HIGH DEGREE PRECISION DECOMPOSITION METHOD FOR THE NONHOMOGENOUS EVOLUTION PROBLEM WITH AN OPERATOR UNDER A SPLIT FORM

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Abstract In the present work symmetrized sequential-parