\in\mathbb{N}} \tag{2.1.16}$$

$\subset \mathbb{C} \times \mathbb{N}$ with (2.1.15) for all $j \in \mathbb{R}$ and $\operatorname{Re} p_j \to -\infty$ as $j \to \infty$ and obtain associated $\mathcal{K}^{s,\gamma}_P(\mathbb{R}_+)$ spaces as projective limits, cf. also the definition for the case $X^\wedge$ below.

Lemma 2.1.15 *Let $p \in \mathbb{C}$, $\gamma \in \mathbb{R}$ with $\gamma < \frac{1}{2} - \operatorname{Re} p$, $k \in \mathbb{N}$, and fix a cut–off function $\omega(r)$. Then $M_\gamma(\omega r^{-p}\log^k r)(z) \in \mathcal{S}(\Gamma_{\frac{1}{2}-\gamma})$ has an extension to a meromorphic function in $\mathbb{C}$ with a pole at p of multiplicity $k+1$, more precisely,*

$$M_\gamma(\omega r^{-p}\log^k r)(z) - (-1)^k k!(z-p)^{-(k+1)} \in \mathcal{A}(\mathbb{C}).$$

Proof. Using the relation $(M_\gamma u)(z) = M(r^{-\gamma}u)(z+\gamma)$ we can reduce the assertion to $\gamma = 0$; let us consider that case. From

$$z^{-1}M\left(-r\frac{\partial}{\partial r}\right)\omega(z) = (M\omega)(z)$$

and $-r\frac{\partial}{\partial r}\omega \in C_0^\infty(\mathbb{R}_+)$ we deduce that $(M\omega)(z)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta)$ for each $\beta \neq 0$, because $(M\varphi)(z)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta)$ for arbitrary $\varphi \in C_0^\infty(\mathbb{R}_+)$ and $\beta \in \mathbb{R}$. Moreover, $(M\varphi)(z)$ is an entire function. Thus $(M\omega)(z)$ extends to a meromorphic function with a simple pole at $z = 0$. Let $\varepsilon(r)$ be the characteristic function of the interval $[0,1]$. Then $\omega(r) - \varepsilon(r)$ has compact support in $\mathbb{R}_+$ and hence $M(\omega - \varepsilon)(z) \in \mathcal{A}(\mathbb{C})$. Thus, in order to calculate the residue of $M(\omega)(z)$ at 0 it suffices to consider $(M\varepsilon)(z)$, but this equals 1. It follows that $(M\omega)(z) - z^{-1} \in M_\gamma(\omega r^{-p}) - (z-p)^{-1}$ extends $\mathcal{A}(\mathbb{C})$, and hence to an entire function. The assertion in general then follows by differentiating this relation with respect to p. □

Lemma 2.1.16 *Let $f(z)$ be a meromorphic function with a pole at $p \in \mathbb{C}$ of multiplicity $m+1$ and*

$$f(z) = \sum_{k=0}^{m} c_k (z-p)^{-(k+1)} + f_0(z)$$

the Laurent expansion ($f_0(z)$ holomorphic near p). Then for $\varepsilon > 0$ so small that $\{z :\ |z-p| \leq \varepsilon\}$ contains no other pole of f we have

$$\frac{1}{2\pi i}\int\limits_{|z-p|=\varepsilon} f(z)h(z)\,dz = \sum_{k=0}^{m}\frac{c_k}{k!}\left(\frac{d}{dz}\right)^k h(z)\bigg|_{z=p} \tag{2.1.17}$$

for every $h \in \mathcal{A}(\mathbb{C})$ (with counter clockwise integration).

Proof. Let $h(z) = \sum_{j=0}^\infty \frac{1}{j!}(z-p)^j((\frac{d}{dz})^j h)(p)$ be the Taylor expansion of h near p. Then $f(z)h(z)$ equals

$$\sum_{\substack{0\leq k\leq m\\ j-(k+1)<0}} c_k\frac{1}{j!}(z-p)^{j-(k+1)}\left(\left(\frac{d}{dz}\right)^j h\right)(p)$$

modulo a holomorphic function. Thus the residue of $f(z)h(z)$ at p equals the right hand side of (2.1.17) and hence the assertion is a consequence of the residue theorem. □

Corollary 2.1.17 *Inserting $h(z) = r^{-z}$ it follows that*

$$\frac{1}{2\pi i} \int\limits_{|z-p|=\varepsilon} f(z) r^{-z}\, dz = \sum_{k=0}^{m} \frac{(-1)^k}{k!} c_k r^{-p} \log^k r.$$

Remark 2.1.18 *Relation* (2.1.17) *can be interpreted in terms of analytic functionals as follows. If $K \subset \mathbb{C}$ is a compact set and $f(z) \in \mathcal{A}(\mathbb{C} \setminus K)$ then*

$$\langle \zeta_f, h \rangle := \frac{1}{2\pi i} \int\limits_{C} f(z) h(z)\, dz$$

for a circle C surrounding K, $h \in \mathcal{A}(\mathbb{C})$, defines an element $\zeta_f \in \mathcal{A}'(\mathbb{C})$ (the dual of $\mathcal{A}(\mathbb{C})$, i.e., an analytic functional). In particular, if $K = \{p\}$ is a pole of f of multiplicity $m+1$, the analytic functional is a finite linear combination of derivatives of the Dirac distribution at p. In particular, for

$$f(z) = \sum_{k=0}^{m} (-1)^k k! c_k (z-p)^{-(k+1)}$$

we get

$$\langle \zeta_f, r^{-z} \rangle = \sum_{k=0}^{m} c_k r^{-p} \log^k r$$

which are the singular functions (up to the cut–off factor ω) of the asymptotics in Definition 2.1.14. *For $f \in \mathcal{A}(\mathbb{C} \setminus K)$ in general we get continuous asymptotics, cf. [53], [58]; the asymptotics in Definition* 2.1.14 *are then called discrete.*

To derive asymptotics for $r \to 0$ of the solution $u(r)$ to equation (2.1.13) we write $u = \mathrm{op}_M(a^{-1})(r^\mu f) = M^{-1}(a^{-1}(z)g(z))$ for $g(z) = M(r^\mu f)(z) = (Mf)(z+\mu)$, cf. the proof of Proposition 2.1.9. Let $\chi_C(z) \in C^\infty(\mathbb{C})$ for $C > 0$ be any fixed function with $\chi_C(z) = 0$ for $|\operatorname{Im} z| < C$, $\chi_C(z) = 1$ for $|\operatorname{Im} z| > 2C$. It is easy to see that $g(z)$ is meromorphic with simple poles at the points $z = -k - \mu$, $k \in \mathbb{N}$, cf. Lemma 2.1.15, and that $\chi_C(z)g(z)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta)$ for each $\beta \in \mathbb{R}$, uniformly in compact β–intervals.

The function $a^{-1}(z)g(z)$ is also meromorphic, where there is a $C > 0$ such that the strip $|\operatorname{Im} z| < C$ contains all poles, and $\chi_C(z)a^{-1}(z)g(z)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta)$

for each $\beta \in \mathbb{R}$, uniformly in compact β–intervals. Now the inverse Mellin transform in the expression for u, namely

$$u(r) = \frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{1}{2}}} r^{-z} a^{-1}(z) g(z)\, dz,$$

can be rewritten in the form

$$u(r) = \frac{1}{2\pi i} \int\limits_{\Gamma_{\frac{1}{2}-\varrho}} r^{-z} a^{-1}(z) g(z)\, dz + \frac{1}{2\pi i} \sum_{j=0}^{N} \int\limits_{|z-p_j|=\varepsilon} r^{-z} a^{-1}(z) g(z)\, dz \tag{2.1.18}$$

for every $\varrho \geq 0$, such that $\Gamma_{\frac{1}{2}-\varrho}$ does not contain a pole of $a^{-1}(z)g(z)$. Here $p_0, \ldots, p_N$ are the poles of $a^{-1}(z)g(z)$ in $\{z : \frac{1}{2} - \varrho < \operatorname{Re} z < \frac{1}{2}\}$ and $\varepsilon > 0$ is chosen so small, such that $|z - p_j| \leq \varepsilon$ does not contain any other pole, for all j. The formula (2.1.18) follows from Cauchy's theorem by a deformation of the integration contour $\Gamma_{\frac{1}{2}}$ to $\Gamma_{\frac{1}{2}-\varrho} \cup \bigcup_{j=0}^{N} \{z : |z - p_j| = \varepsilon\} \cup \bigcup_{j=0}^{N} I_j$, for curves I_j which connect the circles $|z - p_j| = \varepsilon$ with $\Gamma_{\frac{1}{2}-\varrho}$, where $I_j \subset \{z : \operatorname{Re} z \geq \frac{1}{2} - \varrho\}$ and $I_j \cap \{p_l\} = \emptyset$ for all $l \neq j$. We can take I_j as intervals parallel to the real axis, namely $I_j = \{z : \operatorname{Im} z = \operatorname{Im} p_j,\ \frac{1}{2} - \varrho \leq \operatorname{Re} z \leq \operatorname{Re} p_j - \varepsilon\}$ when I_j does not meet the points p_l for all $l \neq j$, otherwise I_j may be chosen as small deformations of such intervals to avoid intersections. Integrations over I_j give zero because they are carried out twice with opposite orientation. Let $u_\varrho(r)$ be the first summand on the right side of (2.1.18). Then we have $u \in r^\varrho L^2(\mathbb{R}_+)$ by (2.1.5). Applying Corollary 2.1.17 it follows that

$$u(r) = u_\varrho(r) + \sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk} r^{-p_j} \log^k r \tag{2.1.19}$$

with certain coefficients c_{jk}, where $m_j + 1$ is the multiplicity of p_j. Multiplying (2.1.19) by a cut–off function $\omega(r)$ we obtain

$$\omega u - \omega(r) \sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk} r^{-p_j} \log^k r \in \mathcal{K}^{0,\varrho}(\mathbb{R}_+).$$

The latter relation remains valid if we omit the factor ω at u. Varying ϱ, this shows altogether $u \in \mathcal{K}_P^{0,0}(\mathbb{R}_+)$ for $P = \{(p_j, m_j)\}_{j \in \mathbb{N}}$.

As we shall see below the operator in (2.1.13) belongs to the cone algebra on the half–axis, cf. Section 2.2.4, and we have elliptic regularity in $\mathcal{K}^{s,\gamma}$–spaces with asymptotics in general, cf. Corollary 2.2.30.

The considerations in connection with Proposition 2.1.10 for equation (2.1.13) can be generalised in various ways. We may generalise the spaces to spaces $\mathcal{H}^{s,(\gamma_0,\gamma_\infty)}(\mathbb{R}_+)$ with different weights γ_0 and γ_∞ at 0 and ∞, respectively, and also solve Fuchs type equations in such spaces as well as in subspaces with asymptotics. This is studied in a systematic manner in a joint paper with Tarkhanov [79] for the case $\mathbb{R}_+ \times X$ in place of $\mathbb{R}_+$ with a closed compact C^∞ manifold X.

Let us generalise asymptotics to the spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$. Denote by $\mathrm{As}(X, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \Theta)$, $\Theta = (\vartheta, 0]$, $-\infty \leq \vartheta < 0$, the set of all sequences

$$P = \{(p_j, m_j, M_j)\}_{j=0,\dots,N} \qquad \text{for } \vartheta > -\infty,$$

$N = N(P)$ (the empty sequences are also admitted) where

$$\pi_{\mathbb{C}} P = \{p_j\}_{j=0,\dots,N} \subset \Gamma_{(\frac{n+1}{2}-\gamma+\vartheta,\frac{n+1}{2}-\gamma)},$$

and

$$P = \{(p_j, m_j, M_j)\}_{j\in\mathbb{N}} \qquad \text{for } \vartheta = -\infty,$$

with $\mathrm{Re}\, p_j < \frac{n+1}{2} - \gamma$, $\mathrm{Re}\, p_j \to -\infty$ as $j \to \infty$, $m_j \in \mathbb{N}$, and $M_j \subset C^\infty(X)$ being finite–dimensional subspaces.

The elements in $\mathrm{As}(X, \boldsymbol{g})$ are called discrete asymptotic types to the weight data $\boldsymbol{g} = (\gamma, \Theta)$. Set

$$\mathcal{K}^{s,\gamma}_{\Theta}(X^\wedge) = \bigcap_{\varepsilon>0} \mathcal{K}^{s,\gamma-\vartheta-\varepsilon}(X^\wedge).$$

Denote by $\mathcal{K}^{s,\gamma}_P(X^\wedge)$ for $P \in \mathrm{As}(X, \boldsymbol{g})$ and finite Θ the subspace of all $u \in \mathcal{K}^{s,\gamma}(X^\wedge)$ such that there are coefficients $c_{jk} \in M_j$, $0 \leq k \leq m_j$, $0 \leq j \leq N(P)$, such that

$$u(r, x) - \omega(r) \sum_{j=0}^{N} \sum_{k=0}^{m_j} c_{jk} r^{-p_j} \log^k r \in \mathcal{K}^{s,\gamma}_{\Theta}(X^\wedge).$$

The space $\mathcal{K}^{s,\gamma}_P(X^\wedge)$ is Fréchet in a canonical way.

Setting

$$\mathcal{E}_P(X^\wedge) = \omega[\![\, c_{jk} r^{-p_j} \log^k r : \ c_{jk} \in M_j, \\ p_j \in \pi_{\mathbb{C}} P, \ 0 \le k \le m_j, \ j = 0, \dots, N \,]\!] \quad (2.1.20)$$

(with $[\![\ \dots\ ]\!]$ denoting the linear span of the functions in the brackets) we can write

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) = \mathcal{K}_\Theta^{s,\gamma}(X^\wedge) + \mathcal{E}_P(X^\wedge)$$

which is a direct sum.

For infinite Θ we form $P_l = \{(p, m, M) \in P : \ \operatorname{Re} p < \frac{n+1}{2} - \gamma - (l+1)\}$, $l \in \mathbb{N}$, and set

$$\mathcal{K}_P^{s,\gamma}(X^\wedge) = \bigcap_{l \in \mathbb{N}} \mathcal{K}_{P_l}^{s,\gamma}(X^\wedge)$$

in the Fréchet topology of the projective limit.

In an analogous manner we can define the spaces $\mathcal{K}_P^{s,\gamma;\varrho}(X^\wedge)$, since asymptotics near $r = 0$ do not affect the properties for large r. We then have continuous embeddings $\mathcal{K}_P^{s',\gamma;\varrho'}(X^\wedge) \hookrightarrow \mathcal{K}_P^{s,\gamma;\varrho}(X^\wedge)$ for all $s' \ge s$, $\varrho' \ge \varrho$. Define

$$\mathcal{S}_P^\gamma(X^\wedge) = \varprojlim_{j \in \mathbb{N}} \mathcal{K}_P^{j,\gamma;j}(X^\wedge) \quad (2.1.21)$$

which is a nuclear Fréchet space in the projective limit topology.

2.1.3 A factorisation of meromorphic functions

In this section we investigate the following problem. Consider an elliptic element $a(z) \in M_R^\mu$, $\mu \in \mathbb{R}$, $R \in \mathbf{As}$. Does there exist a smoothing Mellin symbol $g(z) \in M_S^{-\infty}$ for a certain $S \in \mathbf{As}$ such that $a(z)$ can be decomposed in the form

$$a(z) = h(z)(1 + g(z)) \quad (2.1.22)$$

for some $h(z) \in M_O^\mu$ (which is necessarily elliptic) with the property $h^{-1}(z) \in M_O^{-\mu}$? The answer is positive; the proof is due to Witt [96]. First note that the problem can be reduced to the case $a(z) \in M_O^\mu$. In fact, we have the decomposition $M_R^\mu = M_O^\mu + M_R^{-\infty}$, cf. Theorem 2.1.13, and hence $a(z) =$

$f(z)+r(z)$ for certain $f(z)\in M_O^\mu$, $r(z)\in M_R^{-\infty}$, where $f(z)$ is necessarily elliptic. Because $f^{-1}(z)\in M_P^{-\mu}$ for some $P\in \mathbf{As}$, we get

$$a(z)=f(z)(1+f^{-1}(z)r(z))=f(z)(1+g_0(z)) \tag{2.1.23}$$

with $g_0(z)=f^{-1}(z)r(z)\in M_{S_0}^{-\infty}$ for an $S_0\in\mathbf{As}$. If we assume that $f(z)=h(z)(1+l(z))$ is a factorisation as desired, i.e., $h(z)\in M_O^\mu$, $h^{-1}(z)\in M_O^{-\mu}$, $l\in M_P^{-\infty}$ for some $P\in\mathbf{As}$, then (2.1.22) is an obvious consequence of (2.1.23), with $g(z)=l(z)+g_0(z)+l(z)g_0(z)$.

Remark 2.1.19 *Let $f(z)\in M_O^\mu$ be elliptic and let $f(z)=h(z)+l(z)$ be a decomposition into $h(z)\in M_O^\mu$ with $h^{-1}(z)\in M_O^{-\mu}$ and $l(z)\in M_O^{-\infty}$. Then it follows that $f(z)=h(z)(1+h^{-1}(z)l(z))$ is a corresponding multiplicative decomposition of $f(z)$.*

Lemma 2.1.20 (i) *Let $c(\tau)\in S^0_{\mathrm{cl}}(\mathbb{R})$ and $d(\tau)=e^{c(\tau)}$. Then $d(\tau)\in S^0_{\mathrm{cl}}(\mathbb{R})$ is an elliptic symbol with $d(\tau)\neq 0$ for all $\tau\in\mathbb{R}$.*

(ii) *Let $d(\tau)\in S^0_{\mathrm{cl}}(\mathbb{R})$ be elliptic, $d(\tau)\neq 0$ for all $\tau\in\mathbb{R}$. Then there exists a $c(\tau)\in S^0_{\mathrm{cl}}(\mathbb{R})$ which is unique modulo $2\pi i\mathbb{Z}$ such that $d(\tau)=e^{c(\tau)}$, i.e. $c(\tau)=\log d(\tau)$.*

Proof. To show (i) we define $e^{c(\tau)}$ as a convergent series $\sum_{k=0}^\infty \frac{1}{k!}(c(\tau))^k$, and it is elementary to check that it belongs to $S^0_{\mathrm{cl}}(\mathbb{R})$ and has the desired properties.

Next consider (ii). In order to define $\log d(\tau)$ we first observe that $D=\{d(\tau):\ \tau\in\mathbb{R}\}$ is a smooth curve of finite length (possibly with intersection points), with smoothness up to the end points $d(-\infty)$ and $d(+\infty)$ which comes from the property that $d(\tau)$ is a classical symbol. To every $\varepsilon>0$ there exists an $R_-(\varepsilon)<0$ such that $|d(-\infty)-d(\tau)|<\varepsilon$ for all $\tau\le R_-(\varepsilon)$. Thus, for $\log d(\tau)$, $\tau\le R_-(\varepsilon)$, we can choose a branch of the logarithm where $\log d(\tau)$ is a C^∞ function on $-\infty<\tau<R_-(\varepsilon)$, which is a C^∞ curve $\tilde{C}_-$ smooth up to the end point $c(-\infty)=\log d(-\infty)$. We may regard $\tilde{C}_-$ as a curve on the Riemannian surface of the logarithm function. This curve has a unique extension to a curve $\tilde{C}_1$ on that surface when we continue to define the logarithm $\log d(\tau)$ on the interval $R_-(\varepsilon)\le\tau\le R_1(\varepsilon)$ for a choice of $R_1(\varepsilon)$ such that $|d(R_-(\varepsilon))-d(\tau)|<\varepsilon$ for all $R_-(\varepsilon)\le\tau\le R_1(\varepsilon)$. Further $\tilde{C}_1$ can be continued to a curve $\tilde{C}_2$ by the values $\log d(\tau)$ on a sufficiently small interval $R_1(\varepsilon)\le\tau\le R_2(\varepsilon)$ where $|d(R_1(\varepsilon))-d(\tau)|<\varepsilon$ for all $R_1(\varepsilon)\le\tau\le R_2(\varepsilon)$. After finitely many steps we arrive at an $R_N(\varepsilon)=:R_+(\varepsilon)$ such that $|d(R_+(\varepsilon)) - d(\tau)| < \varepsilon$ for all $R_+(\varepsilon) \leq \tau < \infty$ and we can complete $\tilde{C}_N$ to a unique curve $\tilde{C}$ on the Riemannian surface of the logarithm, only determined by the preceding steps. The projection of the curve into the complex plane is a finite C^∞ curve C, smooth up to the end points $c(-\infty) = \log d(-\infty)$, $c(+\infty) = \log d(+\infty)$, with the logarithm being defined in the described way, and that curve is parametrised by $\tau \in \mathbb{R}$ and represents the value of an element $c(\tau) \in S^0_{\mathrm{cl}}(\mathbb{R})$. Clearly $c(\tau)$ is unique modulo $2\pi i\mathbb{Z}$, but $d(\tau) = e^{c(\tau)}$. □

Lemma 2.1.21 *For a function $b(\tau) \in C^\infty(\mathbb{R})$ the following conditions are equivalent:*

(i) *$b(\tau) = \log a(\tau)$ for an element $a(\tau) \in S^\mu_{\mathrm{cl}}(\mathbb{R})$ which is elliptic and satisfies $a(\tau) \neq 0$ for all $\tau \in \mathbb{R}$.*

(ii) *$b(\tau) = \mu \log\langle\tau\rangle + c(\tau)$ for some $c(\tau) \in S^0_{\mathrm{cl}}(\mathbb{R})$.*

(iii) *$\frac{\partial}{\partial\tau} b(\tau) \in S^{-1}_{\mathrm{cl}}(\mathbb{R})$ and $\frac{\partial}{\partial\tau} b(\tau) \sim \mu\tau^{-1} + \sum_{k=2}^\infty \gamma_k^\pm \tau^{-k}$ for $\tau \to \pm\infty$, for suitable constants $\gamma_k^\pm \in \mathbb{C}$.*

Proof. (i)⇒(ii). Let us write $a(\tau) = \langle\tau\rangle^\mu d(\tau)$ or $d(\tau) = a(\tau)\langle\tau\rangle^{-\mu}$ which belongs to $S^0_{\mathrm{cl}}(\mathbb{R})$ and fulfills the conditions of Lemma 2.1.20, (ii). It follows that there exists $c(\tau) = \log d(\tau) \in S^0_{\mathrm{cl}}(\mathbb{R})$ and then $b(\tau) = \mu \log\langle\tau\rangle + c(\tau)$.

(ii)⇒(i). We have $\mu \log\langle\tau\rangle + c(\tau) = \log(\langle\tau\rangle^\mu e^{c(\tau)})$ where $e^{c(\tau)}$ and hence $a(\tau) = \langle\tau\rangle^\mu e^{c(\tau)}$ is elliptic and $\neq 0$ for all $\tau \in \mathbb{R}$.

(i)⇒(iii). Differentiating the relation $b(\tau) = \mu \log\langle\tau\rangle + c(\tau)$ yields $b'(\tau) = \mu\langle\tau\rangle^{-1}\frac{d}{d\tau}\langle\tau\rangle + \frac{d}{d\tau}c(\tau)$ which gives us the representation (iii); here we use that $\frac{d}{d\tau}S^0_{\mathrm{cl}}(\mathbb{R}) \subset S^{-2}_{\mathrm{cl}}(\mathbb{R})$.

(iii)⇒(i). Relation (iii) shows that there is a symbol $d(\tau) \in S^{-2}_{\mathrm{cl}}(\mathbb{R})$ such that $\frac{\partial}{\partial\tau} b(\tau) = \mu\frac{\tau}{\langle\tau\rangle^2} + d(\tau)$. This yields

$$\int_0^\tau \frac{\partial}{\partial\eta} b(\eta)\, d\eta = \mu \int_0^\tau \frac{\eta}{\langle\eta\rangle^2}\, d\eta + h(\tau)$$

for $h(\tau) = \int_0^\tau d(\eta)\, d\eta$, i.e.,

$$\begin{aligned} b(\tau) &= \mu \log\langle\tau\rangle + h(\tau) + \mathrm{const} \\ &= \mu \log\langle\tau\rangle + \log e^{h(\tau)+\mathrm{const}} \\ &= \log\langle\tau\rangle^\mu e^{h(\tau)+\mathrm{const}} = \log a(\tau). \end{aligned}$$

The symbol $a(\tau) = \langle\tau\rangle^{\mu} e^{h(\tau)+\text{const}}$ is obviously elliptic and we have $a(\tau) \neq 0$ for all $\tau \in \mathbb{R}$. $\square$

Remark 2.1.22 *Let $a(\tau) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R})$ be elliptic and $a(\tau) \neq 0$ for all $\tau \in \mathbb{R}$. Then $\log a(\tau) \in S^{\delta}(\mathbb{R})$ for every $\delta > 0$.*

Theorem 2.1.23 *Let $f(z) \in M^{\mu}_{R}$ be elliptic, $\mu \in \mathbb{R}$, $R \in \mathbf{As}$. Then there exists a $g(z) \in M^{-\infty}_{S}$ for some $S \in \mathbf{As}$ such that $f(z) = h(z)(1 + g(z))$ holds for an elliptic element $h(z) \in M^{\mu}_{O}$ with $h^{-1}(z) \in M^{-\mu}_{O}$.*

Proof. As noted in the beginning it suffices to assume $f(z) \in M^{\mu}_{O}$. The ellipticity of f implies that for every $c < c'$ there is a constant $N(c, c')$ such that $f(z) \neq 0$ for $c \leq \operatorname{Re} z \leq c'$ and $|\operatorname{Im} z| \geq N(c, c')$. Since f is holomorphic, there is only a finite number of zeros in $c \leq \operatorname{Re} z \leq c'$, $|\operatorname{Im} z| < N(c, c')$. Thus there exists a $\beta \in \mathbb{R}$ such that $f(\beta + i\tau) \in S^{\mu}_{\mathrm{cl}}(\mathbb{R}_{\tau})$ is $\neq 0$ for all $\tau \in \mathbb{R}$. Without loss of generality we may assume $\beta = 0$ (otherwise we can apply a corresponding translation of the complex plane). By virtue of Remark 2.1.22 we have $\log(f(z)|_{\Gamma_0}) \in S^{\delta}(\Gamma_0)$. We want to apply the kernel cut–off operator $H(\psi)$, associated with a function $\psi(\varrho) \in C^{\infty}_{0}(\mathbb{R}_{+})$ that equals 1 in a neighbourhood of $\varrho = 1$ (cf. Theorem 2.2.3 and Remark 2.2.9 below). Writing $b(\tau) = \log f(i\tau)$ we have

$$n(z) := (H(\psi)b)(z) = \frac{1}{2\pi i} \int\limits_{0}^{\infty} \int\limits_{\Gamma_0} \varrho^{z - i\tilde{\tau}} \psi(\varrho) \varrho^{-1} b(i\tilde{\tau}) \, d\tau d\varrho$$

which belongs to N^{δ}_{O}, for each $\delta > 0$, cf. Remark 2.2.9 below. We claim that

$$h(z) = e^{n(z)} \in M^{\mu}_{O}$$

is a symbol with the property $h(z) - f(z) \in M^{-\infty}_{O}$. Writing $b(z) = \log(f(z)|_{\Gamma_0})$ we have

$$n(z) := (H(\psi)b)(z) = \frac{1}{2\pi i} \int\limits_{0}^{\infty} \int\limits_{\Gamma_0} \varrho^{z - \tilde{z}} \psi(\varrho) \varrho^{-1} b(\tilde{z}) \, d\tilde{z} d\varrho.$$

This belongs to the space N^{δ}_{O} for each $\delta > 0$, cf. Theorem 2.2.3 below. We want to show the relations $h(z) = e^{n(z)} \in M^{\mu}_{O}$ and $h(z) - f(z) \in M^{-\infty}_{O}$.

First we verify that $n(z)|_{\Gamma_\beta}$ satisfies condition (iii) in Lemma 2.1.21 for each β. In fact, writing $\varphi(z) = \int_0^\infty \varrho^z \psi(\varrho)\varrho^{-1}\,d\varrho$, we have

$$\begin{aligned} n(z)|_{\Gamma_0} &= \frac{1}{2\pi i}\int\limits_{\Gamma_0} \varphi(z-\tilde{z})b(\tilde{z})\,d\tilde{z} \\ &= \frac{1}{2\pi i}\int\limits_{\Gamma_0} \varphi(\tilde{z})b(z-\tilde{z})|_{\Gamma_0}\,d\tilde{z}. \end{aligned}$$

Differentiation with respect to z along Γ_0 gives us

$$\begin{aligned} \frac{d}{dz}n(z)|_{\Gamma_0} &= \frac{1}{2\pi i}\int\limits_{\Gamma_0} \varphi(\tilde{z})\left(\frac{d}{dz}b\right)(z-\tilde{z})|_{\Gamma_0}\,d\tilde{z} \\ &= \frac{1}{2\pi i}\int\limits_{\Gamma_0} \varphi(z-\tilde{z})\frac{d}{d\tilde{z}}b(\tilde{z})\,d\tilde{z}. \end{aligned}$$

This yields $\frac{d}{dz}n(z) = H(\psi)(\frac{d}{dz}b)(z)$. Since $\frac{d}{dz}b(z) \in S^{-1}_{\mathrm{cl}}(\Gamma_0)$ fulfills condition (iii) of Lemma 2.1.21. Since the kernel cut–off operator preserves the homogeneous principal symbol, also $\frac{d}{dz}n(z)$ satisfies condition (iii) on Γ_β for each $\beta \in \mathbb{R}$. Applying Lemma 2.1.21 we get $h(z)|_{\Gamma_\beta} = e^{n(z)}|_{\Gamma_\beta} \in S^{\mu}_{\mathrm{cl}}(\Gamma_\beta)$ for every $\beta \in \mathbb{R}$. This holds uniformly in compact intervals with respect to β; together with the holomorphy of $h(z)$ we obtain $h(z) \in M^{\mu}_{O}$. From the properties of the kernel cut–off operator we know that

$$n(z)|_{\Gamma_0} - b(z)|_{\Gamma_0} \in S^{-\infty}(\Gamma_0).$$

This implies $e^{n(z)}|_{\Gamma_0} - f(z)|_{\Gamma_0} = (h(z)-f(z))|_{\Gamma_0} \in S^{-\infty}(\Gamma_0)$. Hence it follows that $h(z) - f(z) \in M^{-\infty}_{O}$, cf. Remark 2.2.9 below. By construction $h(z)$ has no zeros in the complex plane. Thus, to complete the proof of Theorem 2.1.23, it suffices to apply Remark 2.1.19. $\square$

2.1.4 Quasi–homogeneity

We want to show that differential operators of Fuchs type also appear in connection with quasi–homogeneous functions in the following sense.

Let $f(x) \in C^\infty(\mathbb{R}^n \setminus \{0\})$, $x = (x_1, \dots, x_n)$, and set

$$(\delta_\lambda f)(x) = f(\lambda^{l_1}x_1, \dots, \lambda^{l_n}x_n)$$

for $\lambda \in \mathbb{R}_+$, with fixed $l_j \in \mathbb{N}\setminus\{0\}$, $j = 1, \dots, n$. Let us set $K = 2l_1 l_2 \cdot \ldots \cdot l_n$ and $k_j = K/l_j$, $j = 1, \dots, n$.

Definition 2.1.24 *A function $f \in C^\infty(\mathbb{R}^n\setminus\{0\})$ is called quasi–homogeneous of order $\nu \in \mathbb{R}$ if*

$$(\delta_\lambda f)(x) = \lambda^\nu f(x) \qquad \text{for all } \lambda \in \mathbb{R}_+,\ x \in \mathbb{R}^n \setminus \{0\}.$$

Example 2.1.25 *Let $r = (\sum_{j=1}^n |x_j|^{k_j})^{\frac{1}{K}}$. Then r is quasi–homogeneous of order 1. We also set*

$$|x|_l = r \qquad \text{for } l = (l_1, \dots l_n)$$

and call $|x|_l$ the quasi–homogeneous absolute value of x.

Let $S_l^{n-1} = \{x \in \mathbb{R}^n : \ |x|_l = 1\}$. Then, if $x \neq 0$, we have

$$\varphi := \left(\frac{x_1}{|x|_l^{l_1}}, \dots, \frac{x_n}{|x|_l^{l_n}}\right) \in S_l^{n-1},$$

and hence we get a diffeomorphism

$$\chi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}_+ \times S_l^{n-1},$$

$\chi(x) = (r, \varphi) = (|x|_l, \varphi)$. This allows us to introduce anisotropic polar coordinates.

Remark 2.1.26 *A quasi–homogeneous function $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ of order ν can be written*

$$f(x) = r^\nu f_1(\varphi) \tag{2.1.24}$$

with $\varphi = \varphi(x)$ as above and $f_1(\varphi) = f(\frac{x_1}{|x|_l^{l_1}}, \dots, \frac{x_n}{|x|_l^{l_n}}) \in C^\infty(S_l^{n-1})$.

In particular, for the coordinate function x_j we can write $x_j = r^{l_j}\psi_j(\varphi)$ for certain $\psi_j(\varphi) \in C^\infty(S_l^{n-1})$. Thus $r\frac{\partial}{\partial r}x_j = l_j x_j$. From $r\frac{\partial}{\partial r}f(x) = \sum_{j=1}^n r\frac{\partial x_j}{\partial r}\frac{\partial f}{\partial x_j}$ it follows that

$$r\frac{\partial}{\partial r} = \sum_{j=1}^n l_j x_j \frac{\partial}{\partial x_j}. \tag{2.1.25}$$

Definition 2.1.27 *An operator* $A : C_0^\infty(\mathbb{R}^n \setminus \{0\}) \to C^\infty(\mathbb{R}^n \setminus \{0\})$ *is called quasi–homogeneous of order* μ *if* $A = \lambda^\mu \delta_\lambda A \delta_\lambda^{-1}$ *for all* $\lambda \in \mathbb{R}_+$.

Remark 2.1.28 *If operators* A *and* B *are quasi–homogeneous of order* μ *and* ν, *respectively, then* AB *(provided the composition is defined) is quasi–homogeneous of order* $\mu + \nu$. *The operator of multiplication by a quasi–homogeneous function of order* ν *is quasi–homogeneous of order* $-\nu$. *If* A *is a differential operator, quasi–homogeneous of order* μ *and* f *a* C^∞ *function, quasi–homogeneous of order* ν, *then* Af *is a quasi–homogeneous function of order* $\nu - \mu$.

Proposition 2.1.29 *The operator* $\frac{\partial}{\partial x_i}$ *is quasi–homogeneous of order* l_i. *The operator of multiplication by* x_i *is quasi–homogeneous of order* $-l_i$. *The operator* $\frac{\partial}{\partial r}$ *is quasi–homogeneous of order* 1, *the operator of multiplication by* r *is quasi–homogeneous of order* -1.

The proof is straightforward and left to the reader.

We now introduce anisotropic polar coordinates into a differential operator $A = \sum_{|\alpha| \le m} a_\alpha(x) D_x^\alpha$ with $a_\alpha(x) \in C^\infty(\mathbb{R}^n)$. The functions $(a_\alpha|_{\mathbb{R}^n \setminus \{0\}}) \circ \chi^{-1}$ belong to $C^\infty(\overline{\mathbb{R}}_+ \times S_l^{n-1})$. So it suffices to express the derivatives $\partial/\partial x_i$ in anisotropic polar coordinates.

We have for a function $u(r, \varphi)$

$$\frac{\partial}{\partial x_i} u = \frac{\partial r}{\partial x_i} \frac{\partial u}{\partial r} + \sum_{k=1}^{n-1} \frac{\partial \varphi_k}{\partial x_i} \frac{\partial u}{\partial \varphi_k},$$

where $\varphi = (\varphi_1, \dots \varphi_{n-1})$ are local coordinates on S_l^{n-1}. These may be regarded as functions $\varphi_k(x)$ with $\delta_\lambda \varphi_k = \varphi_k$ for all $\lambda \in \mathbb{R}_+$. Further $\delta_\lambda r = \lambda r$ for $r = r(x)$. By Remark 2.1.28 and Proposition 2.1.29 we have

$$\delta_\lambda \frac{\partial r}{\partial x_i} = \lambda^{1-l_i} \frac{\partial r}{\partial x_i}, \qquad \delta_\lambda \frac{\partial \varphi_k}{\partial x_i} = \lambda^{-l_i} \frac{\partial \varphi_k}{\partial x_i}.$$

Using relation (2.1.24) we get

$$\frac{\partial r}{\partial x_i} = r^{1-l_i} g_i(\varphi), \qquad \frac{\partial \varphi_k}{\partial x_i} = r^{-l_i} h_{ki}(\varphi)$$

for suitable functions $g_i, h_{ki} \in C^\infty(S_l^{n-1})$. It follows that

$$\frac{\partial}{\partial x_i} = r^{-l_i} \left\{ g_i(\varphi) r \frac{\partial}{\partial r} + \sum_{k=1}^{n-1} h_{ki}(\varphi) \frac{\partial}{\partial \varphi_k} \right\}. \tag{2.1.26}$$

We want to apply this, in particular, to a differential operator $A = a(x, D_x)$ whose symbol has the form

$$a(x,\xi) = \sum_{j=0}^{\mu} a_{(\mu-j)}(x,\xi),$$

where $a_{(\mu-j)}(x,\xi)$ is a quasi–homogeneous polynomial in ξ of order $\mu - j$ with coefficients in $C^\infty(\mathbb{R}^n)$. In other words

$$a_{(\mu-j)}(x, \lambda^{l_1}\xi, \dots, \lambda^{l_n}\xi_n) = \lambda^{\mu-j} a_{(\mu-j)}(x,\xi)$$

for $j = 0, \dots, \mu$. Using (2.1.26) we obtain

$$a_{(\mu-j)}(x, D_x) = r^{-\mu+j} \sum_{k=0}^{\mu-j} b_{jk}(r) \left(-r\frac{\partial}{\partial r}\right)^k$$

with coefficients $b_{jr}(r) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j-k}(S_l^{n-1}))$. Thus, in anisotropic polar coordinates, the operator $A = a(x, D_x)$ takes the form

$$\begin{aligned} A &= r^{-\mu} \sum_{j=0}^{\mu} \sum_{k=0}^{\mu-j} r^j b_{jk}(r) \left(-r\frac{\partial}{\partial r}\right)^k \\ &= r^{-\mu} \sum_{j=0}^{\mu} a_j(r) \left(-r\frac{\partial}{\partial r}\right)^j \end{aligned}$$

with coefficients $a_j(r) \in C^\infty(\overline{\mathbb{R}}_+, \mathrm{Diff}^{\mu-j}(S_l^{n-1}))$. This gives us an operator of Fuchs type in the sense of the notation above.

Proposition 2.1.30 *An $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ is quasi–homogeneous of order ν if and only if it satisfies (the anisotropic analogue of Euler's homogeneity relation)*

$$\sum_{j=1}^{n} l_j x_j \frac{\partial}{\partial x_j} f = \nu f. \tag{2.1.27}$$

Proof. $\delta_\lambda f = \lambda^\nu f$ for all $\lambda \in \mathbb{R}_+$ yields relation (2.1.27) when we differentiate with respect to λ. On the other hand, (2.1.27) is equivalent to $r\frac{\partial}{\partial r} f = \nu f$, cf. relation (2.1.25), and hence f has the form (2.1.24) for some $f_1(\varphi) \in C^\infty(S_l^{n-1})$ which means quasi–homogeneity of order ν. $\square$

Let us make some further remarks on polar coordinates and Sobolev spaces. For simplicity we now consider the case $l_1 = \cdots = l_n = 1$. Then the diffeomorphism

$$\chi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}_+ \times S^{n-1}$$

corresponds to standard polar coordinates.

We want to compare the space $\omega H^s(\mathbb{R}^n) = \{\omega u : \ u \in H^s(\mathbb{R}^n)\}$ for a cut–off function ω (say, of the form $\omega(|x|)$ for an $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$ that equals 1 near $r = 0$) with a corresponding weighted space on $\mathbb{R}_+ \times S^{n-1}$ with respect to the Mellin transform in axial direction, for $s \in \mathbb{N}$. For $s = 0$ we have a canonical identification

$$L^2(\mathbb{R}^n) = H^0(\mathbb{R}^n) = r^{-\frac{n-1}{2}} L^2(\mathbb{R}_+ \times S^{n-1}) = \mathcal{H}^{0,0}(\mathbb{R}_+ \times S^{n-1}).$$

The space $\mathcal{H}^{s,\gamma}(\mathbb{R}_+ \times S^{n-1})$, $s \in \mathbb{N}$, can equivalently be described as the set

$$\Big\{ u(r,\varphi) \in r^{\gamma - \frac{n-1}{2}} L^2(\mathbb{R}_+ \times S^{n-1}) : \\ \left(r \frac{\partial}{\partial r} \right)^k D_\varphi^l u \in r^{\gamma - \frac{n-1}{2}} L^2(\mathbb{R}_+ \times S^{n-1}) \text{ for all } k + l \le s \Big\}, \tag{2.1.28}$$

where D_φ^l runs over the set of all differential operators of order l on the sphere S^{n-1}. The space $H^s(\mathbb{R}^n)$, $s \in \mathbb{N}$, coincides with

$$\{ v(x) \in L^2(\mathbb{R}^n) : \ D_x^\alpha u \in L^2(\mathbb{R}^n) \text{ for all } |\alpha| \le s \}. \tag{2.1.29}$$

(2.1.29) is equivalent to

$$\Big\{ v(r,\varphi) \in r^{-\frac{n-1}{2}} L^2(\mathbb{R}_+ \times S^{n-1}) : \ r^{-|\alpha|} \left(r \frac{\partial}{\partial r} \right)^k D_\varphi^l u \in \\ r^{-\frac{n-1}{2}} L^2(\mathbb{R}_+ \times S^{n-1}) \text{ for all } |\alpha| = k + l \le s \Big\}.$$

It is in fact sufficient to require the conditions for all k, l with $k + l = s$:

$$H^s(\mathbb{R}^n) = \{ v(r,\varphi) \in r^{-\frac{n-1}{2}} L^2(\mathbb{R}_+ \times S^{n-1}) : \ r^{-s} (r\partial_r)^k D_\varphi^l v \\ \in r^{-\frac{n-1}{2}} L^2(\mathbb{R}_+ \times S^{n-1}), k + l = s \}, \tag{2.1.30}$$

$$\mathcal{H}^{s,s}(\mathbb{R}_+ \times S^{n-1}) = \{ u(r,\varphi) \in r^{s - \frac{n-1}{2}} L^2(\mathbb{R}_+ \times S^{n-1}) : \\ (r\partial_r)^k D_\varphi^l u \in r^{s - \frac{n-1}{2}} L^2(\mathbb{R}_+ \times S^{n-1}), k + l = s \}. \tag{2.1.31}$$

Thus $\omega\mathcal{H}^{s,s}(\mathbb{R}_+ \times S^{n-1})$ for $s \in \mathbb{N}$ equals the subspace of all $u \in \omega H^s(\mathbb{R}^n)$ (written in polar coordinates) for which $r^{-s}u \in \omega L^2(\mathbb{R}^n)$. The latter relation is equivalent to $u \in \omega\mathcal{H}^{0,s}(\mathbb{R}_+ \times S^{n-1})$.

In fact, $u \in \omega\mathcal{H}^{s,s}(\mathbb{R}_+ \times S^{n-1})$ is of the form $u = \omega r^{s-\frac{n-1}{2}} f$ for $\omega f \in L^2(\mathbb{R}_+ \times S^{n-1})$ and $r^{-s}(r\partial_r)^k D_\varphi^l u \in r^{-\frac{n-1}{2}} L^2(\mathbb{R}_+ \times S^{n-1}) = L^2(\mathbb{R}^n)$ is fulfilled. Thus we have $r^{-s}u \in \omega r^{-\frac{n-1}{2}} L^2(\mathbb{R}_+ \times S^{n-1})$ which implies $u \in \omega r^{-\frac{n-1}{2}} L^2(\mathbb{R}_+ \times S^{n-1})$, and hence, because of the required relation for the derivatives, u belongs to $H^s(\mathbb{R}^n)$.

Conversely, let $u \in \omega H^s(\mathbb{R}^n)$ and $r^{-s}u \in \omega L^2(\mathbb{R}^n)$. Then we have $r^{-s}(r\partial_r)^k D_\varphi^l \omega u \in r^{-\frac{n-1}{2}} L^2(\mathbb{R}_+ \times S^{n-1})$ for all $k+l = s$ and $u \in r^{s-\frac{n-1}{2}} \omega L^2(\mathbb{R} \times S^{n-1})$; those are just the conditions for elements in $\omega\mathcal{H}^{s,s}(\mathbb{R}_+ \times S^{n-1})$.

Proposition 2.1.31 *Let X be a closed compact C^∞ manifold. Then we have*

$$\mathcal{K}^{s,s}(X^\wedge) \subseteq \mathcal{H}^{s,s}(X^\wedge) \tag{2.1.32}$$

for all $s \in \mathbb{N}$.

Proof. For $X = S^{n-1}$ the assertion is an obvious consequence of (2.1.30), (2.1.31). For arbitrary X the equivalence of $\omega u \in \mathcal{H}^{s,s}(X^\wedge)$ and $\omega u \in \mathcal{K}^{s,s}(X^\wedge)$ holds by definition. If $\tilde{U} \subset X$ is a coordinate neighbourhood and $\tilde{U} \to U$ a diffeomorphism to an open set $U \subset S^{n-1}$, it suffices to know that for every $\varphi \in C_0^\infty(U)$ the condition $(1-\omega)\varphi u \in H^s(\mathbb{R}^n)$ implies $(1-\omega)\varphi u \in \mathcal{H}^{s,s}(\mathbb{R}_+ \times S^{n-1})$. But this is true because of the first part of the proof. $\square$

Remark 2.1.32 *The norm in $\mathcal{K}^{s,\gamma}(X^\wedge)$ for $s \in \mathbb{N}$, $\gamma \in \mathbb{R}$, $n = \dim X$, is equivalent to*

$$\left\{ \sum_{j+|\lambda| \le s} \iint |k^{-\gamma}(r)[r]^{-j-|\lambda|}(rD_r)^j D_x^\lambda u(r,x)|^2 r^n \, dr dx \right\}^{\frac{1}{2}}. \tag{2.1.33}$$

Here $k^\gamma(r)$ is a strictly positive function in $C^\infty(\mathbb{R}_+)$ with $k^\gamma(r) = r^\gamma$ near $r = 0$, $k^\gamma(r) = 1$ for $r \ge R$ for an $R > 0$, further $r \to [r]$ is a strictly positive function in $C^\infty(\mathbb{R}_+)$ with $[r] = r$ for $r \ge 1$, and D_x^λ, for a multi–index $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{N}^k$, runs over the system of all differential operators of the form $v_{j_1}^{\lambda_1} \cdot \ldots \cdot v_{j_k}^{\lambda_k}$, $v_{j_k} \in \{v_1, \dots, v_N\}$, where $\{v_1, \dots, v_N\}$ are vector fields on X which span the tangent space T_xX for all $x \in X$.

In fact, let $\{U_1, \dots , U_M\}$ be a covering of X by coordinate neighbourhoods and $\{\varphi_1, \dots , \varphi_M\}$ a subordinate partition of unity. Then we can write

$$\mathcal{K}^{s,\gamma}(X^\wedge) = \sum_{j=1}^{M} [\varphi_j]\mathcal{K}^{s,\gamma}(X^\wedge), \tag{2.1.34}$$

cf. notation (1.1.7), (1.1.8). Applying a diffeomorphism $\kappa_j : U_j \to V_j$, where V_j is an open set in the unit sphere S^n of $\mathbb{R}^{n+1}$, and extending it by homogeneity 0 to a diffeomorphism $\mathbb{R}_+ \times U_j \to \Gamma_j$ where $\Gamma_j = \{\tilde{x} \in \mathbb{R}^{n+1} \setminus \{0\} : \ \tilde{x}/|\tilde{x}| \in V_j\}$, $j = 1, \dots , M$, we have natural isomorphisms

$$[\varphi_j]\mathcal{K}^{s,\gamma}(X^\wedge) \cong [\psi_j]\mathcal{K}^{s,\gamma}(\mathbb{R}^{n+1} \setminus \{0\}),$$

where $\psi_j \in C_0^\infty(V_j)$ is defined by $\varphi_j = \kappa_j^* \psi_j$ and $\mathbb{R}^{n+1} \setminus \{0\}$ regarded as $\mathbb{R}_+ \times S^n$.

This reduces the characterisation of the spaces to $\mathbb{R}^{n+1} \setminus \{0\}$. Now for $s \in \mathbb{N}$ we have

$$\|u\|^2_{\mathcal{H}^{s,s}(X^\wedge)} = \sum_{j+|\lambda|\le s} \iint |r^{-s}(r\partial_r)^j D_x^\lambda u(r,x)|^2 r^n \, dr dx \tag{2.1.35}$$

and

$$\|v\|^2_{H^s(\mathbb{R}^{n+1})} = \sum_{|\alpha|\le s} \int |D_{\tilde{x}}^\alpha v(\tilde{x})|^2 \, d\tilde{x}$$

(equalities of norms are understood in the sense of equivalences). Specialising (2.1.35) to $X = S^n$ we get

$$\begin{aligned} \|u\|_{\mathcal{K}^{s,s}(\mathbb{R}^{n+1}\setminus\{0\})} = \Big\{ &\sum_{j+|\lambda|\le s} \iint |r^{-s}(r\partial_r)^j D_x^\lambda \omega(r) u(r,x)|^2 r^n \, dr dx \\ &+ \sum_{|\alpha|\le s} \int |D_{\tilde{x}}^\alpha (1-\omega(|\tilde{x}|)) u(\tilde{x})|^2 \, d\tilde{x} \Big\}^{\frac{1}{2}}, \end{aligned} \tag{2.1.36}$$

where $\omega(r)$ is a fixed cut–off function. In the first summand of the latter equation we may replace r^{-s} by $k^{-s}(r)$, and the second summand can be expressed in polar coordinates, cf. (2.1.26). It is then clear that (2.1.33) and (2.1.36) give equivalent norms for $X = S^n$. Using (2.1.34) we get the assertion.

Let us consider the case $n = 1$; then S^{n-1} consists of two isolated points, and we may write $\mathbb{R}_+ \times S^{n-1} = \mathbb{R}_- \cup \mathbb{R}_+ = \mathbb{R} \setminus \{0\}$. Thus $\omega\mathcal{H}^{s,s}(\mathbb{R} \setminus \{0\})$ for $s \in \mathbb{N}$ equals the subspace of all $u \in \omega H^s(\mathbb{R})$ for which $r^{-s}u \in \omega L^2(\mathbb{R})$. Now $u \in H^s(\mathbb{R})$ implies that $\partial_r^k u$ is continuous for all $0 \le k \le s-1$; so there exist the values $(\partial_r^k u)(0)$, $0 \le k \le s-1$. Relation $r^{-s}u \in \omega L^2(\mathbb{R})$ requires $u(0) = 0$ (of course, we also have $r^{-j}u \in \omega L^2(\mathbb{R})$ for $0 \le j \le s$). For $s > 1$ we can form $\partial_r u \in \omega\mathcal{H}^{s-1,s-1}(\mathbb{R} \setminus \{0\})$ which also belongs to $\omega H^{s-1}(\mathbb{R})$ and hence by the above criterion $r^{-(s-1)}\partial_r u \in \omega L^2(\mathbb{R})$. This gives us $(\partial_r u)(0) = 0$. We then obtain successively $(\partial_r^k u)(0) = 0$ for all $0 \le k \le s-1$. In other words, $\omega\mathcal{H}^{s,s}(\mathbb{R} \setminus \{0\}) \subseteq \omega H_0^s(\mathbb{R} \setminus \{0\})$. For the converse relation we suppose $u \in H_0^s(\mathbb{R} \setminus \{0\})$. We then have to show $r^{-s}u \in L^2(\mathbb{R})$. This is the consequence of the following lemma.

Lemma 2.1.33 *Let $s \in \mathbb{N}$, $u \in H^s(\mathbb{R})$, and assume $(\partial_r^k u)(0) = 0$ for all $0 \le k \le s-1$. Then $r^{-s}u \in L^2(\mathbb{R})$.*

Proof. Let us first consider the case $s = 1$ and the half–axis $\mathbb{R}_+$. We have the following estimate: If f is a measurable function on $\mathbb{R}_+$ and $1 \le p \le \infty$, $\alpha < \frac{1}{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\left\| r^\alpha \frac{1}{r} \int_0^r f(t)\, dt \right\|_{L^p(\mathbb{R}_+)} \le \left(\frac{1}{p'} - \alpha \right)^{-1} \| r^\alpha f(r) \|_{\mathbb{L}^p(\mathbb{R}_+)}. \tag{2.1.37}$$

The proof follows by substituting $t = xr$, applying Hölder's inequality and then inserting $r = x^{-1}t$ again. Now for $u \in H^1(\mathbb{R}_+)$ and $f := u'$ we apply (2.1.37) for $\alpha = 0$, $p = 2$, and get

$$\left\| \frac{u(r) - u(0)}{r} \right\|_{L^2(\mathbb{R}_+)} = \left\| \frac{1}{r} \int_0^r u'(t)\, dt \right\|_{L^2(\mathbb{R}_+)} \le 2\|u'(r)\|_{L^2(\mathbb{R}_+)} < \infty.$$

Especially for $u(0) = 0$ it follows that $r^{-1}u \in L^2(\mathbb{R}_+)$.

Now assume $u \in H^2(\mathbb{R}_+)$ and $u(0) = u'(0) = 0$. Then

$$\begin{aligned}
\left\| \frac{u(r)}{r^2} \right\|_{L^2(\mathbb{R}_+)} &= \left\| \frac{1}{r^2} \int_0^r u'(t)\, dt \right\|_{L^2(\mathbb{R}_+)} \\
&= \left\| \frac{1}{r} \int_0^r \frac{u'(t)}{r}\, dt \right\|_{L^2(\mathbb{R}_+)} \le \left\| \frac{1}{r} \int_0^r \frac{|u'(t)|}{r}\, dt \right\|_{L^2(\mathbb{R}_+)} \\
&\le \left\| \frac{1}{r} \int_0^r \frac{|u'(t)|}{t}\, dt \right\|_{L^2(\mathbb{R}_+)} \le 2 \left\| \frac{u'(r)}{r} \right\|_{L^2(\mathbb{R}_+)}.
\end{aligned}$$

From the above estimate we get $\left\| \frac{u'(r)}{r} \right\|_{L^2(\mathbb{R}_+)} \le 2 \|u''(r)\|_{L^2(\mathbb{R}_+)}$. Thus $r^{-2}u \in L^2(\mathbb{R}_+)$.

For arbitrary $s \in \mathbb{N}$ the method can be iterated; this shows the assertion for the $\mathbb{R}_+$–case in general. The considerations on $\mathbb{R} \setminus \{0\}$ are completely analogous and left to the reader. □

Summing up we get the following result.

Proposition 2.1.34 *For $s \in \mathbb{N}$ we have*

$$\omega\mathcal{H}^{s,s}(\mathbb{R} \setminus \{0\}) = \{u \in \omega H^s(\mathbb{R}) : \ (\partial_r^k u)(0) = 0 \ \textit{for} \ 0 \le k \le s-1\}.$$

Restriction to $\mathbb{R}_+$ yields $\omega\mathcal{H}^{s,s}(\mathbb{R}_+) = \omega H_0^s(\overline{\mathbb{R}}_+)$.

Remark 2.1.35 *It can be shown, cf. [42], that Proposition* 2.1.34 *holds for all $s \in \mathbb{R}$, $s > -\frac{1}{2}$. This implies $\mathcal{K}^{s,s}(\mathbb{R}_+) = H_0^s(\overline{\mathbb{R}}_+)$ for those s.*

2.2 Pseudo–differential operator–valued meromorphic functions

2.2.1 Kernel cut–off and Mellin quantisation

Let X be a closed compact C^∞ manifold, $n = \dim X \ge 1$. Denote by $L^\mu_{(\mathrm{cl})}(X; \Gamma_\beta)$ the space of all parameter–dependent pseudo–differential operators with parameter z varying on $\Gamma_\beta = \{\operatorname{Re} z = \beta\}$. The case $n = 0$ is also interesting; then we consider the symbol space $S^{\mu}_{(\mathrm{cl})}(\Gamma_\beta)$ with constant coefficients, where the covariable z ranges over $\Gamma_\beta \cong \mathbb{R}$.

If $U \subseteq \mathbb{C}$ is an open set and E a Fréchet space we denote by $\mathcal{A}(U, E)$ the space of all holomorphic functions on U with values in E, endowed with the Fréchet topology from the (completed) projective tensor product $\mathcal{A}(U) \otimes_\pi E$, or equivalently, by the system of semi–norms

$$\sup\{\pi_j(h(z)) : \ z \in K\}, \qquad j \in \mathbb{N}, K \subset\subset U,$$

when $\{\pi_j\}_{j\in\mathbb{N}}$ is a system of semi–norms on E that defines its Fréchet topology.

Definition 2.2.1 *The space $M^{\mu}_{O}(X)$ $(N^{\mu}_{O}(X))$ for $\mu \in \mathbb{R}$ is defined as the subspace of all $h(z) \in \mathcal{A}(\mathbb{C}, L^{\mu}_{\mathrm{cl}}(X))$ $(\in \mathcal{A}(\mathbb{C}, L^{\mu}(X)))$ such that $h(\beta + i\tau) \in L^{\mu}_{\mathrm{cl}}(X; \Gamma_\beta)$ $(\in L^{\mu}(X; \Gamma_\beta))$ for each $\beta \in \mathbb{R}$, and*

$$\{h(\beta + i\tau) : \ c \leq \beta \leq c'\} \tag{2.2.1}$$

is a bounded set in $L^{\mu}_{\mathrm{cl}}(X; \mathbb{R}_\tau)$ $(L^{\mu}(X; \mathbb{R}_\tau))$ for every $c \leq c'$.

If the set (2.2.1) is bounded we also say that the relation

$$h|_{\Gamma_\beta} \in L^{\mu}_{(\mathrm{cl})}(X; \Gamma_\beta) \tag{2.2.2}$$

holds uniformly in $c \leq \beta \leq c'$.

Let $\{\varrho_j\}_{j\in\mathbb{N}}$ be a system of semi–norms on the space $L^{\mu}_{\mathrm{cl}}(X; \mathbb{R})$ $(L^{\mu}(X; \mathbb{R}))$ that defines its Fréchet topology, then $M^{\mu}_{O}(X)$ $(N^{\mu}_{O}(X))$ becomes a Fréchet space in the semi–norm system

$$\sup\{\varrho_j(h(\beta + i\tau)) : \ c \leq \beta \leq c'\}, \qquad j \in \mathbb{N}, c \leq c'.$$

The space $M^{-\infty}_{O}(X) = \bigcap_{\mu\in\mathbb{R}} M^{\mu}_{O}(X) = \bigcap_{\mu\in\mathbb{R}} N^{\mu}_{O}(X)$ will be considered in the Fréchet topology of the projective limit which is nuclear.

Note that

$$M^{-\infty}_{O}(X) = M^{-\infty}_{O} \otimes_\pi L^{-\infty}(X), \tag{2.2.3}$$

where $M^{-\infty}_{O}$ is the corresponding space for $\dim X = 0$, cf. Section 2.1.2, and $L^{-\infty}(X) = \bigcap_{\mu\in\mathbb{R}} L^{\mu}(X)$ is endowed with the Fréchet topology from $C^\infty(X \times X)$ (an identification $L^{-\infty}(X) \cong C^\infty(X \times X)$ is given via a fixed Riemannian metric on X).

In the edge pseudo–differential calculus below we need a parameter–dependent version of the spaces $M^{\mu}_{O}(X)$ and $N^{\mu}_{O}(X)$, repectively, with parameters $\eta \in \mathbb{R}^q$.

Definition 2.2.2 $M_O^\mu(X;\mathbb{R}^q)$ *(*$N_O^\mu(X;\mathbb{R}^q)$*) for* $\mu \in \mathbb{R}$ *is defined as the subspace of all* $h(z,\eta) \in \mathcal{A}(\mathbb{C}, L^\mu_{\mathrm{cl}}(X;\mathbb{R}^q))$ *(*$\in \mathcal{A}(\mathbb{C}, L^\mu(X;\mathbb{R}^q))$*) such that* $h(\beta+i\tau,\eta) \in L^\mu_{\mathrm{cl}}(X;\Gamma_\beta\times\mathbb{R}^q)$ *(*$\in L^\mu(X;\Gamma_\beta\times\mathbb{R}^q)$*) for each* $\beta\in\mathbb{R}$*, and*

$$\{h(\beta+i\tau,\eta):\ c\le\beta\le c'\}$$

is a bounded set in $L^\mu_{\mathrm{cl}}(X;\mathbb{R}_\tau\times\mathbb{R}^q_\eta)$ *(*$L^\mu(X;\mathbb{R}_\tau\times\mathbb{R}^q_\eta)$*).*

Theorem 2.2.3 *To every* $f(z)\in L^\mu_{\mathrm{cl}}(X;\Gamma_\beta)$ *(*$\in L^\mu(X;\Gamma_\beta)$*),* $\beta\in\mathbb{R}$ *fixed, there exists an* $h(z)\in M_O^\mu(X)$ *(*$\in N_O^\mu(X)$*) such that*

$$h(z)|_{\Gamma_\beta} - f(z) \in L^{-\infty}(X;\Gamma_\beta).$$

Theorem 2.2.3 is a special case of a more general result, (including dependence on parameters $\eta\in\mathbb{R}^q$ and C^∞ dependence on further variables). We want to show this in connection with the so–called Mellin quantisation.

Let us illustrate that for the case of Fuchs–type symbols of the form $p(r,\varrho)=\tilde p(r,r\varrho)\in S^\mu_{(\mathrm{cl})}(\mathbb{R}_+\times\mathbb{R})$, $(r,\varrho)\in\mathbb{R}_+\times\mathbb{R}$, where $\tilde p(r,\tilde\varrho)\in S^\mu_{(\mathrm{cl})}(\overline{\mathbb{R}}_+\times\mathbb{R}_{\tilde\varrho})$. The problem is then whether there is a symbol

$$f(r,z)\in S^\mu_{(\mathrm{cl})}(\overline{\mathbb{R}}_+\times\Gamma_{\frac12-\gamma})$$

for given $\gamma\in\mathbb{R}$, such that $\mathrm{op}(p)=\mathrm{op}_M^\gamma(f) \bmod L^{-\infty}(\mathbb{R}_+)$, where op is the pseudo–differential action in $r\in\mathbb{R}_+$ with respect to the Fourier transform.

A choice of a map $m_\gamma: p(r,\varrho)\to f(r,z)$ with this property may be called a Mellin quantisation (or a Mellin operator convention). We shall show that such a choice is possible indeed. Another interesting question is (as we saw in Section 2.1.2 above) whether $f(r,z)$ may be replaced by an $h(r,z)|_{\Gamma_{\frac12-\gamma}}$ for an element $h(r,z)\in C^\infty(\overline{\mathbb{R}}_+, M_O^\mu)$. Then, by Cauchy's theorem, the relation

$$\mathrm{op}(p)=\mathrm{op}_M^\gamma(h) \bmod L^{-\infty}(\mathbb{R}_+)$$

holds for all $\gamma\in\mathbb{R}$. In other words, when we show the existence of an $h(r,z)$ which is holomorphic in $z\in\mathbb{C}$, the choice of the weight γ is unessential, and we may restrict ourselves to a convenient weight. For purposes below we consider symbols that depend on further variables and covariables.

Definition 2.2.4 *Let* $\Sigma\subseteq\mathbb{R}^n$*,* $\Omega\subseteq\mathbb{R}^q$ *be open sets. We denote by* $\tilde S^\mu_{(\mathrm{cl})}(\overline{\mathbb{R}}_+\times\Sigma\times\Omega\times\mathbb{R}^{1+n+q})$ *the subspace of all symbols* $p(r,x,y,\varrho,\xi,\eta)\in S^\mu_{(\mathrm{cl})}(\mathbb{R}_+\times\Sigma\times\Omega\times\mathbb{R}^{1+n+q})$ *who have the form*

$$p(r,x,y,\varrho,\xi,\eta)=\tilde p(r,x,y,r\varrho,\xi,r\eta) \tag{2.2.4}$$

for a symbol $\tilde{p}(r, x, y, \tilde{\varrho}, \xi, \tilde{\eta}) \in S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q}_{\tilde{\varrho},\xi,\tilde{\eta}})$. *Moreover,* $S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ *denotes the space of all functions* $h(r, x, y, z, \xi, \eta) \in C^{\infty}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ *which are holomorphic in* $z \in \mathbb{C}$, *with values in* $S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{n+q})$, *where*

$$h(r, x, y, z, \xi, \eta)|_{\Gamma_\beta} \in S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \Gamma_\beta \times \mathbb{R}^{n+q}_{\xi,\eta})$$

for every $\beta \in \mathbb{R}$, *uniformly for* β *in compact intervals. In a similar manner we define the space* $S^{\mu}_{(\mathrm{cl})}(\mathbb{R}_+ \times \Sigma \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$. *Finally we denote by* $\tilde{S}^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ *the subspace of all* $h(r, x, y, z, \xi, \eta) \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}_+ \times \Sigma \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ *of the form*

$$h(r, x, y, z, \xi, \eta) = \tilde{h}(r, x, y, z, \xi, r\eta) \tag{2.2.5}$$

for an $\tilde{h}(r, x, y, z, \xi, \tilde{\eta}) \in S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q}_{\xi,\tilde{\eta}})$.

The space $S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ is Fréchet in a natural way: If $\{\pi_j\}_{j\in\mathbb{N}}$ is a semi–norm system for the Fréchet topology in $S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}_\varrho \times \mathbb{R}^{n+q})$, then

$$h \to \sup\{\pi_j(h(r, x, y, \beta + i\varrho, \xi, \eta)) : \; c \le \beta \le c'\}$$

for $j \in \mathbb{N}$, $c \le c'$, is a semi–norm system for the Fréchet topology in $S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$. Applying the bijections

$$\tilde{S}^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q}) \to S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q}),$$
$$\tilde{S}^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q}) \to S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q}),$$

defined by relations (2.2.4) and (2.2.5), respectively, we also get Fréchet topologies in the symbol spaces $\tilde{S}^{\mu}_{(\mathrm{cl})}(\ldots)$.

Remark 2.2.5 *By omitting* y *and* η*–variables we get simpler analogues of the spaces of Definition* 2.2.4, *namely* $\tilde{S}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \mathbb{R}^{1+n})$ *as the subspace of all* $p(r, x, \varrho, \xi) \in S_{(\mathrm{cl})}(\mathbb{R}_+ \times \Sigma \times \mathbb{R}^{1+n})$ *with*

$$p(r, x, \varrho, \xi) = \tilde{p}(r, x, r\varrho, \xi)$$

for a symbol $\tilde{p}(r, x, \tilde{\varrho}, \xi) \in S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \mathbb{R}^{1+n}_{\tilde{\varrho},\xi})$. *Moreover, we can introduce the spaces* $S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \mathbb{C} \times \mathbb{R}^n)$ *in an obvious manner.*

Theorem 2.2.6 *Let $p(r,x,y,\varrho,\xi,\eta) \in \tilde{S}^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q})$ be given. Then there exists an element $h(r,x,y,z,\xi,\eta) \in \tilde{S}^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ such that*

$$\mathrm{op}_{r,x}(p)(y,\eta) = \mathrm{op}_M^{\frac{1}{2}}(\mathrm{op}_x(h))(y,\eta) \bmod C^\infty(\Omega, L^{-\infty}(\mathbb{R}_+ \times \Sigma; \mathbb{R}^q)).$$

The symbol $\tilde{h}(r,x,y,i\varrho,\xi,\eta)$ admits the asymptotic expansion

$$\tilde{h}(r,x,y,i\varrho,\xi,\eta) \sim \tilde{p}(r,x,y,-\varrho,\xi,\eta) + \sum_{k=1}^{\infty} \Big\{ \sum_{j=0}^{k} c_{kj} \partial_\varrho^{k+j} \tilde{p}(r,x,y,-\varrho,\xi,\eta) \varrho^j \Big\} \tag{2.2.6}$$

with coefficients $c_{kj} \in \mathbb{R}$ that are independent of $\tilde{p}$ (the asymptotic expansion refers to the space $S^\mu(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q})$). In the case of classical symbols we have the following relation for the homogeneous components

$$\tilde{h}_{(\mu-l)}(r,x,y,i\varrho,\xi,\eta) = \sum_{k=0}^{l} \sum_{j=0}^{k} c_{kj} (\partial_\varrho^{k+j} \tilde{p})_{(\mu-l-j)}(r,x,y,-\varrho,\xi,\eta) \varrho^j,$$

$l \in \mathbb{N}$, in particular,

$$\tilde{h}_{(\mu)}(r,x,y,i\varrho,\xi,\eta) = \tilde{p}_{(\mu)}(r,x,y,-\varrho,\xi,\eta).$$

Proof. Let us set $m(r,r') = (\log r - \log r')(r-r')^{-1}$ for $r, r' \in \mathbb{R}_+$. We have $m \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+)$, $m(r,r') > 0$ for all $r, r' \in \mathbb{R}_+$ and $m(r,r') = m(r',r)$ (further properties of this function can be found in [48], Section 2.4). For $u \in C_0^\infty(\mathbb{R}_+ \times \Sigma)$, regarded as a function $u(r) \in C_0^\infty(\mathbb{R}_+, C^\infty(\Sigma))$, we have

$$\begin{aligned} \mathrm{op}_{r,x}(p)(y,\eta)u(r) &= \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{i(r-r')\varrho} \,\mathrm{op}_x(p)(r,y,\varrho,\eta) u(r')\, dr' đ\varrho \\ &= \int_{-\infty}^{\infty} \int_{0}^{\infty} \Big(\frac{r}{r'}\Big)^{-i\varrho} m(r,r') r' \,\mathrm{op}_x(p)(r,y,-m(r,r')\varrho,\eta) \\ &\quad \times u(r') \,\frac{dr'}{r'} đ\varrho. \end{aligned}$$

In the latter equation we applied the coordinate change $\varrho \to -m(r, r')\varrho$. Then

$$g(r, r', x, y, i\varrho, \xi, \eta) := m(r, r')r'p(r, x, y, -m(r, r')\varrho, \xi, \eta) \\ \in S^{\mu}_{(\mathrm{cl})}(\mathbb{R}_+ \times \mathbb{R}_+ \times \Sigma \times \Omega \times \Gamma_0 \times \mathbb{R}^{n+q}) \quad (2.2.7)$$

and $\mathrm{op}_{r,x}(p)(y, \eta) = \mathrm{op}_M^{\frac{1}{2}}(\mathrm{op}_x(g))(y, \eta)$ for all $(y, \eta) \in \Omega \times \mathbb{R}^q$. We now construct an $h(r, x, y, z, \xi, \eta) \in S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{C}_z \times \mathbb{R}^{n+q}_{\xi,\eta})$ by setting

$$h(r, x, y, z, \xi, \eta) = r^z \,\mathrm{op}_M^{\frac{1}{2}}(\varphi(r'/r)(r')^{-z}g)(x, y, \xi, \eta) \quad (2.2.8)$$

for a function $\varphi \in C_0^{\infty}(\mathbb{R}_+)$ with $\varphi \equiv 1$ in a neighbourhood of 1. In this step we just apply a kernel cut–off construction. It follows that

$$\begin{aligned} h(r, x, y, z, \xi, \eta) &= \int_{-\infty}^{\infty}\int_0^{\infty} \left(\frac{r}{r'}\right)^{-i\varrho+z} \varphi(r'/r)g(r, r', x, y, i\varrho, \xi, \eta)\,\frac{dr'}{r'}đ\varrho \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} s^{i\varrho-z}\varphi(s)g(r, rs, x, y, i\varrho, \xi, \eta)\,\frac{ds}{s}đ\varrho, \end{aligned}$$

where we used the change of variables $r' \to rs$. By virtue of $g(r, rs, x, y, i\varrho, \xi, \eta) = sm(s, 1)\tilde{p}(r, x, y, -m(s, 1)\varrho, \xi, r\eta)$, cf. formula (2.2.7), we get

$$h(r, x, y, z, \xi, \eta) = \iint s^{i\varrho-z}\varphi(s)m(s, 1)\tilde{p}(r, x, y, -m(s, 1)\varrho, \xi, r\eta)\,ds đ\varrho.$$

Recall that the integrals are understood as oscillatory integrals.

We want to show that $h(r, x, y, \beta + i\sigma, \xi, r^{-1}\eta) = \tilde{h}(r, x, y, \beta + i\sigma, \xi, \eta) \in S^{\mu}_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q}_{\sigma,\xi,\eta})$, uniformly in $c_1 \leq \beta \leq c_2$ for arbitrary $c_1 \leq c_2$. Change of variables $\varrho \to \varrho + \sigma$ and integration by parts yields

$$\begin{aligned} \tilde{h}(r, x, y, \beta + i\sigma, \xi, \eta) &= \iint s^{1+i\varrho}\psi_\beta(s)\tilde{p}(r, x, y, -m(s, 1)(\varrho + \sigma), \xi, \eta)\,\frac{ds}{s}đ\varrho \\ &= \iint (1 + i\varrho)^{-N}s^{1+i\varrho}(-s\partial_s)^N \\ &\qquad \{\psi_\beta(s)\tilde{p}(r, x, y, -m(s, 1)(\varrho + \sigma), \xi, \eta)\}\,\frac{ds}{s}đ\varrho \quad (2.2.9) \end{aligned}$$

where $\psi_\beta(s) := s^{-\beta}\varphi(s)m(s,1)$. The latter integral converges for $N \in \mathbb{N}$ large enough. We have

$$\begin{aligned}(-s\partial_s)^N\{\psi_\beta(s)\tilde{p}(r,x,y,-m(s,1)(\varrho+\sigma),\xi,\eta)\} \\ = \sum_{k=0}^{N} c_k(s)(\partial_\varrho^k\tilde{p})(r,x,y,-m(s,1)(\varrho+\sigma),\xi,\eta)(\varrho+\sigma)^k\end{aligned}$$

with certain coefficients $c_k \in C_0^\infty(\mathbb{R}_+)$. In order to show the symbol estimates for $\tilde{h}$ we have to look at the derivatives of $\tilde{p}$ under the integral and to employ the symbol estimates of $\tilde{p}$.

First we have for $l \in \mathbb{N}$, $\alpha \in \mathbb{N}^{n+q}$, $\gamma \in \mathbb{N}^{1+n+q}$

$$\begin{aligned}I_{l,\alpha,\gamma} &:= |\partial_\sigma^l D_{\xi,\eta}^\alpha D_{r,x,y}^\gamma\{(\partial_\varrho^k\tilde{p})(r,x,y,-m(s,1)(\varrho+\sigma),\xi,\eta)(\varrho+\sigma)^k\}| \\ &\le \sum_{j=0}^{l} |c_j\partial_\sigma^j\{(\partial_\varrho^k D_{\xi,\eta}^\alpha D_{r,x,y}^\gamma\tilde{p})(\dots)\}\partial_\sigma^{l-j}(\varrho+\sigma)^k|.\end{aligned}$$

Since

$$\begin{aligned}&|\partial_\sigma^j\partial_\varrho^k D_{\xi,\eta}^\alpha D_{r,x,y}^\gamma\tilde{p}(r,x,y,-m(s,1)(\varrho+\sigma),\xi,\eta)| \\ &\quad = |(\partial_\varrho^{j+k}D_{\xi,\eta}^\alpha D_{r,x,y}^\gamma\tilde{p}(\dots))(-m(s,1))^j| \le c(s)\langle\varrho+\sigma,\xi,\eta\rangle^{\mu-j-k-|\alpha|}\end{aligned}$$

with certain $c(s) \in C^\infty(\mathbb{R}_+)$, and because of $|\partial_\sigma^{l-j}(\varrho+\sigma)^k| \le c\langle\varrho+\sigma,\xi,\eta\rangle^{j+k-l}$, we obtain, using Peetre's inequality,

$$\begin{aligned}I_{l,\alpha,\gamma} &\le \tilde{c}_{l,\alpha,\gamma}(s)\langle\varrho+\sigma,\xi,\eta\rangle^{\mu-l-|\alpha|} \\ &\le c_{l,\alpha,\gamma}(s)\langle\varrho\rangle^{|\mu-l-|\alpha||}\langle\sigma,\xi,\eta\rangle^{\mu-l-|\alpha|},\end{aligned}$$

$\tilde{c}_{l,\alpha,\gamma}, c_{l,\alpha,\gamma} \in C^\infty(\mathbb{R}_+)$. From (2.2.9) we get

$$\begin{aligned}&|\partial_\sigma^j D_{\xi,\eta}^\alpha D_{r,x,y}^\gamma\tilde{h}(r,x,y,\beta+i\sigma,\xi,\eta)| \\ &\quad \le \langle\sigma,\xi,\eta\rangle^{\mu-l-|\alpha|}\iint(1+i\varrho)^{-N}s^{1+i\varrho}P(\psi_\beta(s))\langle\varrho\rangle^{|\mu-l-|\alpha||}\frac{ds}{s}đ\varrho\end{aligned}$$

for a polynomial P in derivatives of $\psi_\beta(s)$ with coefficients in $C^\infty(\mathbb{R}_+)$. N is chosen sufficiently large, such that the integral on the right converges. In view of the fact that $\partial_s^k\psi_\beta$ depends continuously on β for arbitrary $k \in \mathbb{N}$, the symbol estimates for $\tilde{h}$ are uniform in $c_1 \le \beta \le c_2$ for arbitrary $c_1 \le c_2$.

In an analogous manner we can treat the semi–norms of $\tilde{h}$ as a holomorphic function of z with values in $S^{\mu}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{n+q})$.

Let us now derive an asymptotic expansion for $\tilde{h}$, applying Theorem 1.1.20 above to the right hand side of (2.2.8) which refers to the (r, r')–variables in the Mellin version (this can be transformed into the Fourier version by the change of variables $r^{i\varrho} = e^{i\tilde{r}\varrho}$, $(r')^{i\varrho} = e^{i\tilde{r}'\varrho}$, i.e., $\tilde{r} = \log r$, $\tilde{r}' = \log r'$; then the resulting expressions from Theorem 1.1.20 can be transformed again to (r, r')). We obtain

$$\tilde{h}(r,x,y,i\varrho,\xi,\eta) = \sum_{k=0}^{N-1} \frac{1}{k!}(-r'\partial_{r'})^k \partial_\varrho^k \{\varphi(r'/r) g(r,r',x,y,i\varrho,\xi,r^{-1}\eta)\}|_{r'=r} + \int_0^1 \frac{(1-\theta)^{N-1}}{(N-1)!} f_N(r,x,y,\varrho,\xi,\eta,\theta)\, d\theta, \quad (2.2.10)$$

for arbitrary $N \in \mathbb{N}$, $N \geq 1$, where $f_N(r,x,y,\varrho,\xi,\eta,\theta)$ is given by

$$\int_{-\infty}^{\infty}\int_0^{\infty} \left(\frac{r}{r'}\right)^{-i\sigma} (i\sigma)^N \varphi(r'/r)(\partial_\varrho^N g)(r,r',x,y,i(\varrho+\theta\sigma),\xi,r^{-1}\eta)\, \frac{dr'}{r'} đ\varrho.$$

In view of

$$(\partial_\varrho^k g)(r,r',x,y,i\varrho,\xi,r^{-1}\eta) = m(r,r')r'(\partial_\varrho^k \tilde{p})(r,x,y,-m(r,r')r\varrho, \xi,\eta)(-m(r,r')r)^k$$

the sum on the right hand side of (2.2.10) equals

$$\sum_{k=0}^{N-1} \frac{1}{k!}(r'\partial_{r'})^k \{r^k \varphi(r'/r) m(r,r')^{k+1} r'(\partial_\varrho^k \tilde{p})(r,x,y, - m(r,r')r\varrho,\xi,\eta)\}|_{r'=r}. \quad (2.2.11)$$

Moreover, $f_N(r,x,y,\varrho,\xi,\eta,\theta)$ equals

$$\int_{-\infty}^{\infty}\int_0^{\infty} s^{i\sigma}(-i\sigma)^N \varphi(s) m(s,1)^{N+1}(\partial_\varrho^N \tilde{p})(r,x,y,-m(s,1)(\varrho+\theta\sigma),\xi,\eta)\, ds đ\sigma.$$

The latter relation gives us similarly as above that the remainder term in (2.2.10) belongs to $S^{\mu-N}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q}_{\varrho,\xi,\eta})$. This shows that the asymptotic summation of $\tilde{h}$ can be carried out in the space of symbols that are smooth up to $r = 0$.

The expansion (2.2.6) then follows from (2.2.11), using

$$\partial^L_{r'}(u \circ v) = \sum_{j=1}^{L} \sum_{\gamma_1+\ldots+\gamma_j=L} c_j(\partial^j_{r'}u)(v(r'))\partial^{\gamma_1}_{r'}v \cdot \ldots \cdot \partial^{\gamma_j}_{r'}v$$

for $L \geq 1$ and $\gamma_j, c_j \in \mathbb{N}$, and the identity $\partial^k_{r'}m(r,r')|_{r'=r} = (-1)^k \frac{k!}{k+1}\frac{1}{r^{k+1}}$. $\square$

Remark 2.2.7 *There is an obvious (simpler) analogue of Theorem* 2.2.6 *for the* (y,η)*–independent case, cf. Remark* 2.2.5 *above. To every* $p(r,x,\varrho,\xi) \in \tilde{S}^\mu_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \mathbb{R}^{1+n})$ *there exists an element* $h(r,x,z,\xi) \in S^\mu_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \mathbb{C} \times \mathbb{R}^n)$ *with*

$$\mathrm{op}_{r,x}(p) = \mathrm{op}^{\frac{1}{2}}_M(\mathrm{op}_x(h)) \bmod L^{-\infty}(\mathbb{R}_+ \times \Sigma),$$

and there are asymptotic expansions for h *of the same type as in Theorem* 2.2.6 *above.*

Proof of Theorem 2.2.3. Let $\{U_1, \ldots, U_N\}$ be a covering of X by coordinate neighbourhoods, $\{\varphi_1, \ldots, \varphi_N\}$ a subordinate partition of unity and $\psi_j \in C^\infty_0(U_j)$ with $\varphi_j\psi_j = \varphi_j$ for all j. Then we have

$$f = \sum_{j=1}^{N} f_j \bmod L^{-\infty}(X;\Gamma_\beta)$$

for $f_j = \varphi_j f \psi_j$. Assume we can construct an $h_j \in M^\mu_O(X)$ with $h_j|_{\Gamma_\beta} = f_j \bmod L^{-\infty}(X;\Gamma_\beta)$. Then we may set $h = \sum_{j=1}^N \varphi_j h_j \psi_j$. In other words we may restrict ourselves to f_j for a fixed j. First assume $\beta = 0$. Let $a(x,i\varrho,\xi) \in S^\mu_{\mathrm{cl}}(\Sigma \times \Gamma_0 \times \mathbb{R}^n)$ be a local symbol of $f_j(i\varrho) \in L^\mu_{\mathrm{cl}}(X;\Gamma_0)$. Then $a(x,i\varrho,\xi)$ can be regarded as a symbol of the form $\tilde{p}(x,-\varrho,\xi)$ for a $\tilde{p}(x,r\varrho,\xi) \in \tilde{S}^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \mathbb{R}^{1+n})$. So a byproduct of the (y,η)–dependent analogue of Theorem 2.2.6 (cf. Remark 2.2.7) is that there is a $g^0(x,z,\xi) \in S^\mu_{\mathrm{cl}}(\Sigma \times \mathbb{C} \times \mathbb{R}^n)$ with $g^0_{(\mu)}(x,i\varrho,\xi) = a_{(\mu)}(x,i\varrho,\xi)$. In other words, $a^1(x,i\varrho,\xi) := a(x,i\varrho,\xi) - g^0(x,i\varrho,\xi) \in S^{\mu-1}_{\mathrm{cl}}(\Sigma \times \Gamma_0 \times \mathbb{R}^n)$. Applying the same procedure to $a^1(x,i\varrho,\xi)$ we find a $g^1(x,z,\xi) \in S^{\mu-1}_{\mathrm{cl}}(\Sigma \times \mathbb{C} \times \mathbb{R}^n)$ such that

$$a^2(x,i\varrho,\xi) = a^1(x,i\varrho,\xi) - g^1(x,i\varrho,\xi)$$

belongs to $S^{\mu-2}_{\mathrm{cl}}(\Sigma \times \Gamma_0 \times \mathbb{R}^n)$, that is, $a = g^0 + a^1 = g^0 + g^1 + a^2$. This process can be iterated, we can form an asymptotic sum $g \sim \sum_{j=0}^{\infty} g^j$ in $S^{\mu}_{\mathrm{cl}}(\Sigma \times \mathbb{C} \times \mathbb{R}^n)$, and then

$$g(x, i\varrho, \xi) = a(x, i\varrho, \xi) \bmod S^{-\infty}(\Sigma \times \Gamma_0 \times \mathbb{R}^n).$$

If $\omega_j, \tilde{\omega}_j \in C_0^{\infty}(U_j)$ are functions with $\omega_j \psi_j = \tilde{\omega}_j \psi_j = \psi_j$, we set $h_j(z) = (\kappa_j^{-1})_* \omega_j \operatorname{op}_x(g)(z) \tilde{\omega}_j$ (here $\kappa_j : U_j \to \Sigma$ is a corresponding map on X and $(\kappa_j^{-1})_*$ the associated operator push–forward); then $h_j(z) \in M^{\mu}_{O}(X)$ has the desired properties. The assertion for arbitrary β is obtained by an easy translation in the complex plane. □

Remark 2.2.8 *There is an analogue of Theorem* 2.2.3 *of the following form. To each $f(r, z) \in C^{\infty}(\overline{\mathbb{R}}_+, L^{\mu}_{\mathrm{cl}}(X; \Gamma_\beta))$ there exists an element $h(r, z) \in C^{\infty}(\overline{\mathbb{R}}_+, M^{\mu}_{O}(X))$ such that $f(r, z) - h(r, z)|_{\Gamma_\beta} \in C^{\infty}(\overline{\mathbb{R}}_+, L^{-\infty}(X; \Gamma_\beta))$ (and similarly for the non–classical case).*

Remark 2.2.9 *Theorem* 2.2.3 *(or the more general assertion of Remark* 2.2.8*) can also be proved directly by using the kernel cut–off procedure which is as follows: Choose a function $\psi \in C_0^{\infty}(\mathbb{R}_+)$ with $\psi(\varrho) = 1$ in a neighbourhood of $\varrho = 1$. Then, setting*

$$h(z) = M_{\frac{1}{2}-\beta, \varrho \to z}[\psi(\varrho)(M_{\frac{1}{2}-\beta, \tilde{z} \to \varrho} f(\tilde{z}))],$$

we get an $h(z)$ as desired. Details may be found in [58] or [66] (see also [57], where this idea was introduced). This method shows, in particular, that $h \in M^{\mu}_{O}(X)$ $(N^{\mu}_{O}(X))$ and $h|_{\Gamma_\beta} \in L^{\mu-1}_{\mathrm{cl}}(X; \Gamma_\beta)$ for some fixed β implies $h \in M^{\mu-1}_{O}(X)$ $(N^{\mu-1}_{O}(X))$.

2.2.2 Spaces of meromorphic operator functions

In this section we study spaces of meromorphic pseudo–differential operator–valued functions in the complex z–plane, as a generalisation of those of Definition 2.1.11. They play a role in the conormal symbol structure of cone pseudo–differential operators and are responsible for the elliptic regularity with asymptotics on a manifold with conical singularities.

Let X be a closed compact C^{∞} manifold and denote by $\mathbf{As}(X)$ the set of all sequences

$$R = \{(r_j, n_j, L_j)\}_{j \in \mathbb{Z}}$$

with $\pi_{\mathbb{C}}R := \{r_j\}_{j\in\mathbb{Z}} \subset \mathbb{C}$, $|\operatorname{Re} p_j| \to \infty$ as $|j| \to \pm\infty$, $n_j \in \mathbb{N}$, and finite–dimensional subspaces $L_j \subset L^{-\infty}(X)$ of operators of finite rank. Fix a $\pi_{\mathbb{C}}R$–excision function, i.e., a $\chi(z) \in C^\infty(\mathbb{C})$ with $\chi(z) = 0$ in a neighbourhood of $\pi_{\mathbb{C}}R$, $\chi(z) = 1$ for $\operatorname{dist}(z, \pi_{\mathbb{C}}R) > \varepsilon$ for an $\varepsilon > 0$.

Definition 2.2.10 *Let $R \in \mathbf{As}(X)$ and denote by $M_R^{-\infty}(X)$ the space of all meromorphic functions $h(z)$ with poles at the points $z = r_j$ of multiplicities $n_j + 1$ and Laurent coefficients at $(z - r_j)^{-(k+1)}$ in L_j for $0 \le k \le n_j$, $j \in \mathbb{Z}$, and $\chi(z)h(z)|_{\Gamma_\beta} \in L^{-\infty}(X; \Gamma_\beta)$ for every $\beta \in \mathbb{R}$, uniformly in $c \le \beta \le c'$ for arbitrary $c \le c'$. Furthermore, we set*

$$M_R^\mu(X) = M_O^\mu(X) + M_R^{-\infty}(X). \tag{2.2.12}$$

The space $M_R^{-\infty}(X)$ is Fréchet in a natural way, namely with respect to the semi–norm system $\{\pi_{j,K}\}_{K \Subset (\mathbb{C}\setminus\pi_{\mathbb{C}}R), j\in\mathbb{N}} \cup \{\varrho_{l,[c,c']}\}_{c\le c', l\in\mathbb{N}}$, for

$$\begin{aligned}\pi_{j,K}(f) &= \sup\{\pi_j(f(z)) : \ z \in K\},\\ \varrho_{l,[c,c']}(f) &= \sup\{\varrho_l((\chi f)(\beta + i\tau)) : \ c \le \beta \le c'\},\end{aligned}$$

where $\{\pi_j\}_{j\in\mathbb{N}}$ runs over a semi–norm system that defines the Fréchet topology of $L^{-\infty}(X)$ and $\{\varrho_l\}_{l\in\mathbb{N}}$ over a semi–norm system that defines the Fréchet topology of $L^{-\infty}(X; \mathbb{R}_\tau)$. We then endow the space $M_R^\mu(X)$ with the Fréchet topology of the non–direct sum (2.2.12).

Notice that $M_R^{-\infty}(X)$ is nuclear.

Example 2.2.11 *Let $\omega(r)$ be a cut–off function on $\overline{\mathbb{R}}_+$ (i.e., $\omega(r) \in C_0^\infty(\overline{\mathbb{R}}_+)$, $\omega(r) = 1$ for $0 \le r \le \varepsilon$ for an $\varepsilon > 0$). Then, for $p \in \mathbb{C}$, $\operatorname{Re} p < \frac{1}{2}$, $k \in \mathbb{N}$, the function $f(z) = M(\omega(r)r^{-p}\log^k r)l$ with the Mellin transform $M_{r\to z}$ and some finite–dimensional operator $l \in L^{-\infty}(X)$ belongs to $M_P^{-\infty}(X)$ for $P = (p, k, L)$, where L equals the linear span of l.*

Proposition 2.2.12 *$f(z) \in M_P^\mu(X)$, $h(z) \in M_Q^\nu(X)$ for $\mu, \nu \in \mathbb{R}$, $P, Q \in \mathbf{As}(X)$, implies $(fh)(z) \in M_R^{\mu+\nu}(X)$ for a certain $R \in \mathbf{As}(X)$.*

This result is easy to verify, so we omit the details.

Definition 2.2.13 *An element $f(z) \in M_P^\mu(X)$ for $\mu \in \mathbb{R}$, $P \in \mathbf{As}(X)$, is called elliptic if for some $\beta \in \mathbb{R}$ with $\Gamma_\beta \cap \pi_{\mathbb{C}}R = \emptyset$ the operator family $f|_{\Gamma_\beta} \in L_{\mathrm{cl}}^\mu(X; \Gamma_\beta)$ is parameter–dependent elliptic.*

By virtue of Remark 2.2.9 the ellipticity of $f \in M_P^\mu(X)$ implies that $f|_{\Gamma_\delta} \in L_{\mathrm{cl}}^\mu(X;\Gamma_\delta)$ is parameter–dependent elliptic for all $\delta \in \mathbb{R}$ with$\Gamma_\delta \cap \pi_{\mathbb{C}} P = \emptyset$. In particular, if $f \in M_O^\mu(X)$ is elliptic, then $f|_{\Gamma_\delta} \in L_{\mathrm{cl}}^\mu(X;\Gamma_\delta)$ is parameter–dependent elliptic for all $\delta \in \mathbb{R}$.

Remark 2.2.14 *Let $f \in M_P^\mu(X)$ be elliptic. Then to every $c \leq c'$ there exists an $N(c,c') > 0$ such that*

$$f(z) : H^s(X) \to H^{s-\mu}(X) \tag{2.2.13}$$

is an isomorphism for arbitrary $z \in \mathbb{C}$ with $c \leq \operatorname{Re} z \leq c'$ and $|\operatorname{Im} z| \geq N(c,c')$, for all $s \in \mathbb{R}$. More precisely, there exists a countable set $D \subset \mathbb{C}$ with $D \cap \{z : \ c \leq \operatorname{Re} z \leq c'\}$ finite for every $c \leq c'$ such that f is holomorphic in $\mathbb{C} \setminus D$, and (2.2.13) *is an isomorphism for every $z \in \mathbb{C} \setminus D$.*

Theorem 2.2.15 *Let $f \in M_P^\mu(X)$, $\mu \in \mathbb{R}$, $P \in \mathbf{As}(X)$, be elliptic. Then (with the notation of Remark* 2.2.14*) $(f|_{\mathbb{C}\setminus D})^{-1}$ has an extension to an element $f^{-1}(z) \in M_Q^{-\mu}(X)$ for a certain $Q \in \mathbf{As}(X)$.*

A proof of this result may be found in the book [66] (the analogous case of boundary value problems with the transmission property is studied in joint papers [47], [48] with Schrohe).

Corollary 2.2.16 *For every $g \in M_P^{-\infty}(X)$, $P \in \mathbf{As}(X)$ there is an $l \in M_Q^{-\infty}(X)$ for a certain $Q \in \mathbf{As}(X)$ such that $(1 + g(z))^{-1} = 1 + l(z)$.*

The following result of Witt [96] is a generalisation of Theorem 2.1.23 above to the case of operator–valued Mellin symbols.

Theorem 2.2.17 *Let $f \in M_P^\mu(X)$, $\mu \in \mathbb{R}$, $P \in \mathbf{As}(X)$, be elliptic. Then there exist an $h \in M_O^\mu(X)$ which is elliptic with $h^{-1} \in M_O^{-\mu}(X)$ and a $g \in M_S^{-\infty}(X)$ for some $S \in \mathbf{As}(X)$ such that $f = h(1 + g)$.*

Remark 2.2.18 *Under the conditions of Theorem* 2.2.17 *there is an elliptic $h \in M_O^\mu(X)$ with $h^{-1} \in M_O^{-\mu}(X)$ such that for some $l \in M_R^{-\infty}(X)$, $R \in \mathbf{As}(X)$, we have $f = (1 + l)h$. Another equivalent formulation of Theorem* 2.2.17 *is that for elliptic $f \in M_P^\mu(X)$ there exists an $h \in M_O^\mu(X)$ with $h^{-1} \in M_O^{-\mu}(X)$ such that $f - h \in M_P^{-\infty}(X)$.*

2.2.3 Mellin pseudo–differential operators and Green operators with asymptotics

To every $f(z) \in M_R^\mu(X)$ we can form continuous operators

$$\omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(f)\omega_0 : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma}(X^\wedge) \tag{2.2.14}$$

for all $s \in \mathbb{R}$ and all $\gamma \in \mathbb{R}$ with $\pi_{\mathbb{C}} R \cap \Gamma_{\frac{1}{2}-\gamma} = \emptyset$, with $\omega(r)$, $\omega_0(r)$ being fixed cut–off functions. Recall $\operatorname{op}_M^\delta(f) = r^\delta \operatorname{op}_M(T^{-\delta} f) r^{-\delta}$ for $(T^{-\delta}f)(z) = f(z-\delta)$. To each $P \in \operatorname{As}(X, \boldsymbol{g})$, $\boldsymbol{g} = (\gamma, \Theta)$, there exists a $Q \in \operatorname{As}(X, \boldsymbol{g})$ such that (2.2.14) restricts to a continuous operator

$$\omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(f)\omega_0 : \mathcal{K}_P^{s,\gamma}(X^\wedge) \to \mathcal{K}_Q^{s-\mu,\gamma}(X^\wedge).$$

for arbitrary $s \in \mathbb{R}$ (for the proof, cf. also [66], Theorem 2.3.55).

Applying Theorem 2.2.17 we can write $f(z) = h(z)(1 + g(z))$ for an $h(z) \in M_O^\mu(X)$ with $h^{-1}(z) \in M_O^{-\mu}(X)$ and $g(z) \in M_S^{-\infty}(X)$ for a certain $S \in \mathbf{As}(X)$. This gives us

$$\begin{aligned}
\omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(f)\omega_0 &= \omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(h(1+g))\omega_0 \\
&= \omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(h) \operatorname{op}_M^{\gamma-\frac{n}{2}}(1+g)\omega_0 \\
&= \omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)\tilde{\omega} \operatorname{op}_M^{\gamma-\frac{n}{2}}(1+g)\omega_0 \\
&+ \omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)(1-\tilde{\omega}) \operatorname{op}_M^{\gamma-\frac{n}{2}}(1+g)\omega_0
\end{aligned}$$

for cut–off functions ω, ω_0, $\tilde{\omega}$. Choosing $\tilde{\omega}$ in such a way that $(1-\tilde{\omega})\omega_0 = 0$ we get

$$\omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(f)\omega_0 = \omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)\tilde{\omega}_0\tilde{\omega} \operatorname{op}_M^{\gamma-\frac{n}{2}}(1+g)\omega_0 + G, \tag{2.2.15}$$

when $\tilde{\omega}_0$ is a cut–off function with $\tilde{\omega}_0\tilde{\omega} = \tilde{\omega}$, with

$$G = \omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)(1-\tilde{\omega}) \operatorname{op}_M^{\gamma-\frac{n}{2}}(g)\omega_0.$$

It can be proved that G is a Green operator in the sense of the following definition that we formulate for arbitrary pairs $\gamma, \delta \in \mathbb{R}$ of weights.

Definition 2.2.19 *An operator $G \in \bigcap_{s\in\mathbb{R}} \mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{\infty,\delta}(X^\wedge))$ is called a Green operator (in the cone algebra over $X^\wedge$, cf. Section 2.2.4 below) if it induces continuous operators*

$$G : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{S}_P^\delta(X^\wedge), \qquad G^* : \mathcal{K}^{s,-\delta}(X^\wedge) \to \mathcal{S}_Q^{-\gamma}(X^\wedge)$$

for all $s \in \mathbb{R}$, with discrete asymptotic types $P \in \mathrm{As}(X, (\delta, \Theta))$, $Q \in \mathrm{As}(X, (-\gamma, \Theta))$. Here $\Theta = (\vartheta, 0]$ is some weight interval and G^* refers to the sesquilinear pairings

$$\mathcal{K}^{s,\gamma}(X^\wedge) \times \mathcal{K}^{-s,-\gamma}(X^\wedge) \to \mathbb{C}$$

for arbitrary $s, \gamma \in \mathbb{R}$, with respect to the $\mathcal{K}^{0,0}(X^\wedge) = r^{-\frac{n}{2}} L^2(\mathbb{R}_+ \times X)$–scalar product.

Denote $C_G(X^\wedge, (\gamma, \delta, \Theta))$ the set of all Green operators.

Note that a Green operator $G : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\delta}(X^\wedge)$ is compact.

The following result gives a kernel characterisation of Green operators with discrete asymptotics; a proof may be found in J. Seiler [85], cf. also Schulze [68].

Theorem 2.2.20 *For $\Theta = (-\infty, 0]$ the following conditions are equivalent:*

(i) $G \in C_G(X^\wedge, (\gamma, \delta, \Theta))$.

(ii) *$G \in \mathcal{L}(\mathcal{K}^{0,\gamma}(X^\wedge), \mathcal{K}^{0,\delta}(X^\wedge))$ is a Hilbert–Schmidt operator with an integral kernel $g(t, x, t', x') \in \mathcal{S}^{\delta}_{P}(X^\wedge) \otimes_\pi \mathcal{S}^{-\gamma}_{\overline{Q}}(X^\wedge)$ for suitable asymptotic types $P \in \mathrm{As}(X, (\delta, \Theta))$, $\overline{Q} \in \mathrm{As}(X, (-\gamma, \Theta))$.*

Thus every $G \in C_G(X^\wedge, (\gamma, \delta, \Theta))$ has a representation

$$Gu(t, x) = \int\limits_{\mathbb{R}_+ \times X} g(t, x, t', x') u(t', x') (t')^n \, dt' dx'$$

for such a kernel $g(t, x, t', x')$.

Remark 2.2.21 *Let G be a Green operator to the weights $\gamma = \delta$ and assume that*

$$1 + G : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma}(X^\wedge)$$

is invertible for an $s = s_0 \in \mathbb{R}$. Then this operator is invertible for arbitrary $s \in \mathbb{R}$, and we have

$$(1 + G)^{-1} = 1 + G + G(1 + G)^{-1} G$$

where $G + G(1 - G)^{-1} G$ is again a Green operator.

The remainder G in (2.2.15) is a Green operator with weights $\gamma = \delta$ and $\Theta = (-\infty, 0]$.

Consider operators of the form

$$\omega r^{\delta-\gamma} \sum_{j=0}^{k-1} r^j \operatorname{op}_M^{\gamma_j - \frac{n}{2}}(f_j)\omega_0 + G \tag{2.2.16}$$

for $k \in \mathbb{N} \setminus \{0\}$, with G being a Green operator to the weights $\gamma, \delta \in \mathbb{R}$ and $\Theta = (-k, 0]$, and Mellin symbols $f_j \in M_{R_j}^{-\infty}(X)$ for certain $R_j \in \mathbf{As}(X)$ and $\gamma_j \in \mathbb{R}$ with $\gamma_j \leq \gamma \leq \gamma_j + j$, $\pi_{\mathbb{C}} R_j \cap \Gamma_{\frac{1}{2}-\gamma_j} = \emptyset$ for all j. Those operators will be called smoothing Mellin + Green in the cone algebra (with discrete asymptotics and with respect to the weight data (γ, δ, Θ)).

Denote by $C_{M+G}(X^\wedge, (\gamma, \delta, \Theta))$ the set of all operators of the form (2.2.16). These play the role of an ideal in the cone algebra which controls asymptotics of solutions.

Let us illustrate to what extent the smoothing Mellin operators contribute to the index of Fredholm operators. Consider a simplest case of $\mathbb{R}_+$, namely

$$A = 1 + \omega \operatorname{op}_M^\gamma(f)\omega_0 : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \to \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \tag{2.2.17}$$

for an $f \in M_R^{-\infty}$, $\pi_{\mathbb{C}} R \cap \Gamma_{\frac{1}{2}-\gamma} = \emptyset$. It is well–known, cf. Eskin [12], that the condition $1 + f(z) \neq 0$ for all $z \in \Gamma_{\frac{1}{2}-\gamma}$ implies that $1 + \omega \operatorname{op}_M(f)\tilde{\omega}$ is a Fredholm operator and

$$\operatorname{ind}(1 + \omega \operatorname{op}_M(f)\tilde{\omega}) = \frac{1}{2\pi} \Delta \arg(1 + f(z))\Big|_{\Gamma_{\frac{1}{2}-\gamma}} \Big|_{\operatorname{Im} z = -\infty}^{\operatorname{Im} z = \infty}, \tag{2.2.18}$$

where the Δ indicates the change of the arguments of $1 + f(z)$ when z runs from $\operatorname{Im} z = -\infty$ to $\operatorname{Im} z = +\infty$ on the line $\Gamma_{\frac{1}{2}-\gamma}$.

If $k^\gamma(r)$ is an arbitrary strictly positive function on $\mathbb{R}_+$ with $k^\gamma(r) = r^\gamma$ for $0 < r < c_0$, $k^\gamma(r) = 1$ for $c_1 < r$ with constants $c_0 < c_1$, we have $\mathcal{K}^{s,\gamma}(\mathbb{R}_+) = k^\gamma \mathcal{K}^{s,0}(\mathbb{R}_+)$. For sufficiently large c_0 we get

$$A_0 := k^{-\gamma} A k^\gamma = 1 + \omega r^{-\gamma} \operatorname{op}_M^\gamma(f) r^\gamma \omega_0 : \mathcal{K}^{s,0}(X^\wedge) \to \mathcal{K}^{s,0}(X^\wedge).$$

Since $r^{-\gamma} \operatorname{op}_M^\gamma(f) r^\gamma = \operatorname{op}_M(T^{-\gamma} f)$, $(T^{-\gamma} f)(z) = f(z - \gamma)$, it suffices to take the weight $\gamma = 0$. In other words, we consider

$$1 + \omega \operatorname{op}_M(f)\tilde{\omega} : \mathcal{K}^{s,0}(\mathbb{R}_+) \to \mathcal{K}^{s,0}(\mathbb{R}_+). \tag{2.2.19}$$

The following example is due to Witt: Let us choose an arbitrary $f_0(z) \in C^\infty(\Gamma_{\frac{1}{2}})$ with $f_0(z) = 0$ for $\operatorname{Im} z \leq 0$, $f_0(z) = 2\pi i$ for $\operatorname{Im} z \geq 1$, such that $\Delta \arg(1 + f_0(z))|_{\Gamma_{\frac{1}{2}}}|_{\operatorname{Im} z=-\infty}^{\operatorname{Im} z=\infty} = 2\pi$. Then $f_0(z) \in S^0_{\mathrm{cl}}(\Gamma_{\frac{1}{2}})$. By Theorem 2.2.3 we find an $h(z) \in M^0_O$ with $f_0(z) - h(z)|_{\Gamma_{\frac{1}{2}}} \in S^{-\infty}(\Gamma_{\frac{1}{2}})$. It has the property $\Delta \arg(1 + h(z))|_{\Gamma_{\frac{1}{2}}}|_{\operatorname{Im} z=-\infty}^{\operatorname{Im} z=\infty} = 2\pi$, cf. Remarks 1.1.51 and 2.2.11 in [66].

Then we can form $e^{h(z)} = l(z)$ which is also an element in M^0_O with $l^{-1}(z) \in M^0_O$, and

$$\Delta \arg l(z)|_{\Gamma_\beta}|_{\operatorname{Im} z=-\infty}^{\operatorname{Im} z=\infty} = 2\pi \qquad \text{for each real } \beta.$$

Set $f(z) = l(z) - 1$. Then $f(z) \in M^{-\infty}_O$, since $l(\frac{1}{2} + i\tau) \to 1$ for $\tau \to -\infty$, $l(\frac{1}{2} + i\tau) \to 1$ for $\tau \to \infty$ in the sense that $(l(z) - 1)|_{\Gamma_{\frac{1}{2}}} \in \mathcal{S}(\Gamma_{\frac{1}{2}})$. But then we get $(l(z) - 1)|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta)$ for all $\beta \in \mathbb{R}$, cf. Remark 2.2.8.

Summing up we have the following result.

Proposition 2.2.22 *There exists an $f(z) \in M^{-\infty}_O$ with $1 + f(z) \neq 0$ and $(1 + f(z))^{-1} \neq 0$ for all $z \in \mathbb{C}$ such that*

$$1 + \omega \operatorname{op}_M(f)\tilde{\omega} : \mathcal{K}^{s,0}(\mathbb{R}_+) \to \mathcal{K}^{s,0}(\mathbb{R}_+)$$

is a Fredholm operator of index 1.

Note that the index is independent of s and of the choice of the cut–off functions ω, $\tilde{\omega}$.

Similarly to the formula in Remark 2.2.21 we have $(1+f(z))^{-1} = 1+g(z)$ for $g(z) = f(z) + f(z)(1 + f(z))^{-1}f(z) \in M^{-\infty}_O$.

Let us return to operators of the form (2.2.17) for arbitrary $f \in M^{-\infty}_R$ and denote it by A_γ for the given $\gamma \in \mathbb{R}$, $\pi_{\mathbb{C}}R \cap \Gamma_{\frac{1}{2}-\gamma} = \emptyset$. Choose any $\beta \leq \gamma$ with $\pi_{\mathbb{C}}R \cap \Gamma_{\frac{1}{2}-\beta} = \emptyset$ and consider also

$$A_\beta = 1 + \omega \operatorname{op}^\beta_M(f)\omega_0 : \mathcal{K}^{s,\beta}(\mathbb{R}_+) \to \mathcal{K}^{s,\beta}(\mathbb{R}_+).$$

The following result is sometimes referred to as a relative index theorem.

Theorem 2.2.23 *Let $f \in M^{-\infty}_R$ and $\pi_{\mathbb{C}} \cap (\Gamma_{\frac{1}{2}-\gamma} \cup \Gamma_{\frac{1}{2}-\beta}) = \emptyset$. Denote $N_{\beta\gamma}$ the number of zeros and $P_{\beta\gamma}$ the number of poles of $1 + f(z)$ in the strip $\frac{1}{2} - \gamma < \operatorname{Re} z < \frac{1}{2} - \beta$, counted with their multiplicities. Then we have*

$$\operatorname{ind} A_\beta - \operatorname{ind} A_\gamma = N_{\beta\gamma} - P_{\beta\gamma}.$$

Proof. We apply a standard theorem of the complex analysis which is the following. Let $h(z)$ be a meromorphic function in a domain $D \subset \mathbb{C}$ (say simply connected, with a piecewise smooth boundary curve C in counter clockwise orientation) such that $h(z)$ is continuous in $\overline{D}$, $h(z) \neq 0$ on C. Denote by N the number of zeros, P the number of poles of h in D, counted with their multiplicities, and let $\Delta_C \arg h(z)$ be the argument difference of $h(z)$ when z runs along C. Then $N - P = (2\pi)^{-1}\Delta_C \arg h(z)$. This can be applied to $D = \{z : \frac{1}{2} - \gamma < \operatorname{Re} z < \frac{1}{2} - \beta, \ |\operatorname{Im} z| < N + 1\}$ for some sufficiently large $N \in \mathbb{N}$ and $h(z) = 1 + f(z)$. For large N we have $h(z) \neq 0$ on ∂D, and all zeros and poles of h in the strip $\{z : \frac{1}{2} - \gamma < \operatorname{Re} z < \frac{1}{2} - \beta\}$ lie in D. Clearly $\Delta_C \arg h(z)$ equals

$$\Delta \arg(1 + f(z))|_{\Gamma_{\frac{1}{2}-\beta}}\big|_{\operatorname{Im} z=-\infty}^{\operatorname{Im} z=+\infty} - \Delta \arg(1 + f(z))|_{\Gamma_{\frac{1}{2}-\gamma}}\big|_{\operatorname{Im} z=-\infty}^{\operatorname{Im} z=+\infty}$$

and hence, by virtue of (2.2.17), we get the assertion. □

2.2.4 The cone algebra

Classical pseudo–differential operators on a closed compact C^∞ manifold can be motivated by the problem to construct an operator algebra which contains all differential operators (with smooth coefficients) and the parametrices of elliptic elements, cf. Section 1.1.3 above (this program induces in fact a smaller algebra, where homogeneous components of the local symbols are rational functions in the covariables).

Similar problems are reasonable on more general (pseudo–) manifolds such as manifolds with conical singularities or edges. In all these cases the typical differential operators are the starting point. For conical singularities these are operators of Fuchs type, cf. Section 2.1.1.

A typical novelty for conical singularities is that the operators A have two principal symbols, namely the homogeneous principal symbol $\sigma_\psi^\mu(A)$ as usual and an operator–valued symbol, the conormal symbol $\sigma_M^\mu(A)$. If A is given in the form (2.1.2), we can express the operator–valued coefficients $a_j(r)$ in local coordinates $x \in \mathbb{R}^n$ with covariables ξ. Then we have

$$\sigma_\psi^\mu(A)(r, x, \varrho, \xi) = r^{-\mu} \sum_{j+|\alpha|=\mu} c_{j\alpha}(r, x)(r\varrho)^j \xi^\alpha \tag{2.2.20}$$

for certain coefficients $c_{j\alpha}$ that are C^∞ up to $r = 0$. The conormal symbol is defined as the operator function in $z \in \mathbb{C}$

$$\sigma_M^\mu(A)(z) = \sum_{j=0}^{\mu} a_j(0) z^j, \tag{2.2.21}$$

with values in $\mathcal{L}(H^s(X), H^{s-\mu}(X))$, $s \in \mathbb{R}$.

Let us replace $r\varrho$ by ϱ_{f} and omit the weight factor $r^{-\mu}$ in (2.2.20). Then

$$\sigma_{\psi,\mathrm{f}}^\mu(A)(r, x, \varrho_{\mathrm{f}}, \xi) = \sum_{j+|\alpha|=\mu} c_{j\alpha}(r, x) \varrho_{\mathrm{f}}^j \xi^\alpha$$

is called the compressed interior symbol of A. It is invariantly defined as a C^∞ function on the so–called compressed cotangent bundle of $\overline{\mathbb{R}}_+ \times X$ (or more generally of the stretched manifold $\mathbb{B}$ to a manifold B with conical singularities; we mainly consider the case $\overline{\mathbb{R}}_+ \times X$ here).

Notice that the compressed cotangent bundle $T_{\mathrm{f}}^*(\overline{\mathbb{R}}_+ \times X)$ is nothing else than a bundle obtained by the cocycle of transition maps

$$\begin{pmatrix} r^{-1} & 0 \\ 0 & \mathrm{id}_{\mathbb{R}^n} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & {}^T J_{\Sigma\tilde{\Sigma}}^{-1} \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & \mathrm{id}_{\mathbb{R}^n} \end{pmatrix}^{-1} : \begin{matrix} \mathbb{R} \\ \oplus \\ \mathbb{R}^n \end{matrix} \to \begin{matrix} \mathbb{R} \\ \oplus \\ \mathbb{R}^n \end{matrix} \tag{2.2.22}$$

corresponding to trivialisations $(\overline{\mathbb{R}}_+ \times \Sigma) \times \mathbb{R}^{1+n}$ and $(\overline{\mathbb{R}}_+ \times \tilde{\Sigma}) \times \mathbb{R}^{1+n}$, for open sets $\Sigma, \tilde{\Sigma} \subseteq \mathbb{R}^n$ (belonging to charts on X), where ${}^T J_{\Sigma\tilde{\Sigma}}^{-1}$ is the transposed of the inverse of the Jacobian matrix to a coordinate diffeomorphism $\Sigma \to \tilde{\Sigma}$.

Let $\chi : \overline{\mathbb{R}}_+ \times X \to \overline{\mathbb{R}}_+ \times X$ be a diffeomorphism (i.e., the restriction of a diffeomorphism $\mathbb{R} \times X \to \mathbb{R} \times X$ to $\overline{\mathbb{R}}_+ \times X$) and A a differential operator of Fuchs type, with respect to the splitting of variables in (r, x). Set $(t, y) = \chi(r, x)$. Then it is easy to verify that the operator push–forward $\chi_* A$ is of Fuchs type with respect to (t, y) and that we have

$$\sigma_{\psi,\mathrm{f}}^\mu(\chi_* A)(t, y, \tau, \eta) = \sigma_{\psi,\mathrm{f}}^\mu(A)(r, x, \varrho, \xi) \tag{2.2.23}$$

when (t, y, τ, η) and (r, x, ϱ, ξ) denote the variables in $T_{\mathrm{f}}^*(\overline{\mathbb{R}}_+ \times X) \setminus 0$, where (τ, η) is the image of (ϱ, ξ) under the map (2.2.22) over the point (r, x). For the conormal symbols we have

$$\sigma_M^\mu(\chi_* A)(z) = \kappa_* \{\alpha^{\mu+z} \sigma_M^\mu(A)(z) \alpha^{-z}\}, \tag{2.2.24}$$

where $\kappa : X \to X$ is the diffeomorphism that is induced by χ by restriction to $\{0\} \times X$ and $\kappa_* : \mathrm{Diff}^\mu(X) \to \mathrm{Diff}^\mu(X)$, the operator push–forward applied for every fixed $z \in \mathbb{C}$ and $\alpha(x) = (\partial t/\partial r)(0, x)$, cf. [28].

It is obvious that the operator families $\sigma_M^\mu(A)|_{\Gamma_\beta}$ belong to $L_{\mathrm{cl}}^\mu(X; \Gamma_\beta)$ for all β, cf. the notation of Section 1.1.4, here modified in the sense that Λ is replaced by Γ_β, identified with $\mathbb{R}$. Thus $\sigma_M^\mu(A)|_{\Gamma_\beta}$ has a parameter–dependent homogeneous principal symbol $\sigma_{\psi,\mathrm{p}}^\mu \sigma_M^\mu(A)|_{\Gamma_\beta}$ (with $\tau = \mathrm{Im}\, z$ as parameter). It is independent of β. Moreover,

$$(\sigma_{\psi,\mathrm{p}}^\mu \sigma_M^\mu(A)|_{\Gamma_\beta})(x, \tau, \xi)|_{\tau=-\varrho_{\mathrm{f}}} = \sigma_{\psi,\mathrm{f}}^\mu(A)(r, x, \varrho_{\mathrm{f}}, \xi)|_{r=0}. \tag{2.2.25}$$

The latter relation is a compatibility condition between the components of

$$\sigma(A) = (\sigma_{\psi,\mathrm{f}}^\mu(A), \sigma_M^\mu(A)). \tag{2.2.26}$$

Notice that the composition of Fuchs type operators A and B of order μ and ν, respectively, is again of Fuchs type where

$$\sigma(AB) = (\sigma_{\psi,\mathrm{f}}^\mu(A)\sigma_{\psi,\mathrm{f}}^\nu(B), (T^\nu \sigma_M^\mu(A))\sigma_M^\nu(B)), \tag{2.2.27}$$

with $(T^\nu h)(z) = h(z + \nu)$ being the translation in the complex plane (which comes from the weight factor $r^{-\nu}$ in B).

Operators of Fuchs type can be studied on the infinite (stretched) cone $\mathbb{R}_+ \times X$ or on a more general manifold B with conical singularities, where neighbourhoods of the singularities are identified with cylinders whose cross sections X are closed compact C^∞ manifolds or compact C^∞ manifolds with C^∞ boundary (in the case of boundary value problems, cf. joint papers with Schrohe [47], [48]). In this exposition we consider the case of closed compact X and infinite cylinders (the case of arbitrary manifolds with conical singularities is a modification and can be treated by an easy localisation procedure with a corresponding partition of unity).

Concerning Fuchs type differential operators (of order μ), cf. (2.1.2), for convenience we suppose that the coefficients $a_j(r)$ are independent of r for $r > R$ with some $R > 0$ sufficiently large. It is then easy to verify, cf. Proposition 2.1.8 or Proposition 3.1.1 below, that

$$A : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)$$

is continuous for every $s, \gamma \in \mathbb{R}$.

Let us now formulate pseudo–differential operators in the cone algebra. Choose a covering $\{U_1, \dots, U_N\}$ of X by coordinate neighbourhoods, a subordinate partition of unity $\{\varphi_1, \dots, \varphi_N\}$ and functions $\psi_j \in C_0^\infty(U_j)$ with $\varphi_j \psi_j = \varphi_j$ for all j. Further let $\kappa_j : U_j \to \Sigma_j$ be charts, $\Sigma_j \subseteq \mathbb{R}^n$ open. Then a starting point is a system of symbols

$$p_j(r, x, \varrho, \xi) \in \tilde{S}^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma_j \times \mathbb{R}^{1+n}), \qquad j = 1, \dots, N.$$

By definition we have $p_j(r, x, \varrho, \xi) = \tilde{p}_j(r, x, r\varrho, \xi)$ for some $\tilde{p}_j(r, x, \varrho_{\mathrm{f}}, \xi) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma_j \times \mathbb{R}^{1+n}_{\varrho_{\mathrm{f}},\xi})$. Applying Remark 2.2.7 we find elements $h_j(r, x, z, \xi) \in S^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma_j \times \mathbb{C} \times \mathbb{R})$ with

$$\mathrm{op}_{t,x}(p_j) = \mathrm{op}_M^{\frac{1}{2}}(\mathrm{op}_x(h_j)) \bmod L^{-\infty}(\mathbb{R}_+ \times \Sigma_j). \tag{2.2.28}$$

Setting

$$h(r, z) = \sum_{j=1}^N \varphi_j (\kappa_j^{-1})_* \, \mathrm{op}_x(h_j)(r, z) \psi_j \tag{2.2.29}$$

we get an element $h(r, z) \in C^\infty(\overline{\mathbb{R}}_+, M^\mu_O(X))$ which is an operator–valued Mellin symbol. It follows that

$$\sum_{j=1}^N \varphi_j \, \mathrm{op}_r((\kappa_j^{-1})_* \, \mathrm{op}_x(p_j)) \psi_j = \mathrm{op}_M^{\frac{1}{2}}(h) \bmod L^{-\infty}(X^\wedge). \tag{2.2.30}$$

The correspondence $\{p_1, \dots, p_N\} \to h$ is part of the operator convention in the cone pseudo–differential calculus. We can imagine this in a more canonical form when we suppose (without loss of generality) that the local symbols p_j form a compatible system in the sense that

$$\mathrm{op}_r((\kappa_j^{-1})_* \, \mathrm{op}_x(p_j)) = \mathrm{op}_M^{\frac{1}{2}}(h)|_{U_j} \bmod L^{-\infty}(\mathbb{R}_+ \times U_j) \tag{2.2.31}$$

for all j. This is the case if we start with a system $\{p_1, \dots, p_N\}$ which is invariant modulo symbols of order $-\infty$ under the symbol push–forwards (belonging to the operator push–forwards) to coordinate diffeomorphisms $\mathbb{R}_+ \times (\Sigma_j \cap \Sigma_k) \to \mathbb{R}_+ \times (\Sigma_k \cap \Sigma_j)$, $j, k = 1, \dots, N$, cf. Remark 1.1.27. Relation (2.2.30) is to be understood in the sense of pseudo–differential operators acting on $C_0^\infty(X^\wedge)$. However, a choice of an element $h(r, z) \in C^\infty(\overline{\mathbb{R}}_+, M^\mu_O(X))$ allows us to form

$$\omega \, \mathrm{op}_M^{\gamma - \frac{n}{2}}(h) \omega_0 : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma}(X^\wedge)$$

for arbitrary $\gamma \in \mathbb{R}$, in the sense of an extension by continuity, $\omega(r)$, $\omega_0(r)$ are cut–off functions. Notice that by Cauchy's theorem we have

$$\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)u = \mathrm{op}_M^{\tilde{\gamma}-\frac{n}{2}}(h)u$$

for all $u \in C_0^\infty(X^\wedge)$ for arbitrary $\gamma, \tilde{\gamma} \in \mathbb{R}$.

It is clear that $\mathrm{op}_M^{\gamma-\frac{n}{2}}(h) \in L_{\mathrm{cl}}^\mu(X^\wedge)$. Therefore, if $P \in L_{\mathrm{cl}}^\mu(X^\wedge)$ is arbitrary, for instance, of the form of the left hand side of (2.2.30), also the operator

$$A_0 := \omega\, \mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\omega_0 + (1-\omega)P(1-\omega_1) \tag{2.2.32}$$

belongs to $L_{\mathrm{cl}}^\mu(X^\wedge)$ for arbitrary cut–off functions $\omega(r)$, $\omega_0(r)$, $\omega_1(r)$. Assume here $\omega_0\omega = \omega$, $\omega_1\omega = \omega_1$. Setting $P = \sum_{j=1}^N \varphi_j\, \mathrm{op}_r((\kappa_j^{-1})_* \,\mathrm{op}_x(p_j))\psi_j$ we get

$$P = A_0 \bmod L^{-\infty}(X^\wedge)$$

for arbitrary $\gamma \in \mathbb{R}$. Thus $\{p_1, \dots, p_N\} \to A_0$ is an operator convention. An adequate class of smoothing operators for the specific purposes, connected with the conical singularity $r = 0$, are the operators of the form (2.2.16) above, the so–called smoothing Mellin+Green operators. Adequate means that we get an algebra that is closed under parametrix constructions.

The operator A_0 can also be composed with weight factors when they are required in the original operators, cf. (2.1.2). In addition, in order to get continuous operators in $\mathcal{K}^{s,\gamma}(X^\wedge)$–spaces, we impose some natural condition for $r \to \infty$, a so–called exit condition, cf. Parenti [38], Cordes [7] or the joint paper with Dorschfeldt and Grieme [9] and the references there.

In the special case of the unit sphere $X = S^n$ in $\mathbb{R}^{n+1} \ni \tilde{x}$ we have $X^\wedge = \mathbb{R}^{n+1} \setminus \{0\}$. Symbols in $\mathbb{R}^{n+1}$ with exit condition are characterised by the symbol estimates

$$|D_{\tilde{x}}^\alpha D_{\tilde{\xi}}^\beta a(\tilde{x}, \tilde{\xi})| \le c\langle\tilde{\xi}\rangle^{\mu-|\beta|}\langle\tilde{x}\rangle^{\varrho-|\alpha|} \tag{2.2.33}$$

for all multi–indices $\alpha, \beta \in \mathbb{N}^{n+1}$, $\tilde{x}, \tilde{\xi} \in \mathbb{R}^{n+1}$. Denote by $S^{\mu;\varrho}(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ the set of all these symbols, $\mu, \varrho \in \mathbb{R}$. More generally, if $K \subset \mathbb{R}^{n+1} \setminus \{0\}$ is an open conical set (i.e., $\tilde{x} \in K \Rightarrow \lambda\tilde{x} \in K$ for all $\lambda \in \mathbb{R}_+$) we can define $S^{\mu;\varrho}(K \times \mathbb{R}^{n+1})$ by the same kind of estimates for all $(\tilde{x}, \tilde{\xi}) \in K' \times \mathbb{R}^{n+1}$ for arbitrary conical $K' \subset K$ with $\overline{K' \cap S^n} \subset K \cap S^n$ and $|\tilde{\xi}| \ge \mathrm{const}$ for a constant > 0. We are mainly interested in the case $\varrho = 0$. Set $S_{\mathrm{e}}^\mu(K \times \mathbb{R}^{n+1}) =$

$S^{\mu;0}(K \times \mathbb{R}^{n+1}) \cap S^{\mu}_{\mathrm{cl}}(K \times \mathbb{R}^{n+1})$ (the subscript cl refers to the $\tilde{\xi}$–variables). Pseudo–differential operators of the class $L^{\mu}_{\mathrm{e}}(K)$ are defined as operators of the form $\mathrm{Op}(a) + C$ for $a(\tilde{x}, \tilde{\xi}) \in S^{\mu}_{\mathrm{e}}(K \times \mathbb{R}^{n+1})$ and $C \in L^{-\infty}_{\mathrm{e}}(K)$, that are operators with Schwartz kernel $c(\tilde{x}, \tilde{x}') \in \mathcal{S}(K \times K)$. The space $\mathcal{S}(K \times K)$ is defined as the projective tensor product $\mathcal{S}(K) \otimes_{\pi} \mathcal{S}(K)$, where $\mathcal{S}(K) = \omega C^{\infty}(K) + (1-\omega)\mathcal{S}(\mathbb{R}_+, C^{\infty}(K_1))$ for any cut–off function $\omega \in C^{\infty}(K)$ with $\omega \equiv 1$ for $|\tilde{x}| < c_0$, $\omega \equiv 0$ for $|\tilde{x}| > c_1$ for certain $c_0 < c_1$, and K being identified with $\mathbb{R}_+ \times K_1$ for $K_1 = \{\tilde{x} \in K : |\tilde{x}| = 1\}$.

The spaces $L^{\mu}_{\mathrm{e}}(K)$ are invariant under diffeomorphisms $\chi : K \to L$ between conical sets K, L, with $\chi(\lambda \tilde{x}) = \lambda \chi(\tilde{x})$ for all $\lambda \in \mathbb{R}_+$. Then, by a standard procedure, using a partition of unity, we can pass to global spaces $L^{\mu}_{\mathrm{e}}(X^{\wedge})$ for an arbitrary closed compact C^{∞} manifold X. Every $A \in L^{\mu}_{\mathrm{e}}(X^{\wedge})$ in characterised mod $L^{-\infty}(X^{\wedge})$ by a system of symbols $\sigma^{\mu}_{\mathrm{e}}(A)_j(\tilde{x}, \tilde{\xi}) \in S^{\mu}_{\mathrm{e}}(K_j \times \mathbb{R}^{n+1})$ for conical sets $K_j \subset \mathbb{R}^{n+1} \setminus \{0\}$, $j = 1, \dots, N$, associated with charts $\chi_j : \mathbb{R}_+ \times U_j \to K_j$, $\chi_j(\lambda r, x) = \lambda \chi_j(r, x)$ for all $\lambda \in \mathbb{R}_+$, where $\{U_1, \dots, U_N\}$ is a covering of X by coordinate neighbourhoods.

Definition 2.2.24 *The operators of the cone algebra to the weight data* $\boldsymbol{g} = (\gamma, \delta, \Theta)$, $\Theta = (-k, 0]$, $k \in \mathbb{N} \setminus \{0\}$, *are defined as*

$$A = r^{\delta-\gamma} \omega \,\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)\omega_0 + (1-\omega)P(1-\omega_1) + M + G \tag{2.2.34}$$

for an $h(r, z) \in C^{\infty}(\overline{\mathbb{R}}_+, M^{\mu}_{O}(X))$, $P \in L^{\mu}_{\mathrm{e}}(X^{\wedge})$, *and smoothing Mellin + Green operators of the form* (2.2.16). *For* $\Theta = (-\infty, 0]$ *we simply require this form for every finite* $k \in \mathbb{N} \setminus \{0\}$. *Denote* $C^{\mu}(X^{\wedge}, \boldsymbol{g})$ *the class of all these operators.*

Theorem 2.2.25 *Every* $A \in C^{\mu}(X^{\wedge}, \boldsymbol{g})$, $\boldsymbol{g} = (\gamma, \delta, \Theta)$, *induces continuous operators*

$$A : \mathcal{K}^{s,\gamma}(X^{\wedge}) \to \mathcal{K}^{s-\mu,\delta}(X^{\wedge}), \qquad A : \mathcal{K}^{s,\gamma}_P(X^{\wedge}) \to \mathcal{K}^{s-\mu,\delta}_Q(X^{\wedge}),$$

for all $s \in \mathbb{R}$ *and arbitrary* $P \in \mathrm{As}(X, (\gamma, \Theta))$ *with some resulting* $Q \in \mathrm{As}(X, (\delta, \Theta))$.

This result is obvious by the statements above (concerning the continuity of operators with exit property in the standard Sobolev spaces, cf. [9]).

To each $A \in C^{\mu}(X^{\wedge}, \boldsymbol{g})$ we form the pair of symbols $\sigma(A)$, similarly to (2.2.26), where $\sigma^{\mu}_{\psi,\mathrm{f}}(A)$, the compressed interior principal symbol, is defined as $\sigma_\psi^\mu(r^{\gamma-\delta}A)(r,x,r^{-1}\varrho,\xi)$ which is invariantly defined as a C^∞ function on $T_{\mathrm{f}}^*(\overline{\mathbb{R}}_+\times X)\setminus 0$, and

$$\sigma_M^{\gamma-\delta}(A)(z)=h(0,z)+f_0(z),$$

the conormal symbol (to the conormal order $\gamma-\delta$), with $f_0(z)$ from (2.2.16).

There is a third symbol component from the pseudo–differential calculus with exit behaviour for $r\to\infty$, denoted $\sigma_{\mathrm{e}}^0(A)=\{\sigma_{\mathrm{e}}^0(A)_j(\tilde{x},\tilde{\xi})\}_{j=1,\dots,N}$. In fact, the operators in $C^\mu(X^\wedge,\boldsymbol{g})$ belong to $L_{\mathrm{e}}^\mu(X^\wedge)$ (the variables $(r,x)\in\overline{\mathbb{R}}_+\times U_j$ play the role polar coordinates in the corresponding conical sets $K_j\ni\tilde{x}$). Together with the exit symbol component we get

$$\sigma(A)=(\sigma_{\psi,\mathrm{f}}^\mu(A),\sigma_M^{\gamma-\delta}(A),\sigma_{\mathrm{e}}^0(A)).$$

Let us establish an operator convention

$$\mathrm{op}:(p_\psi,p_M,p_{\mathrm{e}})\to A \tag{2.2.35}$$

from the space of symbols, consisting of all occurring triples $(p_\psi,p_M,p_{\mathrm{e}})=(\sigma_{\psi,\mathrm{f}}^\mu(A),\sigma_M^{\gamma-\delta}(A),\sigma_{\mathrm{e}}^0(A))$ for $A\in C^\mu(X^\wedge,\boldsymbol{g})$, into $C^\mu(X^\wedge,\boldsymbol{g})$.

First to the component p_ψ we choose a system of local symbols $\{p_j\}_{j=1,\dots,N}$ of the above kind and then construct $h_j(r,z)$ in the sense of (2.2.28). It follows that

$$\sigma_{\psi,\mathrm{p}}^\mu(h_j)(r,x,\tau,\xi)|_{r=0,\tau=-\varrho_{\mathrm{f}}}=p_{j,(\mu)}(0,x,r^{-1}\varrho_{\mathrm{f}},\xi)$$

(recall that $\sigma_{\psi,\mathrm{p}}^\mu$ denotes the parameter–dependent homogeneous principal symbol, with parameter $\tau=\operatorname{Im}z$; subscript (μ) indicates the homogeneous principal part of order μ). Furthermore, for (2.2.29) we get the relation

$$f_0(z):=p_M(z)-h(0,z)\in M_R^{-\infty}(X)$$

for a certain $R\in\mathbf{As}(X)$.

The compatibility condition (2.2.31) to the system $\{p_1,\dots,p_N\}$ which we impose without loss of generality implies that the functions $p_{j,(\mu)}(r,x,r^{-1}\varrho,\xi)$ are the local representatives of $\sigma_{\psi,\mathrm{f}}^\mu(A)=p_\psi$. We now define the operator A itself by formula (2.2.34), using the data $\{p_1,\dots,p_N\}$, $h(r,z)$ and $f_0(z)$. It is easy to make the choice compatible with p_{e} which is left to the reader. Thus we have (2.2.35) as desired.

Remark 2.2.26 *There is an analogue of the cone algebra in which asymptotic types are ignored (the "cone algebra without asymptotics"). Its invariance under diffeomorphisms* $\mathbb{R}_+ \times X \to \overline{\mathbb{R}} \times X$ *(that respect the manifold structure with exit to infinity) is proved in Schrohe [46]. The case including (discrete) asymptotics is treated in a joint paper with Kapanadze and Witt [28].*

Theorem 2.2.27 $A \in C^\mu(X^\wedge, (\gamma, \delta, \Theta))$, $B \in C^\nu(X^\wedge, (\beta, \gamma, \Theta))$ *implies* $AB \in C^{\mu+\nu}(X^\wedge, (\beta, \delta, \Theta))$, *and we have the symbol rule* (2.2.27)

$$\sigma(AB) = (\sigma^\mu_{\psi,\mathrm{f}}(A)\sigma^\nu_{\psi,\mathrm{f}}(B), (T^{\beta-\gamma}\sigma_M^{\gamma-\delta}(A))\sigma_M^{\beta-\gamma}(B), \\ \{\sigma^0_\mathrm{e}(A)_j \# \sigma^0_\mathrm{e}(B)_j\}_{j=1,\dots,N})$$

with the Leibniz products in the third component.

This theorem is proved in [58], cf. also [66].

Definition 2.2.28 *An operator* $A \in C^\mu(X^\wedge, (\gamma, \delta, \Theta))$ *is called elliptic if*

(i) $\sigma^\mu_{\psi,\mathrm{f}}(A) \neq 0$ *on* $T^*_\mathrm{f}(\overline{\mathbb{R}}_+ \times X) \setminus 0$.

(ii) $\sigma_M^{\gamma-\delta}(A)(z) : H^s(X) \to H^{s-\mu}(X)$ *is an isomorphism for all* $z \in \Gamma_{\frac{n+1}{2}-\gamma}$ *and some* $s \in \mathbb{R}$.

(iii) $\sigma^0_\mathrm{e}(A)_j(\tilde{x}, \tilde{\xi}) \neq 0$ *for all* $(\tilde{x}, \tilde{\xi}) \in K_j \times \mathbb{R}^{n+1}$ *with* $|\tilde{x}|^2 + |\tilde{\xi}|^2 \geq \mathrm{const}$ *for a constant* > 0, $j = 1, \dots, N$.

Theorem 2.2.29 *Let* $A \in C^\mu(X^\wedge, (\gamma, \delta, \Theta))$ *be given; then the following conditions are equivalent:*

(i) *The operator* A *is elliptic.*

(ii) *The operator* A *induces a Fredholm operator*

$$A : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\delta}(X^\wedge) \tag{2.2.36}$$

for some $s \in \mathbb{R}$.

Condition (ii) implies that the operator (2.2.34) *is Fredholm for all* $s \in \mathbb{R}$. *Moreover, if* A *is elliptic there is a parametrix* $B \in C^{-\mu}(X^\wedge, (\delta, \gamma, \Theta))$ *in the sense that the remainders* $AB - I$ *and* $BA - I$ *are Green operators with respect to the weight data* (δ, δ, Θ) *and* (γ, γ, Θ), *respectively.*

A proof of this theorem may be found in [58] or [66].

Notice that $\sigma(B) = \sigma(A)^{-1}$ in the sense of the symbol composition of Theorem 2.2.27.

Corollary 2.2.30 *Let $A \in C^{\mu}(X^{\wedge}, (\gamma, \delta, \Theta)$ be an elliptic operator, and let $u \in \mathcal{K}^{-\infty,\gamma}(X^{\wedge})$, $Au = f$. Then we have*

$$f \in \mathcal{K}^{s-\mu,\delta}(X^{\wedge}) \Rightarrow u \in \mathcal{K}^{s,\gamma}(X^{\wedge}),$$
$$f \in \mathcal{K}_Q^{s-\mu,\delta}(X^{\wedge}) \Rightarrow u \in \mathcal{K}_P^{s,\gamma}(X^{\wedge})$$

for all $s \in \mathbb{R}$ and every $Q \in \mathrm{As}(\delta, \Theta)$ with some resulting $P \in \mathrm{As}(\gamma, \Theta)$. Kernel and cokernel of (2.2.36) *and hence the index of* (2.2.36) *are independent of s.*

The elliptic regularity of solutions u to elliptic equations $Au = f$ (without / with asymptotics) stated in Corollary 2.2.30 follows from a (left) parametrix B that exists by Theorem 2.2.29. In fact, the operator $G = BA - I$ is Green in the sense of Definition 2.2.19. Theorem 2.2.25 gives us $BAu = Bf \in \mathcal{K}^{s,\gamma}(X^{\wedge})$ $(\in \mathcal{K}_R^{s,\gamma}(X^{\wedge}))$ for $f \in \mathcal{K}^{s-\mu,\delta}(X^{\wedge})$ $(\in \mathcal{K}_Q^{s-\mu,\delta}(X^{\wedge}))$; subscripts at the spaces mean suitable asymptotic types. Thus, in view of $BAu = (I + G)u$, we have $u = Bf - Gu$. Applying the mapping properties of G, cf. Definition 2.2.19, we get the elliptic regularity of u.

Remark 2.2.31 *Consider a differential operator*

$$A(\tilde{x}, D_{\tilde{x}}) = \sum_{|\alpha| \le \mu} \tilde{a}_{\alpha}(\tilde{x}) D_{\tilde{x}}^{\alpha} : H^{s}(\mathbb{R}^{n+1}) \to H^{s-\mu}(\mathbb{R}^{n+1}) \tag{2.2.37}$$

with coefficients $\tilde{a}_{\alpha} \in C^{\infty}(\mathbb{R}^{n+1})$ that are independent of $\tilde{x}$ for large $|\tilde{x}|$. Then, if we express A in polar coordinates near $\tilde{x} = 0$, we see that A can be regarded as a continuous operator

$$A : \mathcal{K}^{s,\gamma}(\mathbb{R}^{n+1} \setminus \{0\}) \to \mathcal{K}^{s-\mu,\gamma-\mu}(\mathbb{R}^{n+1} \setminus \{0\}) \tag{2.2.38}$$

for arbitrary $\gamma \in \mathbb{R}$, and we have $A \in C^{\mu}((S^n)^{\wedge}, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma - \mu, (-\infty, 0])$. The operator (2.2.37) *is Fredholm if and only if $\sigma_{\psi}^{\mu}(A)(\tilde{x}, \tilde{\xi}) \neq 0$ for $\tilde{\xi} \neq 0$ and $\sum_{|\alpha| \le \mu} \tilde{a}_{\alpha}(\infty) \tilde{\xi}^{\alpha} \neq 0$ for all $\tilde{\xi} \in \mathbb{R}^{n+1}$, including $\tilde{\xi} = 0$ (this second condition is just the exit ellipticity of* (2.2.37) *which originally means $\sum_{|\alpha| \le \mu} \tilde{a}_{\alpha}(\tilde{x}) \tilde{\xi} \neq 0$ for $|\tilde{x}|^2 + |\tilde{\xi}|^2 > R$ for some $R > 0$). The Fredholm property of* (2.2.38) *is not automatic, it depends on condition (ii) in Definition* 2.2.28, *and it may be violated for special weights γ.*

Note that when

$$A(\tilde{x}, D_{\tilde{x}}, \eta) = \sum_{|\alpha',\alpha''|\leq\mu} \tilde{a}_{\alpha',\alpha''}(\tilde{x}) D_{\tilde{x}}^{\alpha'} \eta^{\alpha''}, \qquad \eta \in \mathbb{R}^q,$$

is parameter–dependent elliptic (i.e., $\sum_{|\alpha',\alpha''|=\mu} \tilde{a}_{\alpha',\alpha''}(\tilde{x}) \tilde{\xi}^{\alpha'} \eta^{\alpha''} \neq 0$ *for* $(\tilde{\xi}, \eta) \neq 0$*) then, for every fixed* $\eta \neq 0$*, the operator* $A(\tilde{x}, D_{\tilde{x}}, \eta) : H^s(\mathbb{R}^{n+1}) \to H^{s-\mu}(\mathbb{R}^{n+1})$ *is Fredholm, since the exit ellipticity is a consequence of the parameter–dependent elliptitcity in* $(\tilde{\xi}, \eta)$.

Operators of the form

$$\omega r^{\delta-\gamma} \sum_{j=1}^{k-1} r^j \operatorname{op}_M^{\gamma_j - \frac{n}{2}}(f_j)\omega_0 + G$$

with the assumption of (2.2.16) are compact as operators $\mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\delta}(X^\wedge)$. Thus, without changing the index of (2.2.36), we may pass to

$$A_\gamma = r^{\delta-\gamma} \omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(h + f_0)\omega_0 + A_\infty : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\delta}(X^\wedge),$$

$A_\infty = (1 - \omega)P(1 - \omega_1)$, cf. formula (2.2.34).

Remark 2.2.32 *Let an element* $A \in C^\mu(X^\wedge, (\gamma, \delta, \Theta))$ *be given in the form*

$$A = r^{\delta-\gamma}\omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)\omega_0 + (1-\omega)P(1-\omega_1) + r^{\delta-\gamma}\omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(f_0)\omega_0$$

with the notation from Definition 2.2.24*, where* $f_0 \in M_O^{-\infty}(X)$*. Then we can replace* h *by* $h + f_0$*, and omit the third summand. The operator* A *induces continuous operators*

$$A_\beta = r^{\delta-\gamma}\omega \operatorname{op}_M^{\gamma+\beta}(h)\omega_0 + (1-\omega)P(1-\omega_1) : \mathcal{K}^{s,\gamma+\beta}(X^\wedge) \to \mathcal{K}^{s-\mu,\delta+\beta}(X^\wedge) \tag{2.2.39}$$

for all $s \in \mathbb{R}$ *and all* $\beta \in \mathbb{R}$*. Now assume that* A_β *for* $\beta = 0$ *fulfills conditions (i) and (ii) of Definition* 2.2.28*. Then* A_β *fulfills the conditions for all* β*. Further assume that* $\sigma_M^{\gamma-\delta}(A_\beta)(z) : H^s(X) \to H^{s-\mu}(X)$ *is an isomorphism for all* $z \in \Gamma_{\frac{n+1}{2}-(\gamma+\beta)}$ *for all* $c \leq \beta \leq c'$ *for certain* $c \leq c'$*. Then, by Theorem* 2.2.29*, the operator* (2.2.39) *is Fredholm for all* $c \leq \beta \leq c'$ *and the index is independent of* β *(more precisely,* $\ker A_\beta$ *and* $\operatorname{coker} A_\beta$ *are independent of* β *for* $c \leq \beta \leq c'$*).*

The asserted Fredholm property of (2.2.39) for all $c \le \beta \le c'$ is a consequence of the ellipticity for all weights $\gamma + c \le \gamma + \beta \le \gamma + c'$, cf. Definition 2.2.28 (ii). Changing β in the interval gives us a homotopy through Fredholm operators and thus an index independent of β. The result that even $\ker A_\beta$ (and hence $\operatorname{coker} A_\beta$) are independent of β is due to the fact that the Green operators $G_\beta = B_\beta A_\beta - I$ (with B_β being a parametrix of A_β) produce the same gain of weights for all β in the interval. In other words $A_\beta u_\beta = 0$ for $u_\beta \in \mathcal{K}^{s,\gamma+\beta}(X^\wedge)$ implies $B_\beta A_\beta u_\beta = u_\beta + G_\beta u_\beta = 0$, but $G_\beta u_\beta \in \mathcal{K}^{\infty,\gamma+c'}(X^\wedge)$ for all $c \le \beta \le c'$ and hence $u_\beta \in \mathcal{K}^{s,\gamma+c'}(X^\wedge)$ for all $c \le \beta \le c'$.

We want to compare the Fredholm indices for different weights $\gamma \in \mathbb{R}$ also in more general cases.

For any other weight β with $\pi_{\mathbb{C}} R_0 \cap \Gamma_{\frac{1}{2}-\beta} = \emptyset$ we can form

$$A_\beta = r^{\zeta-\beta} \omega \operatorname{op}_M^{\beta-\frac{n}{2}}(h + f_0)\omega_0 + A_\infty : \mathcal{K}^{s,\beta}(X^\wedge) \to \mathcal{K}^{s-\mu,\zeta}(X^\wedge),$$

for $\zeta = \beta - \gamma + \delta$ which belongs to the class $C^\mu(X^\wedge, (\beta, \zeta, \Theta))$ (recall that $R_0 \in \mathbf{As}(X)$ is the asymptotic type of f_0); clearly $\zeta - \beta = \delta - \gamma$.

The conormal symbol $\sigma_M^{\gamma-\delta}(A) =: f$ belongs to $M_R^\mu(X)$ for some $R \in \mathbf{As}(X)$. From Theorem 2.2.17 we get a decomposition $f = h_0(1 + g)$ for an $h_0 \in M_O^\mu(X)$ with $h_0^{-1} \in M_O^{-\mu}(X)$ and an element $g \in M_S^{-\infty}(X)$ for some $S \in \mathbf{As}(X)$. Let us define an operator

$$M_\gamma = 1 + \omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(g)\omega_0$$

which belongs to $C_{M+G}(X^\wedge, (\gamma, \gamma, \Theta))$ and set

$$A_{0,\gamma} = r^{\delta-\gamma} \omega \operatorname{op}_M^{\gamma-\frac{n}{2}}(h_0)\omega_0 + A_\infty.$$

Then we have

$$A_\gamma = A_{0,\gamma} M_\gamma + C_\gamma$$

for a $C_\gamma \in C_{M+G}(X^\wedge, (\gamma, \delta, \Theta))$. Because of the composition rule for conormal symbols, cf. Theorem 2.2.27, we have $\sigma_M^0(C) = 0$. Thus C_γ is a compact operator, and it follows that

$$\operatorname{ind} A_\gamma = \operatorname{ind} A_{0,\gamma} + \operatorname{ind} M_\gamma$$

(clearly, $A_{0,\gamma}$ and M_γ are elliptic such that their index is finite).

In a similar manner we can proceed for A_β when we set $M_\beta = 1 + \omega \operatorname{op}_M^{\beta-\frac{n}{2}}(g)\omega_0$ and choose β in such a way that condition (ii) in Definition 2.2.28 is fulfilled. We then obtain

$$A_\beta = A_{0,\beta} M_\beta + C_\beta, \qquad A_{0,\beta} = r^{\zeta-\beta}\omega \operatorname{op}_M^{\beta-\frac{n}{2}}(h_0)\omega_0 + A_\infty,$$

for a $C_\beta \in C_{M+G}(X^\wedge, (\beta, \zeta, \Theta))$ whose principal conormal symbol vanishes, i.e., C_β is compact, and thus

$$\operatorname{ind} A_\beta = \operatorname{ind} A_{0,\beta} + \operatorname{ind} M_\beta.$$

Assume without loss of generality $\beta \leq \gamma$ and set $\gamma_t = (1-t)\beta + t\gamma$ for $0 \leq t \leq 1$. Then

$$A_{0,\gamma_t} = r^{\delta-\gamma}\omega \operatorname{op}_M^{\gamma_t-\frac{n}{2}}(h_0)\omega_0 + A_\infty, \qquad 0 \leq t \leq 1,$$

is a homotopy through Fredholm operators

$$A_{0,\gamma_t} : \mathcal{K}^{s,\gamma_t}(X^\wedge) \to \mathcal{K}^{s-\mu,\delta_t}(X^\wedge)$$

for $\delta_t = (1-t)\zeta + t\delta$ (recall $\delta - \gamma = \zeta - \beta$). Hence, for their index it follows that $\operatorname{ind} A_{0,\beta} = \operatorname{ind} A_{0,\gamma}$. We thus obtain

$$\operatorname{ind} A_\gamma - \operatorname{ind} A_\beta = \operatorname{ind} M_\gamma - \operatorname{ind} M_\beta.$$

2.2.5 Further remarks on Fuchs type operators

In this section we make some additional remarks on operators of Fuchs type on $X^\wedge$

$$A = r^{-\mu} \sum_{j=0}^{\mu} a_j(r) \left(-r\frac{\partial}{\partial r}\right)^j \tag{2.2.40}$$

with coefficients $a_j \in C^\infty(\overline{\mathbb{R}}_+, \operatorname{Diff}^{\mu-j}(X))$, $j = 0, \dots, \mu$.

Given a function $v(t) \in C^\infty(\mathbb{R})$ we have $-r\frac{\partial}{\partial r}v(-\log r) = (\frac{\partial v}{\partial t})(-\log r)$. In other words the substitution $\chi(r) = -\log r$, $\chi : \mathbb{R}_+ \to \mathbb{R}$, induces an operator push–forward χ_*, where

$$\chi_*\left(r^{-\mu}\sum_{j=0}^{\mu} a_j(r)\left(-r\frac{\partial}{\partial r}\right)^j\right) = e^{\mu t}\sum_{j=0}^{\mu} a_j(e^{-t})\left(\frac{\partial}{\partial t}\right)^j. \tag{2.2.41}$$

In this representation of $\chi_* A$ the coefficients $a_j(e^{-t})$ stabilise for $t \to \infty$. Writing $\chi_* A = e^{\mu t} \sum_{j=0}^{\mu} c_j(t) D_t^j$ with $D_t = \frac{1}{i}\partial_t$, the operator function

$$\sum_{j=0}^{\mu} c_j(\infty) w^j : H^s(X) \to H^{s-\mu}(X),$$

$w \in \mathbb{C}$, plays the role of the principal conormal symbol. The correspondence between $-r\frac{\partial}{\partial r}$ and $z \in \mathbb{C}$ turns to a correspondence between D_t and $w = -iz \in \mathbb{C}$. Weight lines $\{z : \operatorname{Re} z = \beta\}$ are transformed to $\{w : \operatorname{Im} w = -\beta\}$, weighted Sobolev spaces $\mathcal{H}^{s,\gamma}(X^\wedge)$ to $e^{-(\gamma - \frac{n}{2})t} H^s(\mathbb{R} \times X)$, cf. also Section 2.1.1 above. The theory of elliptic operators on a manifold with conical singularities can be interpreted as a theory on a manifold B with cylindrical exit to infinity where a collar neighbourhood of the boundary $\partial\mathbb{B}$ of the stretched manifold is replaced by an infinite cylinder. This point of view is fruitful for many purposes:

The cone operator theory in the special case $\dim X = 0$ with the Mellin transform on $\mathbb{R}_+ \ni r$ and the ellipticity with respect to the conormal symbol turns out to be an aspect of the ellipticity in the pseudo–differential calculus with exits in the sense of Parenti [38] or Cordes [7], cf. also Dorschfeldt, Grieme and Schulze [9] and the references there. In fact, symbols of the calculus with respect to the Mellin transform $f(r,z) \in S_{\mathrm{cl}}^{\mu}(\overline{\mathbb{R}}_+ \times \Gamma_0)$ that are smooth up to $r = 0$ turn to symbols $a(t,\tau) \in S_{\mathrm{cl}}^{\mu;0}(\mathbb{R} \times \mathbb{R})$ in the sense of the "exit symbol estimates" (2.2.33) (suppose for simplicity that f vanishes for $r > R$ for some $R > 0$). For instance, for $f(r,z) = \sum_{j=0}^{\mu} a_j(r) z^j$ we get $a(t,\tau) = \sum_{j=0}^{\mu} a_j(e^{-t})(i\tau)^j$. Ellipticity in the exit symbol class (say for $t \to \infty$) means ellipticity as usual and in addition $a(t,\tau) \neq 0$ for all (t,τ) with $|t,\tau| > T$ for some $T > 0$. The second condition is equivalent to $a(\infty,\tau) \neq 0$ for all $\tau \in \mathbb{R}$. In fact, usual ellipticity up to $t = \infty$ implies $a(t,\tau) \neq 0$ for all (t,τ) with $t > T_1$, $|\tau| > T_2$ for some $T_1, T_2 > 0$. Moreover $a(\infty,\tau) \neq 0$ for all $\tau \in \mathbb{R}$ gives us a constant $T_0 > 0$ with $a(t,\tau) \neq 0$ for all $t > T_0$, $|\tau| \leq T_2$. Thus $a(t,\tau) \neq 0$ for $t > \max(T_0, T_1)$, $|\tau| > T_2$, i.e., $a(t,\tau) \neq 0$ for $|t,\tau| > \max(T_0, T_1, T_2)$.

Conversely $a(t,\tau) \neq 0$ for $|t,\tau| > T$ and usual ellipticity obviously imply $a(\infty,\tau) \neq 0$ for all $\tau \in \mathbb{R}$. In other words, when we return to $f(r,z)$, the ellipticity of f in the usual sense together with the conormal ellipticity $f(r,z) \neq 0$ on $\Gamma_0 = \{z : \operatorname{Re} z = 0\}$ is a special case of the exit ellipticity, transformed to $\mathbb{R}$ (for $t \to \infty$). Since the exit context is more general (because smoothness up to $t = \infty$ is not necessary there) the Mellin calculus has a natural generalisation to symbols $f(r,z) \in C^{\infty}_{\mathrm{bf}}(\mathbb{R}_+, S^{\mu}_{\mathrm{cl}}(\Gamma_0))$ where $C^{\infty}_{\mathrm{bf}}(\mathbb{R}_+)$ is the subspace of all $u \in C^{\infty}(\mathbb{R}_+)$ for which $(r\partial_r)^j u$ is bounded for all $j \in \mathbb{N}$.

Analogous considerations are possible for cone pseudo–differential operators on $X^\wedge$, with an arbitrary non–trivial base X. Relations of this kind are employed in anisotropic form in a joint work with Krainer [32] to treat parabolic equations in an infinite space–time cylinder.

Let us also note that when X itself has conical singularities, $X^\wedge$ has edges, and $t = \infty$ corresponds to a corner singularity. Such an exit calculus with cone operator–valued and edge symbols is established in Seiler [84].

More general assumptions on the coefficients lead to cusp versions of the calculi as it is systematically studied in joint papers with Tarkhanov, cf. [78] and the references there.

The analysis of operators on cones is also employed for studying boundary value problems for differential operators. If $\mathbb{B}$ is a compact C^∞ manifold with C^∞ boundary X and D a differential operator on $\mathbb{B}$ of order μ (with smooth coefficients up to the boundary) we can extend $\mathbb{B}$ to a non–compact C^∞ manifold $\tilde{\mathbb{B}}$ by gluing $\overline{\mathbb{R}}_+ \times X$ to $\partial\mathbb{B}$ such that $\{0\} \times X$ is identified with $\partial\mathbb{B}$ in a canonical way and D to a differential operator $\tilde{D}$ on $\tilde{\mathbb{B}}$ with C^∞ coefficients that are defined in terms of their restrictions to $\partial\mathbb{B}$ and then extended to $\mathbb{R}_+ \times X$ by pull–back with respect to the projection $\mathbb{R}_+ \times X \to X$ (up to a small modification near $\partial\mathbb{B}$ that gives smoothness, not only continuity there). The definition of $\tilde{D}$ can be chosen in a way that $\tilde{D}$ on $\mathbb{R}_+ \times X \ni (t,x)$ has the form (2.2.41) (up to the weight factor). The operator $\tilde{D}$ then belongs to a class with a behaviour near the cylindrical exit as described above. Ellipticity implies that the solutions of $\tilde{D}u = f$ in weighted Sobolev spaces (with power weights for $t \to \infty$, with f being in such a space) are strongly decreasing as $t \to \infty$ (like $C^\infty(X)$–valued Schwartz functions). Transforming $\tilde{D}$ back to a stretched manifold $\mathbb{B}_1$ with conical singularities (which equals in fact $\mathbb{B}$, up to an attached cylinder $[1,0] \times X$) gives us a Fuchs type operator A on $\mathbb{B} \cup ([1,0] \times X) = \mathbb{B}_1$. This relation between boundary value problems for standard differential operators (without any degeneracy near the boundary) and Fuchs type operators on the same manifold, interpreted as the stretched manifold of a manifold with conical singularities, plays a role in the recent development of the index theory for Dirac operators who do not possess elliptic Shapiro–Lopatinskij boundary conditions.

Chapter 3

Edge pseudo–differential calculus

3.1 Edge–degenerate operators

3.1.1 Manifolds with edges

A manifold W with edges Y is (roughly speaking) a topological space, where $Y \subset W$ is a subspace, such that both $W \setminus Y$ and Y are C^∞ manifolds of dimension $n+1+q$ and q, respectively, for certain $n, q \in \mathbb{N}$, $q \geq 1$, where each point $y \in Y$ has a neighbourhood U in W which is homeomorphic to a wedge $\{(\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)\} \times \Omega$ for a (closed) compact C^∞ manifold X, $n = \dim X$, and an open set $\Omega \subseteq \mathbb{R}^q$. This homeomorphism restricts to diffeomorphisms $U \setminus Y \to X^\wedge \times \Omega$, $X^\wedge = \mathbb{R}_+ \times X$, and $U \cap Y \to \Omega$. We call $(\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)$ the model cone of the wedge. Precise definitions may be found in [66], [68], especially on the nature of transition maps between the local wedges. For the purposes of this exposition it may be sufficient to imagine either wedges or other configurations, embedded in $\mathbb{R}^N$, like polyhedra near faces of a fixed codimension. Another special case is the Euclidean space $\mathbb{R}^N$ with $Y \subset \mathbb{R}^N$ being a C^∞ submanifold of dimension $1 < q < N$. Let us also note that C^∞ manifolds with C^∞ boundary are regarded as manifolds with edges. In this case the boundary is the edge and the inner normal (with respect to a chosen Riemannian metric) the model cone (the base X is in this case of dimension 0). Many ideas from the edge pseudo–differential calculus as they are first published in [56] may be read off from a suitably formulated pseudo–differential theory of boundary value problems, cf. [61], [65]. This concerns the case of "regular" model cones. A joint article with N. Tarkhanov [78] studies the case of cuspidal edges.

3.1.2 The typical differential operators

Operators and associated symbols on a manifold with edges are locally near edge points expressed with respect to a splitting of coordinates $(r,x,y) \in \mathbb{R}_+ \times X \times \Omega = X^\wedge \times \Omega$. A differential operator $A \in \mathrm{Diff}^\mu(X^\wedge \times \Omega)$ is called edge–degenerate if it has the form

$$A = r^{-\mu} \sum_{j+|\alpha|\leq\mu} a_{j\alpha}(r,y)\left(-r\frac{\partial}{\partial r}\right)^j (rD_y)^\alpha \tag{3.1.1}$$

with coefficients $a_{j\alpha}(r,y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, \mathrm{Diff}^{\mu-(j+|\alpha|)}(X))$. The weight factor $r^{-\mu}$ is rather essential for our formalism with operator–valued symbols. Up to that factor any such operator can be generated as a polynomial in vector fields that are in local coordinates $x \in \Sigma$ for open $\Sigma \subseteq \mathbb{R}^n$ of the form

$$v_0(r,x,y)r\frac{\partial}{\partial r} + \sum_{k=1}^{n} v_k(r,x,y)\frac{\partial}{\partial x_k} + \sum_{k=n+1}^{n+q} v_k(r,x,y)r\frac{\partial}{\partial y_k}$$

with coefficients $v_k(r,x,y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega)$, $k = 0,\ldots,n+q$.

Given a family of Riemannian metrics $g_X(r,y)$ on X, C^∞ dependent (and non–degenerate) on $(r,y) \in \overline{\mathbb{R}}_+ \times \Omega$, we can define

$$dr^2 + r^2 g_X(r,y) + dy^2$$

which is a Riemannian metric on the wedge $X^\wedge \times \Omega$. The associated Laplace–Beltrami operator is of the form (3.1.1) for $\mu = 2$ (including the weight factor). Other examples of edge–degenerate operators (with weight factors) are obtained from differential operators of order μ in $\mathbb{R}^{n+1}_{\tilde{x}} \times \Omega$ with C^∞ coefficients by inserting polar coordinates $\mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}_+ \times S^n$ with respect to the $\tilde{x}$–variables, cf. Section 2.1.4 above. Of course, edge–degenerate operators over $\mathbb{R}_+ \times S^n \times \Omega$ are much more general than those induced by polar coordinates in operators with smooth coefficients in $\mathbb{R}^{n+1} \times \Omega$.

Let us look at a particularly simple example of the latter observation. If

$$\tilde{A} = \sum_{k+|\beta|\leq\mu} \tilde{a}_{k\beta}(r,y) D_r^k D_y^\beta \tag{3.1.2}$$

is a differential operator in the half–space $\overline{\mathbb{R}}_+ \times \Omega$ with coefficients $\tilde{a}_{k\beta}(r,y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega)$, then $\tilde{A}$ can equivalently be written in the form

$$A = r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(r,y) \left(-r\frac{\partial}{\partial r}\right)^j (rD_y)^\alpha \tag{3.1.3}$$

with $a_{j\alpha}(r,y) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega)$. Also in this case not every such A follows from a corresponding $\tilde{A}$. The program to establish a calculus of pseudo–differential boundary value problems which covers the differential operators (3.1.2) gives rise to an operator algebra with the transmission property, cf. Boutet de Monvel [4], Rempel and Schulze [42]. If we wish to include operators (3.1.3) in general we have to drop the transmission condition and obtain a calculus without the transmission property in the sense of the book [61] of the author. A similar problem is meaningful on manifolds with edges in general, where the calculus covers all edge–degenerate operators of the form (3.1.1). This theory is developed by the author in [66], cf. also [11], [58], Seiler [84], and joint papers with Gil and Seiler [20], and Dorschfeldt and Grieme [9].

Here we only discuss a few elements and certain new aspects of the theory.

Assume for convenience that the coefficients in the operator (3.1.1) are independent of r for $r > R$ for a constant $R > 0$. Let us set

$$a(y,\eta) = r^{-\mu} \sum_{j+|\alpha|\le\mu} a_{j\alpha}(r,y) \left(-r\frac{\partial}{\partial r}\right)^j (r\eta)^\alpha. \tag{3.1.4}$$

Proposition 3.1.1 *For every fixed $y \in \Omega$, $\eta \in \mathbb{R}^n$ the operator $a(y,\eta) : C_0^\infty(X^\wedge) \to C_0^\infty(X^\wedge)$ extends to a continuous map*

$$a(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \tag{3.1.5}$$

for arbitrary $s, \gamma \in \mathbb{R}$. More precisely, we have

$$a(y,\eta) \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)) \tag{3.1.6}$$

for all $s, \gamma \in \mathbb{R}$. If the coefficients $a_{j\alpha}$ are independent of r then

$$a(y,\eta) \in S^\mu_{\mathrm{cl}}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)), \tag{3.1.7}$$

and we have $\sigma^\mu_\wedge(a)(y,\eta) = a(y,\eta)$.

Proof. It suffices to consider the summands

$$p_{j\alpha}(y,\eta) = r^{-\mu} a_{j\alpha}(r,y) \left(-r\frac{\partial}{\partial r}\right)^j (r\eta)^\alpha$$

separately, $j + |\alpha| \leq \mu$. In order to show that (3.1.5) is continuous we write

$$p_{j\alpha}(y,\eta) = \omega(r) p_{j\alpha}(y,\eta)\omega_0(r) + (1-\omega(r))p_{j\alpha}(y,\eta)(1-\omega_1(r))$$

for cut–off functions $\omega, \omega_0, \omega_1$ which satisfy the conditions

$$\omega\omega_0 = \omega, \qquad \omega\omega_1 = \omega_1. \tag{3.1.8}$$

It is then obvious that

$$\omega p_{j\alpha}(y,\eta)\omega_0 : \mathcal{H}^{s,\gamma}(X^\wedge) \to \mathcal{H}^{s-\mu,\gamma-\mu}(X^\wedge)$$

is continuous.

For the second summand we choose an open covering $\{U_1, \dots, U_N\}$ of X by coordinate neighbourhoods, a subordinate partition of unity $\{\varphi_1, \dots, \varphi_N\}$ and a system of functions $\{\psi_1, \dots, \psi_N\}$, $\psi_j \in C_0^\infty(U_j)$, with $\varphi_j\psi_j = \varphi_j$ for all j. It follows that

$$(1-\omega)p_{j\alpha}(1-\omega_1) = \sum_{k=1}^N \varphi_k (1-\omega)p_{j\alpha}(1-\omega_1)\psi_k.$$

Without loss of generality we may assume that U_k is diffeomorphic to an open set $\tilde{U}_k$ of the unit sphere S^n. Let $\chi_k : U_k \to \tilde{U}_k$ such a diffeomorphism and define $\chi_k^\wedge : \mathbb{R}_+ \times U_k \to V_k = \{\tilde{x} \in \mathbb{R}^{n+1} \setminus \{0\} : \ \tilde{x}/|\tilde{x}| \in \tilde{U}_k\}$ by $\chi_k^\wedge(r,x) = r\chi_k(x)$. Then it suffices to show that the operator push–forward of $\varphi_k(1-\omega)p_{j\alpha}(1-\omega_1)\psi_k$ under $\chi_k^\wedge$ induces continuous operators $H^s(\mathbb{R}^{n+1}) \to H^{s-\mu}(\mathbb{R}^{n+1})$. This reduces the proof to the case that X itself is the unit sphere, embedded in $\mathbb{R}^{n+1}$. In other words, after an obvious change of notation, it suffices to show the continuity of

$$(1-\omega)p_{j\alpha}(y,\eta)(1-\omega_1) : H^s(\mathbb{R}^{n+1}) \to H^{s-\mu}(\mathbb{R}^{n+1}).$$

Here $r = |\tilde{x}|$ and $r\frac{\partial}{\partial r} = \sum_{l=1}^{n+1} \tilde{x}_l \frac{\partial}{\partial \tilde{x}_l}$, cf. relation (2.1.25). Thus

$$\begin{aligned}(1-\omega)&p_{j\alpha}(y,\eta)(1-\omega_1) = \\ &(1-\omega(|\tilde{x}|))|\tilde{x}|^{-\mu} a_{j\alpha}(|\tilde{x}|,y) \left(-\sum_{l=1}^{n+1} \tilde{x}_l \frac{\partial}{\partial \tilde{x}_l}\right)^j |\tilde{x}|^{|\alpha|}\eta^\alpha (1-\omega_1(|\tilde{x}|)).\end{aligned} \tag{3.1.9}$$

By assumption $a_{j\alpha}$ is independent of r for large r. Thus we can choose the cut–off functions ω, ω_1 in such a way that $a_{j\alpha}(|\tilde{x}|, y) = a_{j\alpha}(\infty, y)$ on $\mathrm{supp}(1-\omega_1)$. The coefficient $a_{j\alpha}(\infty, y)$ is a differential operator of order $\mu-(j+|\alpha|)$ in the angular variables $\sigma = (\sigma_1, \dots, \sigma_n)$ on S^n. Since we localised the operators in a coordinate neighbourhood $\tilde{U}_k$ of S^n, we can express them in derivatives $\frac{\partial}{\partial\sigma_m}$, $m = 1, \dots, n$. They have the form

$$\frac{\partial}{\partial\sigma_m} = \sum_{l=1}^{n+1} \tau_{ml}(\tilde{x}) \frac{\partial}{\partial\tilde{x}_l}$$

in the conical set V_k in $\mathbb{R}^{n+1} \setminus \{0\}$, with coefficients $\tau_{ml}(\tilde{x}) \in C^\infty(V_k)$, $\tau_{ml}(\lambda\tilde{x}) = \tau_{ml}(\tilde{x})$ for all $\lambda \in \mathbb{R}_+$.

It follows that (3.1.9) is a sum of expressions of the form

$$(1-\omega(|\tilde{x}|))|\tilde{x}|^{-\mu+|\alpha|+j}\beta_\iota(\tilde{x}) D_{\tilde{x}}^\iota \eta^\alpha (1-\omega_1(|\tilde{x}|)), \tag{3.1.10}$$

$\iota \in \mathbb{N}^{k+1}$, $|\iota| = j$, $\beta_\iota(\tilde{x}) \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$, $\beta_\iota(\lambda\tilde{x}) = \beta_\iota(\tilde{x})$ for all $\lambda \in \mathbb{R}_+$. We have $|\iota| \le \mu$, so the $\tilde{x}$–derivatives act as continuous operators $H^s(\mathbb{R}^{n+1}) \to H^{s-\mu}(\mathbb{R}^{n+1})$, and the coefficients are multiplication operators who are continuous in Sobolev spaces because of the homogeneity of β_ι and $-\mu + |\alpha| + j \le 0$. Summing up we proved the continuity of (3.1.5) for all $s, \gamma \in \mathbb{R}$. It is trivial that the operators are C^∞ dependent on $y \in \Omega$, $\eta \in \mathbb{R}^q$. For the symbol property we first look at the case of r–independent coefficients. In that case we have $a(y, \lambda\eta) = \lambda^\mu \kappa_\lambda a(y,\eta)\kappa_\lambda^{-1}$ for all $\lambda \in \mathbb{R}_+$, $(y, \eta) \in \Omega \times (\mathbb{R}^q \setminus \{0\})$ which shows relation (3.1.7) and the last assertion. For (3.1.6) we can apply a tensor product argument in the sense of Theorem 1.1.7, cf. also [58] or [61]. □

Remark 3.1.2 *Let $a(y,\eta)$ be an operator function of the form* (3.1.4)*. Then for $b(y,\eta) = r^\delta a(y,\eta)$, $\delta \in \mathbb{R}$, we have*

$$b(y,\eta) \in S^{\mu-\delta}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma;\varrho}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu+\delta;\varrho+\delta}(X^\wedge))$$

for all $s, \gamma, \varrho \in \mathbb{R}$. If the coefficients $a_{j\alpha}$ are independent of r then

$$b(y,\eta) \in S_{\mathrm{cl}}^{\mu-\delta}(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma;\varrho}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu+\delta;\varrho+\delta}(X^\wedge))$$

for all $s, \gamma, \varrho \in \mathbb{R}$, and $\sigma_\wedge^{\mu-\delta}(b)(y,\eta) = b(y,\eta)$. This can be proved in an analogous manner as Proposition 3.1.1.

3.1.3 Weighted wedge Sobolev spaces

Next we want to formulate a scale of weighted Sobolev spaces on wedges (and then on manifolds with edge singularities) in which the edge–degenerate operators are continuous and where we may expect a transparent description of elliptic regularity (including asymptotics near the edges).

An idea (for the stretched wedge $\mathbb{R}_+ \times X \times \mathbb{R}^q$) may be to define a space $\mathcal{H}^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ for $s \in \mathbb{N}$, $\gamma \in \mathbb{R}$ as the set of all $u(r,x,y) \in r^{\gamma-\frac{n}{2}}L^2(\mathbb{R}_+ \times X \times \mathbb{R}^q)$ $(n = \dim X)$ such that

$$(r\partial_r)^j D_x^l (rD_y)^\beta u \in r^{\gamma-\frac{n}{2}}L^2(\mathbb{R}_+ \times X \times \mathbb{R}^q) \tag{3.1.11}$$

for all $j+l+|\beta| \leq s$. Here D_x^l runs over all differential operators of order l on X. Then a definition for all $s \in \mathbb{R}$ may be obtained by duality and interpolation. It follows that $\mathcal{H}^{s,\gamma+\delta}(X^\wedge \times \mathbb{R}^q) = r^\delta \mathcal{H}^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ for all $s, \gamma, \delta \in \mathbb{R}$. If $\omega(r)$ is a cut–off function we have $\omega\mathcal{H}^{s',\gamma'}(X^\wedge \times \mathbb{R}^q) \subseteq \omega\mathcal{H}^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ for $s' \geq s$, $\gamma' \geq \gamma$.

The spaces $\mathcal{H}^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ are widely used in the literature. We intend to employ another scale, namely

$$\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) = \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)),$$

$s, \gamma \in \mathbb{R}$, cf. Definition 1.1.2 above, with $\{\kappa_\lambda\}_{\lambda \in \mathbb{R}_+}$ acting on $\mathcal{K}^{s,\gamma}(X^\wedge)$ by $(\kappa_\lambda u)(r,x) = \lambda^{\frac{n+1}{2}} u(\lambda r, x)$, $\lambda \in \mathbb{R}_+$, cf. Remark 2.1.7. The motivation for our scale is that pseudo–differential operators on a manifold with edges are locally of the form of pseudo–differential operators along the edge with operator–valued symbols, with values in the cone algebra over $X^\wedge$ (and adapted to twisted homogeneity). The symbols are understood in the sense of Definition 1.1.12. The advantage of that approach is that classical (pseudo–differential) boundary value problems are automatically a subclass of edge problems, including a corresponding natural interpretation of the boundary conditions as particular edge conditions, as it is described in the books [61], [66], cf. also joint papers with Schrohe [47], [48].

Although it can be proved that

$$H^s_{\mathrm{comp}}(X^\wedge \times \mathbb{R}^q) \subset \mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) \subset H^s_{\mathrm{loc}}(X^\wedge \times \mathbb{R}^q)$$

holds (and a similar relation for $\mathcal{H}^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ which is trivial) the spaces $\mathcal{H}^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ and $\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ do not coincide, except for certain special s, γ.

Proposition 3.1.3 *We have*

$$\mathcal{W}^{s,s}(X^\wedge \times \mathbb{R}^q) \subseteq \mathcal{H}^{s,s}(X^\wedge \times \mathbb{R}^q) \tag{3.1.12}$$

for all $s \in \mathbb{N}$. Moreover, if $\omega(r)$ is an arbitrary cut–off function (i.e., $\omega \in C_0^\infty(\overline{\mathbb{R}}_+)$, $\omega \equiv 1$ near $r = 0$) we have

$$\omega\mathcal{W}^{s,s}(X^\wedge \times \mathbb{R}^q) = \omega\mathcal{H}^{s,s}(X^\wedge \times \mathbb{R}^q) \tag{3.1.13}$$

for all $s \in \mathbb{N}$.

Proof. The case $s = 0$ is clear; then

$$\mathcal{H}^{0,0}(X^\wedge \times \mathbb{R}^q) = \mathcal{W}^{0,0}(X^\wedge \times \mathbb{R}^q) = r^{-\frac{n}{2}}L^2(X^\wedge \times \mathbb{R}^q).$$

For arbitrary $s \in \mathbb{N}$ we can form the set of all operator–valued symbols

$$a(\eta) = r^{-s}(r\partial_r)^j D_x^\lambda (r\eta)^\beta$$

for arbitrary $j + |\lambda| + |\beta| \leq s$. Here we use the notation from Remark 2.1.32 above. By Proposition 3.1.1 they belong to $S^s_{\mathrm{cl}}(\mathbb{R}^q; \mathcal{K}^{s,s}(X^\wedge), \mathcal{K}^{0,0}(X^\wedge))$. Applying Theorem 1.2.2 we get

$$\mathrm{Op}(a) = r^{-s}(r\partial_r)^j D_x^{|\lambda|}(rD_y)^\beta : \mathcal{W}^{s,s}(X^\wedge \times \mathbb{R}^q) \to \mathcal{W}^{0,0}(X^\wedge \times \mathbb{R}^q) \tag{3.1.14}$$

for all $j + l + |\beta| \leq s$. Applying the continuous embedding

$$\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) \hookrightarrow \mathcal{W}^{s',\gamma'}(X^\wedge \times \mathbb{R}^q)$$

for arbitrary $s \geq s'$, $\gamma \geq \gamma'$ (which are easy to obtain) to the case $s = \gamma$ and $s' = \gamma' = 0$ we get $\mathcal{W}^{s,s}(X^\wedge \times \mathbb{R}^q) \subseteq r^{-\frac{n}{2}}L^2(X^\wedge \times \mathbb{R}^q)$. Comparing (3.1.14) with (3.1.11) yields (3.1.12).

To show (3.1.13) it suffices to consider elements $u(r,x,y) \in \mathcal{H}^{s,s}(X^\wedge \times \mathbb{R}^q)$ with $u(r,x,y) = 0$ for $r > R$ for some $R > 0$. By assumption we have (up to equivalence of norms)

$$\|u\|^2_{\mathcal{H}^{s,s}(X^\wedge \times \mathbb{R}^q)} = \sum_{j+|\lambda|+|\beta| \leq s} \iiint |r^{-s+\frac{n}{2}}(r\partial_r)^j D_x^\lambda (rD_y)^\beta u(r,x,y)|^2 \, dr dx dy.$$

The right hand side can be replaced by

$$\sum \iiint |r^{-s+\frac{n}{2}}(r\partial_r)^j D_x^\lambda (r\eta)^\beta u(r,x,\hat{\eta})|^2 \, dr dx d\eta$$

where $u(r,x,\hat\eta)=(F_{y\to\eta}u)(r,x,\eta)$ (the sum is always taken over all $j+|\lambda|+|\beta|\le s$). This is equivalent to

$$\sum\iiint |r^{-s+\frac{n}{2}}(r\partial_r)^j D_x^\lambda (r\langle\eta\rangle)^{|\beta|}u(r,x,\hat\eta)|^2\,drdxd\eta. \tag{3.1.15}$$

Substituting $\tilde r=r\langle\eta\rangle$, $d\tilde r=\langle\eta\rangle dr$, (3.1.15) gives us

$$\begin{aligned}&\sum\iiint \langle\eta\rangle^{2s-n-1}|\tilde r^{-s+\frac{n}{2}}(\tilde r\partial_{\tilde r})^j D_x^\lambda \tilde r^{|\beta|}u(\tilde r\langle\eta\rangle^{-1},x,\hat\eta)|^2\,d\tilde r dxd\eta\\ &=\int\langle\eta\rangle^{2s}\Big\{\iint\sum|r^{-s}(r\partial_r)^jD_x^\lambda r^{|\beta|}\langle\eta\rangle^{-\frac{n+1}{2}}u(r\langle\eta\rangle^{-1},x,\hat\eta)|^2r^n\,drdx\Big\}\,d\eta.\end{aligned}$$

On the other hand we have

$$\begin{aligned}\|u\|^2_{\mathcal{W}^{s,s}(X^\wedge\times\mathbb{R}^q)}=\int\langle\eta\rangle^{2s}\Big\{\iint\sum_{j+|\lambda|\le s}|k^{-s}(r)[r]^{-j-|\lambda|}(r\partial_r)^jD_x^\lambda\langle\eta\rangle^{-\frac{n+1}{2}}\\ u(r\langle\eta\rangle^{-1},x,\hat\eta)|^2r^n\,drdx\Big\}\,d\eta,\end{aligned}$$

cf. Remark 2.1.32. This shows a formal similarity to the norm of $\mathcal{H}^{s,s}(X^\wedge\times\mathbb{R}^q)$ (though both norms are not equivalent). Let us reformulate the $\mathcal{W}^{s,s}(X^\wedge\times\mathbb{R}^q)$–norm by substituting $\tilde r=r\langle\eta\rangle^{-1}$, $d\tilde r=\langle\eta\rangle^{-1}dr$. Then

$$\begin{aligned}\|u\|^2_{\mathcal{W}^{s,s}(X^\wedge\times\mathbb{R}^q)}=\iiint\langle\eta\rangle^{2s}\sum_{j+|\lambda|\le s}|k^{-s}(r\langle\eta\rangle)[r\langle\eta\rangle]^{-j-|\lambda|}(r\partial_r)^jD_x^\lambda\\ u(r,x,\hat\eta)|^2r^n\,drdxd\eta.\end{aligned}$$

By assumption u vanishes for $r>R$ for a given $R>0$ which is taken so large that

$$[r\langle\eta\rangle]=|r\langle\eta\rangle|\qquad\text{for } r\langle\eta\rangle\ge R. \tag{3.1.16}$$

We show that for such u we have

$$\|u\|_{\mathcal{W}^{s,s}(X^\wedge\times\mathbb{R}^q)}\le c\|u\|_{\mathcal{H}^{s,s}(X^\wedge\times\mathbb{R}^q)}$$

for a constant $c>0$.

Let us set $v_{j\lambda}(r,x,\hat{\eta}) = (r\partial_r)^j D_x^\lambda u(r,x,\hat{\eta})$. Then

$$\|u\|^2_{\mathcal{W}^{s,s}(X^\wedge\times\mathbb{R}^q)} = \iiint \sum_{j+|\lambda|\le s} |\langle\eta\rangle^s k^{-s}(r\langle\eta\rangle)[r\langle\eta\rangle]^{-j-|\lambda|} v_{j\lambda}|^2 r^n \, dr dx d\eta,$$

$$\|u\|^2_{\mathcal{H}^{s,s}(X^\wedge\times\mathbb{R}^q)} = \iiint \sum_{j+|\lambda|+|\beta|\le s} |r^{-s}(r\langle\eta\rangle)^{|\beta|} v_{j\lambda}|^2 r^n \, dr dx d\eta.$$

We shall show that the expressions

$$W = \sum_{j+|\lambda|\le s} |\langle\eta\rangle^s k^{-s}(r\langle\eta\rangle)[r\langle\eta\rangle]^{-j-|\lambda|} v_{j\lambda}|^2$$

and

$$H = \sum_{j+|\lambda|+|\beta|\le s} |r^{-s}(r\langle\eta\rangle)^{|\beta|} v_{j\lambda}|^2$$

fulfill the estimates $W \le cH$ for a constant $c > 0$ which proves the assertion. Without loss of generality we may choose the function $k^{-s}(\cdot)$ in such a way that

$$k^{-s}(r\langle\eta\rangle) = (r\langle\eta\rangle)^{-s} \qquad \text{for } 0 < r\langle\eta\rangle \le R. \tag{3.1.17}$$

In our estimate we distinguish the cases

(i) $0 < r\langle\eta\rangle \le R$,

(ii) $r\langle\eta\rangle \ge R$.

For (i) we have $[r\langle\eta\rangle] \le$ constant and thus, using (3.1.17),

$$\begin{aligned} W &\le c \sum_{j+|\lambda|\le s} |\langle\eta\rangle^s (r\langle\eta\rangle)^{-s} v_{j\lambda}|^2 \\ &\le c\Big\{ \sum_{j+|\lambda|\le s} |r^{-s} v_{j\lambda}|^2 + \sum_{\substack{j+|\lambda|+|\beta|\le s \\ |\beta|\ne 0}} |r^{-s}(r\langle\eta\rangle)^{|\beta|} v_{j\lambda}|^2 \Big\}. \end{aligned}$$

Here and in the sequel c denotes different positive constants. For (ii) we use relation (3.1.16) and get

$$\begin{aligned} W &= \sum_{j+|\lambda|\le s} |\langle\eta\rangle^s k^{-s}(r\langle\eta\rangle) r^{-j-|\lambda|}\langle\eta\rangle^{-j-|\lambda|} v_{j\lambda}|^2 \\ &\le c \sum_{j+|\lambda|\le s} |\langle\eta\rangle^{s-(j+|\lambda|)} r^{-j-|\lambda|} v_{j\lambda}|^2 \\ &\le c \sum_{j+|\lambda|+|\beta|\le s} |r^{-s}(r\langle\eta\rangle)^{|\beta|} v_{j\lambda}|^2. \end{aligned} \qquad \square$$

Remark 3.1.4 *Consider the operator* $m_\delta : u(r,x) \to r^\delta u(r,x)$ *for* $\delta \in \mathbb{R}$, $u \in \mathcal{K}^{s,\gamma}(X^\wedge)$. *It is then clear that* $m_\delta : \mathcal{K}^{s,\gamma}(X^\wedge) \to r^\delta \mathcal{K}^{s,\gamma}(X^\wedge)$ *is an isomorphism with the inverse* $m_{-\delta}$, *cf. also formula* (2.1.12). *We have*

$$m_\delta = \lambda^{-\delta}\kappa_\lambda m_\delta \kappa_\lambda^{-1} \qquad \textit{for all } \lambda \in \mathbb{R}_+.$$

Therefore we get $m_\delta \in S^{-\delta}_{\mathrm{cl}}(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge), r^\delta\mathcal{K}^{s,\gamma}(X^\wedge))$ *for arbitrary* $s,\gamma \in \mathbb{R}$. *It follows that*

$$\mathrm{Op}(m_\delta) : \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)) \to \mathcal{W}^{s+\delta}(\mathbb{R}^q, r^\delta\mathcal{K}^{s,\gamma}(X^\wedge))$$

is an isomorphism for all $s,\gamma \in \mathbb{R}$. *In other words, we have*

$$r^\delta \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)) = \mathcal{W}^{s+\delta}(\mathbb{R}^q, r^\delta\mathcal{K}^{s,\gamma}(X^\wedge)),$$

cf. Remark 1.2.4.

Remark 3.1.5 *Consider the operator of multiplication* $k_\beta(\eta) : u(r,x) \to k^\beta(r[\eta])u(r,x)$, $\beta \in \mathbb{R}$, *where* $k^\beta(r)$ *is a fixed function in* $C^\infty(\mathbb{R}_+)$ *with* $k^\beta(r) > 0$ *for all* $r \in \mathbb{R}_+$ *and* $k^\beta(r) = r^\beta$ *for* $0 < r < c_0$, $k^\beta(r) = 1$ *for* $c_1 < r < \infty$, *with constants* $c_0 < c_1$. *Then*

$$k_\beta(\lambda\eta) = \kappa_\lambda k_\beta(\eta)\kappa_\lambda^{-1} \qquad \textit{for all } \lambda > c \textit{ for a constant } c > 0 \textit{ and } |\eta| \ge 1. \tag{3.1.18}$$

The multiplication by k^β *induces isomorphisms* $k^\beta : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s,\gamma+\beta}(X^\wedge)$. *From* (3.1.18) *it follows that* $k_\beta(\eta) \in S^0_{\mathrm{cl}}(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s,\gamma+\beta}(X^\wedge))$, *and*

$$\mathrm{Op}(k_\beta) : \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge)) \to \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma+\beta}(X^\wedge))$$

is an isomorphism for all $s,\gamma,\beta \in \mathbb{R}$.

3.1.4 Discrete edge asymptotics

The wedge Sobolev spaces $\mathcal{W}^{s,\gamma}(X^\wedge \times \mathbb{R}^q) = \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$ contain an interesting class of subspaces with asymptotics. The choice of asymptotics (discrete or continuous ones) depends on how general we want to formulate the theory of edge pseudo–differential problems. We restrict ourselves here to the constant discrete case; continuous edge asymptotics are studied in [58], [66]. Let E be a Fréchet space, $E = \varprojlim_{j\in\mathbb{N}} E^j$, with Hilbert spaces E^j and continuous embeddings $E^{j+1} \hookrightarrow E^j$ for all j, where a strongly continuous group of isomorphisms $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ acts on E^0 which induces (by restriction) strongly continuous groups of isomorphisms on E^j for all j. Then we can form the spaces $\mathcal{W}^s(\mathbb{R}^q, E^j)$, where $\mathcal{W}^s(\mathbb{R}^q, E^{j+1}) \hookrightarrow \mathcal{W}^s(\mathbb{R}^q, E^j)$ are continuous embeddings for all j, and we then set $\mathcal{W}^s(\mathbb{R}^q, E) = \varprojlim_{j\in\mathbb{N}} \mathcal{W}^s(\mathbb{R}^q, E^j)$; this is a Fréchet space in the projective limit topology. In particular, if $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$ consists of identity maps for all $\lambda \in \mathbb{R}_+$, we get the spaces $H^s(\mathbb{R}^q, E) = \varprojlim_{j\in\mathbb{N}} H^s(\mathbb{R}^q, E^j)$, $s \in \mathbb{R}$.

Applying this construction to $E = \mathcal{K}^{s,\gamma}_\Theta(X^\wedge)$ for a weight interval $\Theta = (\vartheta, 0]$, $-\infty \le \vartheta < 0$, we obtain the space

$$\mathcal{W}^{s,\gamma}_\Theta(X^\wedge \times \mathbb{R}^q) = \mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_\Theta(X^\wedge))$$

of edge flatness Θ (with respect to the weight γ). More generally we can insert $E = \mathcal{K}^{s,\gamma}_P(X^\wedge)$ for a discrete asymptotic type $P \in \mathrm{As}(X, (\gamma, \Theta))$, where we use that $\mathcal{K}^{s,\gamma}_P(X^\wedge)$ can be written as a projective limit $\varprojlim_{j\in\mathbb{N}} E^j$ in the above sense.

For finite $\Theta = (\vartheta, 0]$ and $P = \{(p_l, m_l, M_l)\}_{l=0,\dots,N}$ we can set

$$E^j = \mathcal{K}^{s,\gamma-\vartheta-(j+1)^{-1}}(X^\wedge) + \mathcal{E}_{P_j}(X^\wedge)$$

for $P_j = \{(p, m, M) \in P : \; \frac{n+1}{2} - (\gamma - \vartheta - (j+1)^{-1}) < \mathrm{Re}\, p\}$, cf. formula (2.1.20) (the involved cut–off function is fixed). For $\Theta = (-\infty, 0]$ we can set

$$E^j = \mathcal{K}^{s,\gamma+j}(X^\wedge) + \mathcal{E}_{P_j}(X^\wedge)$$

for $P_j = \{(p, m, M) \in P : \; \frac{n+1}{2} - (\gamma + j) < \mathrm{Re}\, p\}$. We then define

$$\mathcal{W}^{s,\gamma}_P(X^\wedge \times \mathbb{R}^q) = \varprojlim_{j\in\mathbb{N}} \mathcal{W}^s(\mathbb{R}^q, E^j).$$

Similarly we have the spaces

$$H^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}_P(X^\wedge)) = \varprojlim_{j\in\mathbb{N}} H^s(\mathbb{R}^q, E^j).$$

It follows that

$$\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}_P^{s,\gamma}(X^\wedge)) = T^{-1} H^s(\mathbb{R}^q, \mathcal{K}_P^{s,\gamma}(X^\wedge)),$$

cf. formula (1.1.6).

Notice that the spaces $\mathcal{E}_P(X^\wedge)$ for $P \in \mathrm{As}(X, (\gamma, \Theta))$, Θ finite, are not preserved under the κ_λ–action. Nevertheless, we can form $H^s(\mathbb{R}^q, \mathcal{E}_P(X^\wedge))$ where

$$\mathcal{W}_P^{s,\gamma}(X^\wedge \times \mathbb{R}^q) = \mathcal{W}_\Theta^{s,\gamma}(X^\wedge \times \mathbb{R}^q) + T^{-1} H^s(\mathbb{R}^q, \mathcal{E}_P(X^\wedge)) \tag{3.1.19}$$

(clearly $T^{-1}H^s(\mathbb{R}^q, \mathcal{E}_P(X^\wedge))$ is a subspace of $\mathcal{W}_P^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ and (3.1.19) holds as a non–direct sum). The restriction (3.1.19) gives us the singular functions of the discrete edge asymptotics who are (by definition) the elements of the second summand in (3.1.19). Each $u \in \mathcal{W}_P^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ can be written

$$u = u_{\mathrm{flat}} + u_{\mathrm{sing}} \tag{3.1.20}$$

for $u_{\mathrm{flat}} \in \mathcal{W}_\Theta^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$ and

$$(F_{y\to\eta} u_{\mathrm{sing}})(r, x, \eta) = \sum_{j=0}^{N} \sum_{k=0}^{m_j} [\eta]^{-\frac{n+1}{2}} \omega(r[\eta]) c_{jk}(x, \eta)(r[\eta])^{-p_j} \log^k (r[\eta])$$

for certain coefficients $c_{jk}(x, \eta) \in M_j \otimes F_{y\to\eta} H^s(\mathbb{R}^q)$, $0 \le k \le m_j$. The decomposition (3.1.20) is very similar to (1.2.13), (1.2.14) for standard Sobolev spaces.

Another relation to the considerations in Section 1.2.3 is that

$$s_{jk}(\eta) : c \to [\eta]^{-\frac{n+1}{2}} \omega(r[\eta]) c_{jk}(x)(r[\eta])^{-p_j} \log^k (r[\eta]) c, \tag{3.1.21}$$

$c \in \mathbb{C}$, $c_{jk} \in M_j$, $0 \le k \le m_j$, is an operator–valued symbol of the class

$$S_{\mathrm{cl}}^0(\mathbb{R}^q; \mathbb{C}, \mathcal{S}_P^\gamma(X^\wedge))$$

and that $T^{-1}H^s(\mathbb{R}^q, \mathcal{E}_P(X^\wedge))$ is the linear span of all elements $\mathrm{Op}(s_{jk})v_{jk}$ for $v_{jk} \in H^s(\mathbb{R}^q)$ and arbitrary s_{jk} of the form (3.1.21).

Remark 3.1.6 *The space*

$$\mathcal{W}_P^{\infty,\gamma}(X^\wedge \times \mathbb{R}^q) = \bigcap_{s\in\mathbb{R}} \mathcal{W}_P^{s,\gamma}(X^\wedge \times \mathbb{R}^q)$$

is independent of the choice of the group action and equals $H^\infty(\mathbb{R}^q, \mathcal{K}_P^{\infty,\gamma}(X^\wedge))$*. The singular functions (for a finite weight interval) belong to* $H^s(\mathbb{R}^q, \mathcal{E}_P(X^\wedge))$*; they have the form of finite linear combinations of functions*

$$\omega(r)c_{jk}(x,y)r^{-p_j}\log^k r, \quad c_{jk} \in M_j \otimes H^\infty(\mathbb{R}^q),\ 0 \le k \le m_j,\ j = 0, \dots, N.$$

Remark 3.1.7 *We have*

$$\mathcal{W}_\Theta^{\infty,\gamma}(X^\wedge \times \mathbb{R}^q) = H^\infty(\mathbb{R}^q, \mathcal{K}_\Theta^{\infty,\gamma}(X^\wedge)).$$

It follows that for $\Theta = (\vartheta, 0]$*,* $-\infty < \vartheta < 0$

$$\mathcal{W}_\Theta^{\infty,\gamma}(X^\wedge \times \mathbb{R}^q) = \bigcap_{\varepsilon > 0} k^{-\vartheta-\varepsilon}(r)\mathcal{W}^{\infty,\gamma}(X^\wedge \times \mathbb{R}^q)$$

with the function $k^\beta(r)$*,* $\beta \in \mathbb{R}$*, defined in Remark* 3.1.5*. For infinite* Θ *we have*

$$\mathcal{W}_\Theta^{\infty,\gamma}(X^\wedge \times \mathbb{R}^q) = \bigcap_{\vartheta < 0} \mathcal{W}_{(\vartheta,0]}^{\infty,\gamma}(X^\wedge \times \mathbb{R}^q) = \mathcal{W}^{\infty,\infty}(X^\wedge \times \mathbb{R}^q)$$

3.2 The edge algebra

3.2.1 Edge–degenerate symbols

Symbols of edge–degenerate differential operators (3.1.1) in local coordinates with respect to a chart $U \to \Sigma$ on X are of the form $r^{-\mu}p(r,x,y,\tau,\xi,\eta)$ for $p(r,x,y,\tau,\xi,\eta) \in \tilde{S}^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q})$, cf. Definition 2.2.4. We call the symbols of the class $r^{-\mu}\tilde{S}^\mu_{(\mathrm{cl})}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q})$ (classical) edge–degenerate. If $p(r,x,y,\varrho,\xi,\eta)$ is written

$$p(r,x,y,\varrho,\xi,\eta) = \tilde{p}(r,x,y,\varrho_{\mathrm{f}},\xi,\eta_{\mathrm{w}})|_{\varrho_{\mathrm{f}}=r\varrho, \eta_{\mathrm{w}}=r\eta},$$

we call the homogeneous principal part of order μ in $(\varrho_{\mathrm{f}},\xi,\eta_{\mathrm{w}}) \ne 0$ its compressed principal symbol. This is an invariant C^∞ function on $T^*_{\mathrm{w}}(\overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q) \setminus 0$, where $T^*_{\mathrm{w}}(\overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q)$ is the compressed cotangent bundle of the stretched wedge $\overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q$. It may be defined by a special cocycle of transition maps of $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^q$, that of the usual cotangent bundle of $T^*(\mathbb{R}_+ \times X \times \mathbb{R}^q)$ (near $r = 0$) conjugated with

$$\begin{pmatrix} r^{-1} & 0 & 0 \\ 0 & \mathrm{id}_{\mathbb{R}^n} & 0 \\ 0 & 0 & r^{-1}\,\mathrm{id}_{\mathbb{R}^q} \end{pmatrix},$$

cf. similarly Section 2.2.4, formula (2.2.22).

Thus $T^*_{\mathrm{w}}(\overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q)$ is isomorphic to $T^*(\overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q)$. Denote $\sigma^{\mu}_{\psi,\mathrm{w}}(A)$ the compressed principal symbol of the differential operator (3.1.1) (given by the invariant system of local representatives $\tilde{p}_{(\mu)}(r, x, y, \varrho_{\mathrm{f}}, \xi, \eta_{\mathrm{w}})$). Then, if A is elliptic in the sense $\sigma^{\mu}_{\psi,\mathrm{w}}(A)(r, x, y, \varrho_{\mathrm{f}}, \xi, \eta_{\mathrm{w}}) \neq 0$ for all $(r, x, y, \varrho_{\mathrm{f}}, \xi, \eta_{\mathrm{w}}) \in T^*_{\mathrm{w}}(\overline{\mathbb{R}}_+ \times X \times \mathbb{R}^q) \setminus 0$, we can form the system of inverses with respect to the Leibniz multiplication

$$r^{-\mu}p(r, x, y, \varrho, \xi, \eta) \#_{r,x,y} \{r^{\mu}a(r, x, y, \varrho, \xi, \eta)\} \\ \sim \sum_{\alpha} \frac{1}{\alpha!}(r^{-\mu}D^{\alpha}_{\varrho,\xi,\eta}p)(\partial^{\alpha}_{r,x,y}r^{\mu}a)(r, x, y, \varrho, \xi, \eta).$$

The corresponding asymptotic sum for the symbol a can be carried out in a way that $a(r, x, y, \varrho, \xi, \eta) \in \tilde{S}^{-\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma \times \Omega \times \mathbb{R}^{1+n+q})$. It can easily be proved that in this sense the Leibniz inversion (i.e., inversion with respect to the Leibniz multiplication) respects the $(r^{-\mu}\tilde{S}^{\mu}_{(\mathrm{cl})})$–symbol classes also in general (not only for polynomial symbols). In other words, starting with the operator A and a system of charts $U_j \to \Sigma_j$ on X, $j = 1, \dots, N$, we get by Leibniz inversion a system of local symbols $r^{\mu}\tilde{S}^{-\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma_j \times \Omega \times \mathbb{R}^{1+n+q})$, $j = 1, \dots, N$. In order to construct a parametrix of A, the problem is now to associate with this system an operator that acts in the (local with respect to $y \in \Omega$) weighted wedge Sobolev spaces $\mathcal{W}^s_{\mathrm{loc}(y)}(\Omega, \mathcal{K}^{s,\gamma}(X^{\wedge}))$.

More precisely, the problem of an operator convention in the pseudo–differential edge calculus is as follows. Given a system

$$r^{-\mu}p_j(r, x, y, \varrho, \xi, \eta) \in r^{-\mu}\tilde{S}^{\mu}_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma_j \times \Omega \times \mathbb{R}^{1+n+q}) \tag{3.2.1}$$

(to an atlas $U_j \to \Sigma_j$, $j = 1, \dots, N$, on X), find an associated continuous operator

$$P : \mathcal{W}^s_{\mathrm{comp}(y)}(\Omega, \mathcal{K}^{s,\gamma}(X^{\wedge})) \to \mathcal{W}^{s-\mu}_{\mathrm{loc}(y)}(\Omega, \mathcal{K}^{s-\mu,\gamma-\mu}(X^{\wedge})) \tag{3.2.2}$$

(for all $s \in \mathbb{R}$) whose local symbols on $\mathbb{R}_+ \times \Sigma_j \times \Omega$ are the given ones (modulo smoothing symbols).

In the case that P is an edge–degenerate differential operator (with coefficients $a_{j\alpha}(r, y)$ being independent of r of large r, cf. (3.1.1)), then by Proposition 3.1.1 and Theorem 1.2.1 we get continuous operators (3.2.2). However, for pseudo–differential operators this is not clear. But the idea is again to first form operator–valued symbols $a(y, \eta) \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$ and then to pass to $\mathrm{Op}(a)$.

3.2.2 Operator–valued edge symbols

First observe that the operator family (3.1.4) satisfies $a(y, \eta) \in C^\mu(X^\wedge, \boldsymbol{g})$ for every fixed $y \in \Omega$, $\eta \in \mathbb{R}^q$, and $\boldsymbol{g} = (\gamma, \gamma - \mu, (-(k+1), 0])$ with arbitrary $\gamma \in \mathbb{R}$, $k \in \mathbb{N}$. The proof is contained in the arguments for Proposition 3.1.1, as far as it concerns the exit aspect for large r. Setting $\tilde{h}(r, y, z, \eta) = \sum_{j+|\alpha|\le\mu} a_{j\alpha}(r, y) z^j \eta^\alpha$, we get

$$a(y, \eta) = r^{-\mu} \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)(y, \eta)$$

for $h(r, y, z, \eta) = \tilde{h}(r, y, z, r\eta)$. Choosing cut–off functions $\omega(r)$, $\omega_0(r)$, $\omega_1(r)$ with $\omega\omega_0 = \omega$, $\omega\omega_1 = \omega_1$, we can write

$$a(y, \eta) = \omega(r[\eta]) r^{-\mu} \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)(y, \eta)\omega_0(r[\eta]) \\ + (1 - \omega(r[\eta]))P(y, \eta)(1 - \omega_1(r[\eta])) \quad (3.2.3)$$

where

$$P(y, \eta) = r^{-\mu} \operatorname{op}_M^{\gamma-\frac{n}{2}}(h)(y, \eta) \quad (3.2.4)$$

$\in L^\mu_{\mathrm{e}}(X^\wedge)$ for every fixed $(y, \eta) \in \Omega \times \mathbb{R}^q$. The combination $r[\eta]$ in the cut–off factors is motivated by the interpretation of our families as operator–valued symbols with κ_λ–homogeneity. In fact, we have

$$\kappa_\lambda \omega(r[\eta]) \kappa_\lambda^{-1} = \omega(r[\eta])$$

for all $\lambda \geq 1$ and all η with $|\eta| \geq$ const for a constant > 0. It is not always necessary to control the exit aspect up to $r = \infty$ (except for the case of homogeneous operator–valued symbols). Therefore, in the operator convention we content ourselves with expressions of the form $\sigma(r)a(y, \eta)\sigma_0(r)$

for cut–off functions $\sigma(r)$, $\sigma_0(r)$ (global operators on manifolds with edges are defined by localisations; so they contain, in particular, expressions localised near edges, that is, for small r).

Starting point of the operator convention is a system of local symbols (3.2.1) to an open convering $\{U_1, \dots, U_N\}$ of X by coordinate neighbourhoods and charts $\kappa_j : U_j \to \Sigma_j$, $j = 1, \dots, N$. Let $\{\varphi_1, \dots, \varphi_N\}$ be a subordinate partition of unity on X and $\{\psi_1, \dots, \psi_N\}$ be a system of functions $\psi_j \in C_0^\infty(U_j)$ with $\varphi_j\psi_j = \varphi_j$ for all j. By Theorem 2.2.6 to each p_j we can choose an $h_j(r, x, y, t, \xi, \eta) \in \tilde{S}^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma_j \times \Omega \times \mathbb{C} \times \mathbb{R}^{n+q})$ such that

$$\mathrm{op}_{t,x}(p_j)(y, \eta) = \mathrm{op}_M^{\frac{1}{2}}(\mathrm{op}_x(h_j))(y, \eta) \bmod C^\infty(\Omega, L^{-\infty}(\mathbb{R}_+ \times \Sigma_j; \mathbb{R}^q)). \tag{3.2.5}$$

Setting

$$h(r, y, z, \eta) = \sum_{j=1}^N \varphi_j\{(\kappa_j^{-1})_* \,\mathrm{op}_x(h_j)(y, \eta)\}\psi_j \tag{3.2.6}$$

and $\tilde{h}(r, y, z, \eta) = h(r, y, z, r^{-1}\eta)$, cf. also formula (2.2.5), we get

$$\tilde{h}(r, y, z, \eta) \in C^\infty(\overline{\mathbb{R}}_+ \times \Omega, M_O^\mu(X; \mathbb{R}^q)).$$

Let us form an operator function (3.2.3) with $P(y, \eta)$ being defined by (3.2.4). Since we compose from both sides with cut–off factors, it is not necessary to ensure the properties of the exit pseudo–differential set–up.

Proposition 3.2.1 *Let*

$$r^{-\mu}p_j(r, x, y, \varrho, \xi, \eta) \in r^{-\mu}\tilde{S}^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma_j \times \Omega \times \mathbb{R}^{1+n+q}),$$

$j = 1, \dots, N$, *be a system of local edge–degenerate symbols, choose* $h_j(r, x, y, z, \xi, \eta) \in \tilde{S}^\mu_{\mathrm{cl}}(\overline{\mathbb{R}}_+ \times \Sigma_j \times \mathbb{C} \times \mathbb{R}^{n+q})$ *according to* (3.2.5), *form* $h(r, y, z, \eta)$ *by* (3.2.6) *and* $P(y, \eta)$ *by* (3.2.4). *Then, for every pair of cut–off functions* $\sigma(r)$, $\sigma_0(r)$ *we have* $\sigma(r)a(y, \eta)\sigma_0(r) \in S^\mu(\Omega \times \mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$ *for all* $s \in \mathbb{R}$, *for*

$$a(y, \eta) = r^{-\mu}\omega(r[\eta])\,\mathrm{op}_M^{\gamma-\frac{n}{2}}(h)(y, \eta)\omega_0(r[\eta]) \\ + (1 - \omega(r[\eta]))P(y, \eta)(1 - \omega_1(r[\eta])).$$

Moreover,

$$r^{-\mu}\sigma(r)\,\mathrm{Op}\Big\{\sum_{j=1}^{N}\varphi_j[(\kappa_j^{-1})_*\,\mathrm{op}_{t,x}(p_j)(y,\eta)]\psi_j\Big\}\sigma_0(r)=\mathrm{Op}(\sigma a\sigma_0)$$

$\mathrm{mod}\,L^{-\infty}(X^\wedge\times\mathbb{R}^q)$ *with* Op *denoting the pseudo–differential action with respect to the* y*–variables.*

Proposition 3.2.1 is proved in [66]. Assume $\sigma\sigma_0=\sigma$ and choose another cut–off function $\sigma\sigma_1=\sigma_1$. As a corollary we see that there is an operator convention

$$\{r^{-\mu}p_j\}_{j=1,\dots,N}\to\mathrm{Op}(\sigma a\sigma_0)+R=:A$$

for $R=(1-\sigma(r))r^{-\mu}\,\mathrm{Op}\{\sum_{j=1}^{N}\varphi_j[(\kappa_j^{-1})_*\,\mathrm{op}_{t,x}(p_j)(y,\eta)]\psi_j\}(1-\sigma_1(r))$, where P (given by (3.2.4)) equals A mod $L^{-\infty}(X^\wedge\times\Omega)$, and

$$\mathrm{Op}(\sigma a\sigma_0):\mathcal{W}^s_{\mathrm{comp}(y)}(\Omega,\mathcal{K}^{s,\gamma}(X^\wedge))\to\mathcal{W}^{s-\mu}_{\mathrm{loc}(y)}(\Omega,\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge))$$

is continuous for all $s\in\mathbb{R}$.

Although the above $\sigma(r)a(y,\eta)\sigma_0(r)$ is not a classical operator–valued symbol we can single out a kind of principal edge symbol. We set

$$\begin{aligned}\sigma^\mu_\wedge(\sigma a\sigma_0)(y,\eta)=r^{-\mu}\omega(r|\eta|)\,\mathrm{op}_M^{\gamma-\frac{n}{2}}(h_0)(y,\eta)\omega_0(r|\eta|)\\+(1-\omega(r|\eta|))P_0(y,\eta)(1-\omega_1(r|\eta|)),\end{aligned}\tag{3.2.7}$$

for $y\in\Omega$, $\eta\neq0$, where $h_0(r,y,z,\eta)=\tilde h(0,y,z,r\eta)$, while $P_0(y,\eta)$ is defined by the same rule as $P(y,\eta)$, but with symbols $r^{-\mu}\tilde p_j(0,x,y,r\varrho,\xi,r\eta)$, $j=1,\dots,N$, instead of (3.2.1), $p_j(r,x,y,\varrho,\xi,\eta)=\tilde p_j(r,x,y,r\varrho,\xi,r\eta)$. Notice that (3.2.7) is an operator family in $C^\mu(X^\wedge,\boldsymbol{g})$ for $\boldsymbol{g}=(\gamma,\gamma-\mu,\Theta)$, $\Theta=(-\infty,0]$, parametrised by $(y,\eta)\in T^*\Omega\setminus0$, with the homogeneity

$$\sigma^\mu_\wedge(\sigma a\sigma_0)(y,\lambda\eta)=\lambda^\mu\kappa_\lambda\sigma^\mu_\wedge(\sigma a\sigma_0)(y,\eta)\kappa_\lambda^{-1}$$

for all $\lambda>0$.

Moreover, introduce the compressed interior symbol $\sigma^\mu_{\psi,\mathrm{w}}(\sigma a\sigma_0)$ as the (invariantly defined on $T^*_{\mathrm{w}}((\overline{\mathbb{R}}_+\times X\times\Omega)\setminus0)$) function whose local representatives are

$$\{\sigma(r)\sigma_0(r)\tilde p_{j,(\mu)}(r,x,y,\varrho_{\mathrm{f}},\xi,\eta_{\mathrm{w}})\}_{j=1,\dots,N}.$$

Our next objective is to complete the operator families $\sigma(r)a(y,\eta)\sigma_0(r)$ from Proposition 3.2.1 by the so–called smoothing Mellin + Green edge symbols, who are (y,η)–dependent families of operators in $C^{\mu}_{M+G}(X^\wedge,\boldsymbol{g})$ for $\boldsymbol{g}=(\gamma,\gamma-\mu,\Theta)$. We saw in the cone algebra that the smoothing Mellin + Green operators contain asymptotics as they are produced in parametrix constructions and in the elliptic regularity. In the (y,η)–dependent case we have to expect y–dependent asymptotic types (η–dependent ones are not generated). In order to single out a particularly simple case we formulate the operators for the case of constant (in y) discrete asymptotics.

Definition 3.2.2 *An operator family*

$$g(y,\eta)\in\bigcap_{s\in\mathbb{R}}C^\infty(\Omega\times\mathbb{R}^q,\mathcal{L}(\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{K}^{\infty,\delta}(X^\wedge)))$$

for $\gamma,\delta\in\mathbb{R}$ is called a Green symbol of order μ, with constant discrete asymptotics (of types $P\in\mathrm{As}(X,(\delta,\Theta))$, $Q\in\mathrm{As}(X,(-\gamma,\Theta))$ for a given weight interval $\Theta=(\vartheta,0]$, $-\infty\le\vartheta<0$), if

$$g(y,\eta)\in\bigcap_{s\in\mathbb{R}}S^{\mu}_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{S}^{\delta}_{P}(X^\wedge)),$$
$$g^*(y,\eta)\in\bigcap_{s\in\mathbb{R}}S^{\mu}_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;\mathcal{K}^{s,-\delta}(X^\wedge),\mathcal{S}^{-\gamma}_{Q}(X^\wedge)).$$

Here g^* is the pointwise adjoint with respect to the $\mathcal{K}^{0,0}(X^\wedge)$–scalar product that induces non–degenerate sesquilinear pairings $(\cdot,\cdot):\mathcal{K}^{s,\beta}(X^\wedge)\times\mathcal{K}^{-s,-\beta}(X^\wedge)\to\mathbb{C}$ for all $s,\beta\in\mathbb{R}$. Denote by $R^{\mu}_{G}(\Omega\times\mathbb{R}^q,\boldsymbol{g})$ for $\boldsymbol{g}=(\gamma,\delta,\Theta)$ the set of all Green edge symbols of order μ, with the weight data $\boldsymbol{g}$.

Definition 3.2.3 *Let $\gamma\in\mathbb{R}$ and $\Theta=(-k,0]$, $k\in\mathbb{N}\setminus\{0\}$. Then an operator family of the form*

$$m(y,\eta)=\omega(r[\eta])r^{-\mu}\sum_{j=0}^{k-1}\sum_{|\alpha|\le j}r^j\,\mathrm{op}_M^{\gamma_{j\alpha}-\frac{n}{2}}(f_{j\alpha})(y)\eta^\alpha\omega_0(r[\eta])\tag{3.2.8}$$

with $f_{j\alpha}(y,z)\in C^\infty(\Omega,M^{-\infty}_{R_{j\alpha}}(X))$ is called a smoothing Mellin edge symbol. Here $R_{j\alpha}\in\mathbf{As}(X)$, $\pi_{\mathbb{C}}R_{j\alpha}\cap\Gamma_{\frac{1}{2}-\gamma_{j\alpha}}=\emptyset$ for $\gamma_{j\alpha}\in\mathbb{R}$, $\gamma_{j\alpha}\le\gamma\le\gamma_{j\alpha}+j$, $j=0,\dots,k-1$, and ω, ω_0 are cut–off functions.

Remark 3.2.4 *Every $m(y,\eta)$ of the form (3.2.8) is an element of $\bigcap_{s\in\mathbb{R}} S^{\mu}_{\mathrm{cl}}(\Omega \times\mathbb{R}^q;\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{K}^{\infty,\gamma-\mu}(X^\wedge))$. In addition, to each $P\in \mathrm{As}(X,(\gamma,\Theta))$ there exists a $Q\in\mathrm{As}(X,(\gamma-\mu,\Theta))$ such that*

$$m(y,\eta)\in\bigcap_{s\in\mathbb{R}} S^{\mu}_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;\mathcal{K}^{s,\gamma}_P(X^\wedge),\mathcal{S}^{\gamma-\mu}_Q(X^\wedge)).$$

Denote by $R^{\mu}_{M+G}(\Omega\times\mathbb{R}^q,\boldsymbol{g})$ for $\boldsymbol{g}=(\gamma,\gamma-\mu,\Theta)$, $\Theta=(-k,0]$, the space of all operator families $m(y,\eta)+g(y,\eta)$ for arbitrary smoothing Mellin edge symbols $m(y,\eta)$ and Green edge symbols $g(y,\eta)$ to the weight data $\boldsymbol{g}$. Set

$$\sigma^{\mu-j}_M(m+g)(y,z,\eta)=\sum_{|\alpha|\le j} f_{j\alpha}(y,z)\eta^{\alpha}, \tag{3.2.9}$$

$j=0,\ldots,k-1$. The functions (3.2.9) are called the conormal symbols of $(m+g)(y,\eta)$ (they only depend on $m(y,\eta)$).

Proposition 3.2.5 *Let $(m+g)(y,\eta)$ and $(\tilde m+\tilde g)(y,\eta)$ be elements in $R^{\mu}_{M+G}(\Omega\times\mathbb{R}^q,\boldsymbol{g})$ and assume that $\sigma^{\mu-j}_M(m+g)=\sigma^{\mu-j}_M(\tilde m+\tilde g)$ for all $j=0,\ldots,k-1$. Then we have $(m+g)-(\tilde m+\tilde g)\in R^{\mu}_G(\Omega\times\mathbb{R}^q,\boldsymbol{g})$ (in the definition of m and $\tilde m$ the cut–off functions may be different as well as the weights in the Mellin operators).*

For a proof, see, for instance, [66], Lemmas 3.3.15, 3.3.16.

By virtue of

$$R^{\mu}_{M+G}(\Omega\times\mathbb{R}^q,\boldsymbol{g})\subset\bigcap_{s\in\mathbb{R}} S^{\mu}_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;\mathcal{K}^{s,\gamma}(X^\wedge),\mathcal{K}^{\infty,\gamma-\mu}(X^\wedge))$$

the elements in R^{μ}_{M+G} also have homogeneous components

$$\sigma^{\mu-j}_{\wedge}(m+g)(y,\eta)=\sigma^{\mu-j}_{\wedge}(m)(y,\eta)+\sigma^{\mu-j}_{\wedge}(g)(y,\eta), \tag{3.2.10}$$

$j\in\mathbb{N}$, cf. the notation after Definition 1.1.14, where

$$\sigma^{\mu-l}_{\wedge}(m)(y,\eta)=\omega(r|\eta|)r^{-\mu+l}\sum_{\substack{0\le|\alpha|\le j\le k-1\\ j-|\alpha|=l}}\mathrm{op}_M^{\gamma_{j\alpha}-\frac{n}{2}}(f_{j\alpha})(y)(r\eta)^{\alpha}\omega_0(r|\eta|).$$

In order to formulate additional edge conditions on the level of operator–valued edge symbols we extend Definition 3.2.2 to a class of block matrix–valued symbols $\mathcal{R}^{\mu}_{G}(\Omega\times\mathbb{R}^q,\boldsymbol{g};N_-,N_+)\ni g(y,\eta)$ for $N_-,N_+\in\mathbb{N}$ by requiring

$$g(y,\eta)\in\bigcap_{s\in\mathbb{R}}S^{\mu}_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;\mathcal{K}^{s,\gamma}(X^\wedge)\oplus\mathbb{C}^{N_-},\mathcal{S}^{\delta}_{P}(X^\wedge)\oplus\mathbb{C}^{N_+}),$$
$$g^*(y,\eta)\in\bigcap_{s\in\mathbb{R}}S^{\mu}_{\mathrm{cl}}(\Omega\times\mathbb{R}^q;\mathcal{K}^{s,-\delta}(X^\wedge)\oplus\mathbb{C}^{N_+},\mathcal{S}^{-\gamma}_{Q}(X^\wedge)\oplus\mathbb{C}^{N_-})$$

(according to Remark 1.1.13 the symbol spaces are defined with respect to groups of isomorphisms that act as identity operators on finite–dimensional spaces, e.g., $\kappa_\lambda(u(r,x)\oplus v)=(\lambda^{\frac{n+1}{2}})u(\lambda r,x)\oplus v$ for $u\oplus v\in\mathcal{K}^{s,\gamma}(X^\wedge)\oplus\mathbb{C}^{N_-}$), cf. also Remark 1.1.13.

Definition 3.2.6 *Let $\mu\in\mathbb{Z}$, $\gamma\in\mathbb{R}$ and $\boldsymbol{g}=(\gamma,\gamma-\mu,\Theta)$, $\Theta=(-k,0]$, $k\in\mathbb{N}\setminus\{0\}$. Then an operator family of the form*

$$\mathfrak{a}(y,\eta)=\begin{pmatrix}\sigma a(y,\eta)\sigma_0+m(y,\eta) & 0\\ 0 & 0\end{pmatrix}+g(y,\eta):\begin{matrix}\mathcal{K}^{s,\gamma}(X^\wedge)\\ \oplus\\ \mathbb{C}^{N_-}\end{matrix}\to\begin{matrix}\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)\\ \oplus\\ \mathbb{C}^{N_+}\end{matrix},\tag{3.2.11}$$

$s\in\mathbb{R}$, for arbitrary $\sigma a(y,\eta)\sigma_0$ from Proposition 3.2.1 *and $g\in\mathcal{R}^{\mu}_{G}(\Omega\times\mathbb{R}^q,\boldsymbol{g};N_-,N_+)$ is called an edge symbol of the class $\mathcal{R}^{\mu}(\Omega\times\mathbb{R}^q,\boldsymbol{g};N_-,N_+)$. Especially for $N_-=N_+=0$ we set*

$$R^{\mu}(\Omega\times\mathbb{R}^q,\boldsymbol{g})=\mathcal{R}^{\mu}(\Omega\times\mathbb{R}^q,\boldsymbol{g};0,0).$$

We omit Ω in the notation when we mean the subclasses of y–independent elements.

In particular, if we write $g=(g_{ij})_{i,j=1,2}$, we call $g_{21}(y,\eta)$ the trace symbol, $g_{12}(y,\eta)$ the potential symbol of (3.2.11), while $g_{22}(y,\eta)$ is an $N_+\times N_-$–matrix of elements in $S^{\mu}_{\mathrm{cl}}(\Omega\times\mathbb{R}^q)$.

We write $\sigma^{\mu}_{\psi,\mathrm{w}}(\mathfrak{a})=\sigma^{\mu}_{\psi,\mathrm{w}}(\sigma a\sigma_0)$, cf. Section 3.2.1, and set

$$\sigma^{\mu}_{\wedge}(\mathfrak{a})(y,\eta)=\begin{pmatrix}\sigma^{\mu}_{\wedge}(\sigma a\sigma_0+m)(y,\eta) & 0\\ 0 & 0\end{pmatrix}+\sigma^{\mu}_{\wedge}(g)(y,\eta),$$

cf. (3.2.7), (3.2.10), which are operator families

$$\sigma_\wedge^\mu(\mathfrak{a})(y,\eta): \begin{array}{c}\mathcal{K}^{s,\gamma}(X^\wedge)\\ \oplus \\ \mathbb{C}^{N_-}\end{array} \to \begin{array}{c}\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge)\\ \oplus \\ \mathbb{C}^{N_+}\end{array} \tag{3.2.12}$$

with

$$\sigma_\wedge^\mu(\mathfrak{a})(y,\lambda\eta) = \lambda^\mu \begin{pmatrix}\kappa_\lambda & 0\\ 0 & 1\end{pmatrix} \sigma_\wedge^\mu(\mathfrak{a})(y,\eta) \begin{pmatrix}\kappa_\lambda & 0\\ 0 & 1\end{pmatrix}^{-1}$$

for all $\lambda \in \mathbb{R}_+$, $(y,\eta) \in T^*\Omega \setminus 0$. Let us set

$$\sigma(\mathfrak{a}) = (\sigma_{\psi,\mathrm{w}}^\mu(\mathfrak{a}), \sigma_\wedge^\mu(\mathfrak{a})).$$

Theorem 3.2.7 *Let* $\mathfrak{a}(y,\eta) \in \mathcal{R}^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g}; N_-, N_0)$, $\mathfrak{b}(y,\eta) \in \mathcal{R}^\nu(\Omega \times \mathbb{R}^q, \boldsymbol{h}; N_0, N_+)$ *for* $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$, $\boldsymbol{h} = (\gamma-\mu, \gamma-\mu-\nu, \Theta)$, $\gamma, \mu, \nu \in \mathbb{R}$, *and* $N_-, N_0, N_+ \in \mathbb{N}$. *Then we have* $(\mathfrak{b}\mathfrak{a})(y,\eta) \in \mathcal{R}^{\mu+\nu}(\Omega \times \mathbb{R}^q, \mathbf{c}; N_-, N_+)$ *for* $\mathbf{c} = (\gamma, \gamma-\mu-\nu, \Theta)$, *and* $\sigma(\mathfrak{a}\mathfrak{b}) = \sigma(\mathfrak{a})\sigma(\mathfrak{b})$ *(with componentwise composition).*

The proof may be found in [66], Theorem 3.3.45. An alternative technique is developed in a joint paper with Gil and Seiler [20].

Definition 3.2.8 *An element* $\mathfrak{a}(y,\eta) \in \mathcal{R}^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g}; N_-, N_+)$ *for* $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$, $\gamma, \mu \in \mathbb{R}$, *is called elliptic if*

(i) $\sigma_{\psi,\mathrm{w}}^\mu(\mathfrak{a}) \neq 0$ *on* $T_\mathrm{w}^*(\overline{\mathbb{R}}_+ \times X \times \Omega)|_{[0,\varepsilon)\times X\times\Omega} \setminus 0$ *for some* $\varepsilon > 0$,

(ii) *the principal edge symbol* (3.2.12) *is an isomorphism for all* $(y,\eta) \in T^*\Omega \setminus 0$.

The restriction to a neighbourhood of the edge Ω in condition (i) comes from the fact that because of the cut–off factors σ, σ_0 in Definition 3.2.6 we avoid to describe the interior symbols for large r up to ∞. This could be done, of course, by a corresponding modification of Definition 3.2.6. However, since the global calculus works with partitions of unity who cut off the operators near edges as well as outside neighbourhoods of the edges, it is easier to formulate ellipticity far from the edges separately, cf. Definition 3.2.14, below.

The asymptotic types of Mellin symbols and those in the image spaces of Green symbols are motivated by the fact that the inversion of families

$\sigma^\mu_\wedge(\mathfrak{a})(y,\eta)$, who originally have no specific asymptotic types (e.g. when the Mellin and Green symbols are zero) automatically produces asymptotic information via the nature of the conormal symbols

$$\sigma^\mu_M \sigma^\mu_\wedge(\mathfrak{a})(y,z). \tag{3.2.13}$$

In fact, the inversion of $\sigma^\mu_\wedge(\mathfrak{a})(y,\eta)$ within the (y,η)–dependent cone algebra on $X^\wedge$ (say for the case of upper left corners) required the inversion of (3.2.13) which is a special aspect in the parametrix construction for cone operators, cf. Theorem 2.2.15 and Theorem 2.2.29. Asymptotics are due to the distribution of poles of $(\sigma^\mu_M \sigma^\mu_\wedge(\mathfrak{a})(y,z))^{-1}$. The involved asymptotic types may depend on $y \in \Omega$. This is not admitted in the Mellin and Green objects in Definition 3.2.6. However the definition can be extended in a way that dependence of discrete asymptotic types on y is included, cf. Schulze [62], [64] and the joint paper with Witt [81]. For the present exposition this would be too voluminous. Therefore, in the ellipticity we impose an extra condition, namely that

$$\{\text{the asymptotic types in } (\sigma^\mu_M \sigma^\mu_\wedge(\mathfrak{a})(y,z))^{-1} \text{ are independent of } y\}. \tag{3.2.14}$$

If we drop this condition we reach the context of [62], [64], [81], or we have the possibility to extend the notion of asymptotics by admitting continuous ones, cf. [53], [58]. Another (simpler) idea is to shrink the length of the weight interval in the inverted objects which has the effect that some asymptotic information is ignored. In all these cases we obtain the Fredholm aspects of elliptic edge operators, cf. Theorem 3.2.15 below. The following result is proved, for instance, in [66], Theorem 3.5.9.

Theorem 3.2.9 *Let $\mathfrak{a}(y,\eta) \in \mathcal{R}^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g}; N_-, N_+)$ for $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$ be elliptic, and assume condition* (3.2.14). *Then there exists a $\mathfrak{p}(y,\eta) \in \mathcal{R}^{-\mu}(\Omega \times \mathbb{R}^q, \boldsymbol{g}^{-1}; N_+, N_-)$ for $\boldsymbol{g}^{-1} = (\gamma-\mu, \gamma, \Theta)$ such that $\sigma^\mu_{\psi,\mathrm{w}}(\mathfrak{a})^{-1} = \sigma^{-\mu}_{\psi,\mathrm{w}}(\mathfrak{p})$ over $[0,\varepsilon) \times X \times \Omega$ for an $\varepsilon > 0$ and $\sigma^\mu_\wedge(\mathfrak{a})^{-1} = \sigma^{-\mu}_\wedge(\mathfrak{p}))$ for all $(y,\eta) \in T^*\Omega \setminus 0$.*

Let us now turn to some examples and remarks. Condition (ii) in Definition 3.2.8 is not easy to verify for a concrete operator. On the other hand it can be proved that for every choice of $N_-, N_+ \in \mathbb{N}$ and $\mu, \gamma \in \mathbb{R}$ there exists an element $\mathfrak{a}(y,\eta) \in \mathcal{R}^\mu(\Omega \times \mathbb{R}^q, \boldsymbol{g}; N_-, N_+)$ which is elliptic in the sense of Definition 3.2.8, cf. Schulze and Tarkhanov [77] or Dorschfeldt [8]. The structure of these examples is that every interior edge–degenerate elliptic symbol is admitted and that then by a suitable choice of smoothing Mellin and Green objects the required isomorphism of Definition 3.2.8 (ii) can be organised. It is sometimes not desirable to pass to such pseudo–differential corrections (even though they are smoothing for $r > 0$ and not compact in general), if we start with an edge–degenerate elliptic differential symbol. Therefore our following example (elaborated in joint discussion with N. Tarkhanov) is of the kind of a "pure" differential operator.

Set $\mathcal{K}^{s,s}(X^\wedge, \mathbb{C}^L) = \mathcal{K}^{s,s}(X^\wedge) \otimes \mathbb{C}^L$ for $L \in \mathbb{N}$ and consider an η–dependent family of differential operators of first order

$$a(\eta) : \mathcal{K}^{1,1}(X^\wedge) \to \mathcal{K}^{0,0}(X^\wedge, \mathbb{C}^L), \tag{3.2.15}$$

where $a(\eta)$ is given as a column vector of the form

$$a(\eta) = r^{-1} \begin{pmatrix} r\partial_r \\ c \\ (v_j)_{j=1,\dots,N} \\ r\eta \\ re \end{pmatrix} \tag{3.2.16}$$

with constants $c, e \in \mathbb{R}$, the vector fields $v_1, \dots, v_N$ from Remark 2.1.32, and $r\eta$ interpreted as a column vector with components $r\eta_l$, $l = 1, \dots, q$. Thus $L = 3 + N + q$.

From Proposition 2.1.31 and formula (2.1.31) in Section 2.1.4 we see that (3.2.15) is an injective operator as soon as $c \neq 0$, for all $\eta \in \mathbb{R}^q$. Let us now pass to the η–wise adjoint $a^*(\eta)$ with respect to the scalar product of the space $\mathcal{K}^{0,0}(X^\wedge, \mathbb{C}^L)$,

$$a^*(\eta) : \mathcal{K}^{0,0}(X^\wedge, \mathbb{C}^L) \to \mathcal{K}^{-1,-1}(X^\wedge).$$

Then we consider the operator function

$$a^*(\eta)a(\eta) : \mathcal{K}^{1,1}(X^\wedge) \to \mathcal{K}^{-1,-1}(X^\wedge). \tag{3.2.17}$$

We have $a^*(\eta)a(\eta) \in C^2(X^\wedge, \boldsymbol{g})$ for $\boldsymbol{g} = (\gamma, \gamma - 2, \Theta)$, $\Theta = (-\infty, 0]$ for all $\eta \in \mathbb{R}^q$, $\gamma \in \mathbb{R}$. For each $\eta \in \mathbb{R}^q$ the operator is of Fuchs type with an iterior elliptic symbol. In other words, condition (i) in Definition 2.2.28 is fulfilled. For (ii) we observe that the conormal symbol $\sigma_M^\mu(a^*a)(z)$ is independent of

η (it may be calculated for $\eta = 0$). But $(a^*a)(z,c) : H^s(X) \to H^{s-2}(X)$ is parameter–dependent elliptic, i.e., in the class $L^2_{\mathrm{cl}}(X;\Gamma_\beta \times \mathbb{R})$ with the parameters $(z,c) \in \Gamma_\beta \times \mathbb{R}$ for arbitrary β. Thus, for every fixed finite interval $[\gamma_0,\gamma_1]$, $\gamma_0 \le \gamma_1$, and suitable choice of $c(\gamma_0,\gamma_1) > 0$ sufficiently large we get a family of isomorphisms for all $z \in \Gamma_\beta$, $\gamma_0 \le \beta \le \gamma_1$, cf. Theorem 1.1.41. To show that condition (iii) of Definition 2.2.28 is fulfilled we express any left symbol of $a^*(\eta)a(\eta)$ in the covariables $(\tilde{\xi},\eta) \in \mathbb{R}^{1+n+q}$ which can be done by the same kind of calculations as in the proof of Proposition 3.1.1. We then obtain that this symbol is parameter–dependent elliptic in $\mathbb{R}^{1+n+q}$ with parameter $e \in \mathbb{R}$. It follows that every left symbol of $a^*(\eta)a(\eta)$ is $\neq 0$ for all $(\tilde{\xi},\eta) \in \mathbb{R}^{1+n+q}$ when $e \neq 0$. Thus we fix $e \neq 0$ and see that $a^*(\eta)a(\eta)$ has a left symbol in $(\tilde{x},\tilde{\xi})$ that never vanishes for large $|\tilde{x}|^2 + |\tilde{\xi}|^2$ for all $\eta \in \mathbb{R}^q$. Thus $a^*(\eta)a(\eta)$ is elliptic in the sense of Definition 2.2.28 for all $\eta \in \mathbb{R}^q$. Theorem 2.2.29 shows that (3.2.17) is a Fredholm operator. Since it is injective by the above arguments, Lemma 1.2.15 shows that it is an isomorphism. It is obvious that $a^*(\eta)a(\eta)$ has the form of an edge symbol $\sigma^2_\wedge(\mathfrak{a})(\eta)$ of some elliptic $\mathfrak{a}(\eta) \in R^2(\mathbb{R}^q,\boldsymbol{g})$ for $\boldsymbol{g} = (1,-1,(-\infty,0])$.

The above discussion gives us more information on the possible weights. For an arbitrary interval $[\gamma_0,\gamma_1]$ we can choose the constant c so large that $\sigma^\mu_M(a^*a)(z) : H^s(X) \to H^{s-2}(X)$ is an isomorphism for all $z \in \Gamma_{\frac{n+1}{2}-\gamma}$, $\gamma_0 \le \gamma \le \gamma_1$. Thus, by Theorem 2.2.29, we get a Fredholm operator

$$a^*(\eta)a(\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-2,\gamma-2}(X^\wedge) \tag{3.2.18}$$

for all $s \in \mathbb{R}$, $\gamma_0 \le \gamma \le \gamma_1$ and all $\eta \in \mathbb{R}^q$. The operators $a^*(\eta)a(\eta)$ satisfy the assumptions of Remark 2.2.32 above. In particular, kernel and cokernel of (3.2.18) are independent of γ in the interval $\gamma_0 \le \gamma \le \gamma_1$. Since we proved that (3.2.18) is an isomorphism for $\gamma = 1$, we see that (3.2.18) is an isomorphism for $\gamma_0 \le \gamma \le \gamma_1$.

In a similar way we can argue for the case of higher order operators, i.e., when we replace (3.2.16) by a vector containing the derivatives and powers of r and $r\eta$ up to an order $k \in \mathbb{N}$. We then obtain the following result:

Theorem 3.2.10 *For every $k \in \mathbb{N}$ and $\gamma_0 \le \gamma_1$ there exists an operator family*

$$b^{2k}(\eta) = r^{-2k} \sum_{j+|\beta|\le 2k} g_{j\beta}(-r\partial_r)^j (r\eta)^\beta$$

with coefficients $g_{j\beta} \in \mathrm{Diff}^{\mu-(j+|\beta|)}(X)$ *which defines isomorphisms*

$$b^{2k}(\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \to \mathcal{K}^{s-2k,\gamma-2k}(X^\wedge)$$

for all $s \in \mathbb{R}$, $\gamma_0 \leq \gamma \leq \gamma_1$, *and all* $\eta \in \mathbb{R}^q$.

Remark 3.2.11 *The operator functions* $b^{2k}(\eta)$ *of Theorem* 3.2.10 *belong to* $S^{\mu}_{\mathrm{cl}}(\mathbb{R}^q; \mathcal{K}^{s,\gamma}(X^\wedge), \mathcal{K}^{s-2k,\gamma-2k}(X^\wedge))$ *for all* $s \in \mathbb{R}$, $\gamma_0 \leq \gamma \leq \gamma_1$, *and we have* $(b^{2k}(\eta))^{-1} \in \sigma_\wedge^{-2k}(R^{-2k}(\mathbb{R}^q; \boldsymbol{g}))$ *for* $\boldsymbol{g} = (\gamma - 2k, \gamma, \Theta)$, $\Theta = (-\infty, 0]$.

The first assertion is immediate consequence of the homogeneity

$$b^{2k}(\lambda\eta) = \lambda^{2k}\kappa_\lambda b^{2k}(\eta)\kappa_\lambda^{-1} \qquad \text{for all } \lambda \in \mathbb{R}_+$$

(together with the fact that the operators are C^∞ functions in η). The second one comes from the fact that bijective operator functions who occur as edge symbols can be inverted within this class, cf. Theorem 2.2.29.

3.2.3 Edge pseudo–differential operators

We now pass to pseudo–differential operators on a manifold W with edges Y. The precise definitions may be found in [66]. For a first understanding it suffices to say that both $Y \subset W$ and $W \setminus Y$ are C^∞ manifolds, and every $y \in Y$ has a neighbourhood V in W which is a wedge $\{\text{cone}\} \times \Omega$, where $\Omega \subseteq \mathbb{R}^q$ (for $q = \dim Y$) is open and $\{\text{cone}\} = (\overline{\mathbb{R}}_+ \times X)(\{0\} \times X)$ for a closed compact C^∞ manifold X. We always pass to stretched neighbourhoods $\mathbb{V} = \overline{\mathbb{R}}_+ \times X \times \Omega$ with local coordinates (r, x, y); the precise definitions contain the natural coordinate diffeomorphisms (roughly speaking $(r, x, y) \to (\tilde{r}, \tilde{x}, \tilde{y})$ is required to be C^∞ up to $r = 0$). It is convenient to speak about the stretched manifold $\mathbb{W}$ to W, where the neighbourhoods V are replaced by $\mathbb{V}$ and attached to the remaining part of the space. Especially for $W = \{(\overline{\mathbb{R}}_+ \times X)/(\{0\} \times X)\} \times \Omega$ we have $\mathbb{W} = \overline{\mathbb{R}}_+ \times X \times \Omega$. The stretched manifold $\mathbb{W}$ is a C^∞ manifold with C^∞ boundary $\partial\mathbb{W}$, and $\partial\mathbb{W}$ is a bundle over Y with fibre X.

Assume for simplicity that $\mathbb{W}$ is compact (which implies that also Y is compact). Then, using a corresponding invariance of the wedge Sobolev spaces $\mathcal{W}^s_{\mathrm{loc}(y)}(\Omega, \mathcal{K}^{s,\gamma}(X^\wedge))$ we can define the spaces $\mathcal{W}^{s,\gamma}(\mathbb{W}) \subset H^s_{\mathrm{loc}}(\mathrm{int}\,\mathbb{W})$ which localise to $\mathcal{W}^s_{\mathrm{loc}(y)}(\Omega, \mathcal{K}^{s,\gamma}(X^\wedge))$ over $\mathrm{int}\,\mathbb{V} = X^\wedge \times \Omega$ for each V. Let us fix a Riemannian metric on $\mathbb{W}$ which is near $\partial\mathbb{W}$ the product metric between chosen metrics on $\partial\mathbb{W}$ and a "normal interval" to $\partial\mathbb{W}$. Then, with respect to the reference scalar product of $\mathcal{W}^{0,0}(\mathbb{W}) = k^{-\frac{n}{2}}L^2(\mathbb{W})$ (with $n = \dim X$ and a strictly positive C^∞ function k on $\operatorname{int}\mathbb{W}$ with $k(\cdot) = \operatorname{dist}(\cdot, \partial\mathbb{W})$) we have non–degenerate sesquilinear pairings

$$\mathcal{W}^{s,\gamma}(\mathbb{W}) \times \mathcal{W}^{-s,-\gamma}(\mathbb{W}) \to \mathbb{C}$$

for all $s, \gamma \in \mathbb{R}$.

Similarly to Section 3.1.4 we can also consider subspaces $\mathcal{W}^{s,\gamma}_P(\mathbb{W}) \subset \mathcal{W}^{s,\gamma}(\mathbb{W})$ for (discrete) asymptotic types $P \in \operatorname{As}(X, (\gamma, \Theta))$, locally described by $W^s_{\mathrm{loc}(y)}(\Omega, \mathcal{K}^{s,\gamma}_P(X^\wedge))$ over $\operatorname{int}\mathbb{V}$, for all V. In such a definition we need some invariance discussion under coordinate changes that we omit here. Let us content ourselves with the case that the coordinate diffeomorphisms for $\mathbb{W}$ near $\partial\mathbb{W}$ only depend on the y–variables in a neighbourhood of the boundary. In this case the space $\mathcal{W}^{s,\gamma}_P(\mathbb{W})$ is well–defined (in the general case we simply have to replace P by $P_V \in \operatorname{As}(X, (\gamma, \Theta))$, and then the asymptotic type P in $\mathcal{W}^{s,\gamma}_P(\mathbb{W})$ is nothing else than the collection of all these local P_V).

Definition 3.2.12 *The space of edge problems $\mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; N_-, N_+)$ for $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, $\gamma, \mu \in \mathbb{R}$, $\Theta = (-k, 0]$, $k \in \mathbb{N} \setminus \{0\}$, $N_\pm \in \mathbb{N}$, is defined as the set of all operator block matrices*

$$\mathcal{A} = \begin{pmatrix} A_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} + \mathcal{C} : \begin{array}{c} C_0^\infty(\operatorname{int}\mathbb{W}) \\ \oplus \\ C^\infty(Y, \mathbb{C}^{N_-}) \end{array} \to \begin{array}{c} C^\infty(\operatorname{int}\mathbb{W}) \\ \oplus \\ C^\infty(Y, \mathbb{C}^{N_+}) \end{array} \tag{3.2.19}$$

with $A_{11} = A + M + G_{11}$, where $A \in L^\mu_{\mathrm{cl}}(\operatorname{int}\mathbb{W})$ is of the form $A = A_Y + A_{\mathrm{int}}$, $A_Y = \sigma A \sigma_0$, $A_{\mathrm{int}} = (1 - \sigma)A(1 - \sigma_1)$ for certain $\sigma, \sigma_0, \sigma_1 \in C^\infty(\mathbb{W})$ which are equal 1 near Y, vanish outside a neighbourhood of Y and fulfill $\sigma\sigma_0 = \sigma$, $\sigma\sigma_1 = \sigma_1$. The operator $A_Y + M$ is locally given by $\operatorname{Op}(\sigma a \sigma_0 + m)$ with operator–valued symbols $\sigma a(y, \eta)\sigma_0 + m(y, \eta)$ as they occur in (3.2.11), and $\mathcal{G} = (G_{ij})_{i,j=1,2}$ is locally of the form $\operatorname{Op}(g)$ for some $g(y, \eta) \in \mathcal{R}^\mu_G(\Omega \times \mathbb{R}^q, \boldsymbol{g}; N_-, N_+)$, over $\operatorname{int}\mathbb{V} = X^\wedge \times \Omega$ for every V. Finally $\mathcal{C}$ is a smoothing operator in the sense that

$$\mathcal{C} : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W}) \\ \oplus \\ H^s(Y, \mathbb{C}^{N_-}) \end{array} \to \begin{array}{c} \mathcal{W}^{\infty,\gamma-\mu}_P(\mathbb{W}) \\ \oplus \\ H^\infty(Y, \mathbb{C}^{N_+}) \end{array}, \qquad \mathcal{C}^* : \begin{array}{c} \mathcal{W}^{s,-\gamma+\mu}(\mathbb{W}) \\ \oplus \\ H^s(Y, \mathbb{C}^{N_+}) \end{array} \to \begin{array}{c} \mathcal{W}^{\infty,-\gamma}_Q(\mathbb{W}) \\ \oplus \\ H^\infty(Y, \mathbb{C}^{N_-}) \end{array}$$

are continuous for all s and certain asymptotic types P, Q. The space of all such smoothing operators is denoted by $\mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}; N_-, N_+)$.

Theorem 3.2.13 *An operator $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; N_-, N_+)$ for $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$ induces continuous operators*

$$\mathcal{A} : \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W}) \\ \oplus \\ H^s(Y, \mathbb{C}^{N_-}) \end{array} \to \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}) \\ \oplus \\ H^{s-\mu}(Y, \mathbb{C}^{N_+}) \end{array} \tag{3.2.20}$$

for all $s \in \mathbb{R}$. Moreover, for every $P \in \mathrm{As}(X, (\gamma, \Theta))$ there is a $Q \in \mathrm{As}(X, (\gamma - \mu, \Theta))$ such that (3.2.20) *has a restriction to a continuous operator*

$$\mathcal{A} : \begin{array}{c} \mathcal{W}_P^{s,\gamma}(\mathbb{W}) \\ \oplus \\ H^s(Y, \mathbb{C}^{N_-}) \end{array} \to \begin{array}{c} \mathcal{W}_Q^{s-\mu,\gamma-\mu}(\mathbb{W}) \\ \oplus \\ H^{s-\mu}(Y, \mathbb{C}^{N_+}) \end{array}$$

for each $s \in \mathbb{R}$.

The proof is essentially a consequence of Theorem 1.2.1 and of the fact that $\sigma a(y,\eta)\sigma_0 + m(y,\eta)$ as well as $g(y,\eta)$ are operator–valued symbols, both for $\mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{N_-}$ and $\mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \oplus \mathbb{C}^{N_+}$ as parameter spaces as well as for corresponding subspaces with asymptotics.

To each $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; N_-, N_+)$ we can form a pair of principal symbols

$$\sigma(\mathcal{A}) = (\sigma^\mu_{\psi,\mathrm{w}}(\mathcal{A}), \sigma^\mu_\wedge(\mathcal{A}))$$

where $\sigma^\mu_{\psi,\mathrm{w}}(\mathcal{A})$ is determined by the upper left corner A_{11} as the compressed homogeneous principal interior symbol which is a C^∞ function on $T^*_{\mathrm{w}}\mathbb{W} \setminus 0$, and $\sigma^\mu_\wedge(\mathcal{A})$ is the homogeneous principal edge symbol, a family of maps

$$\sigma^\mu_\wedge(\mathcal{A})(y,\eta) : \begin{array}{c} \mathcal{K}^{s,\gamma}(X^\wedge) \\ \oplus \\ \mathbb{C}^{N_-} \end{array} \to \begin{array}{c} \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \\ \oplus \\ \mathbb{C}^{N_+} \end{array}, \tag{3.2.21}$$

$s \in \mathbb{R}$, parametrised by $(y,\eta) \in T^*Y \setminus 0$, and homogeneous in the sense

$$\sigma^\mu_\wedge(\mathcal{A})(y,\lambda\eta) = \lambda^\mu \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & \mathrm{id}_{\mathbb{C}^{N_+}} \end{pmatrix} \sigma^\mu_\wedge(\mathcal{A})(y,\eta) \begin{pmatrix} \kappa_\lambda & 0 \\ 0 & \mathrm{id}_{\mathbb{C}^{N_-}} \end{pmatrix}^{-1}$$

for all $\lambda \in \mathbb{R}_+$.

Definition 3.2.14 *An operator $\mathcal{A} \in \mathcal{Y}^\mu(\mathbb{W}, \boldsymbol{g}; N_-, N_+)$ for $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$ is called elliptic if*

(i) $\sigma^{\mu}_{\psi,\mathrm{w}}(\mathcal{A}) \neq 0$ *on* $T^*_{\mathrm{w}}(\mathbb{W}) \setminus 0$,

(ii) *the principal edge symbol* (3.2.21) *is an isomorphism for all* $(y, \eta) \in T^*Y \setminus 0$.

Theorem 3.2.15 *For an operator* $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; N_-, N_+)$, $\boldsymbol{g} = (\gamma, \gamma-\mu, \Theta)$, *the following conditions are equivalent:*

(i) $\mathcal{A}$ *is elliptic,*

(ii) (3.2.20) *is a Fredholm operator for an* $s \in \mathbb{R}$.

If $\mathcal{A}$ *is elliptic then* (3.2.20) *is a Fredholm operator for all* $s \in \mathbb{R}$, *and (under condition* (3.2.14)*)* $\mathcal{A}$ *has a parametrix* $\mathcal{P} \in \mathcal{Y}^{-\mu}(\mathbb{W}, \boldsymbol{g}^{-1}; N_+, N_-)$ *for* $\boldsymbol{g}^{-1} = (\gamma - \mu, \gamma, \Theta)$, *i.e.,* $\mathcal{AP} - \mathcal{I} \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}_r; N_+, N_+)$, $\mathcal{PA} - \mathcal{I} \in \mathcal{Y}^{-\infty}(\mathbb{W}, \boldsymbol{g}_l; N_-, N_-)$ *for* $\boldsymbol{g}_r = (\gamma - \mu, \gamma - \mu, \Theta)$, $\boldsymbol{g}_l = (\gamma, \gamma, \Theta)$, *where*

$$\sigma^{-\mu}_{\psi,\mathrm{w}}(\mathcal{P}) = \sigma^{\mu}_{\psi,\mathrm{w}}(\mathcal{A})^{-1}, \qquad \sigma^{-\mu}_{\wedge}(\mathcal{P}) = \sigma^{\mu}_{\wedge}(\mathcal{A})^{-1}.$$

The proof is given in [66], Section 3.5.2 as far as it concerns (i)⇒(ii). The converse direction (ii)⇒(i) may be found in [8]. We also have elliptic regularity with and without asymptotics in the wedge Sobolev spaces, cf. [66], Theorem 3.5.16.

Theorem 3.2.16 *Let* $\mathcal{A} \in \mathcal{Y}^{\mu}(\mathbb{W}, \boldsymbol{g}; N_-, N_+)$, $\boldsymbol{g} = (\gamma, \gamma - \mu, \Theta)$, *be elliptic. Then (under condition* (3.2.14)*)* $\mathcal{A}u = f \in \mathcal{W}^{s-\mu,\gamma-\mu}_{Q}(\mathbb{W}) \oplus H^{s-\mu}(Y, N_+)$ *for an asymptotic type* $Q \in \mathrm{As}(X, (\gamma-\mu, \Theta))$ *and* $u \in \mathcal{W}^{-\infty,\gamma}(\mathbb{W}) \oplus H^{-\infty}(Y, N_-)$ *imply*

$$u \in \mathcal{W}^{s,\gamma}_{P}(\mathbb{W}) \oplus H^{s}(Y, N_+),$$

for a $P \in \mathrm{As}(X, (\gamma, \Theta))$; *this holds for all* $s \in \mathbb{R}$. *A similar result holds for the spaces without subscripts* P, Q.

The proof is formally analogous to that of Corollary 2.2.30 and based on the existence of a (left parametrix) $\mathcal{P}$ of $\mathcal{A}$ together with Theorem 3.2.13 and the corresponding continuity of the remainders in $\mathcal{Y}^{-\infty}(\dots)$.

Remark 3.2.17 *The definitions and results on pseudo–differential operators on manifolds withe edges have a natural generalisation to spaces of distributional sections in vector bundles* $E, F \in \mathrm{Vect}(\mathbb{W})$ *and* $J^-, J^+ \in \mathrm{Vect}(Y)$, *such that the operators are continuous as maps*

$$\mathcal{A}: \begin{array}{c} \mathcal{W}^{s,\gamma}(\mathbb{W}, E) \\ \oplus \\ H^s(Y, J^-) \end{array} \rightarrow \begin{array}{c} \mathcal{W}^{s-\mu,\gamma-\mu}(\mathbb{W}, F) \\ \oplus \\ H^{s-\mu}(Y, J^+) \end{array}.$$

Also the case $E = F = \mathbb{W} \times \mathbb{C}$ *requires, in general, non–trivial bundles* $J^\mp$ *in place of* $Y \times \mathbb{C}^{N_\mp}$*; our notation with trivial bundles was chosen to avoid lengthy explanations on cocycles and invariance.*

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List of Symbols

Index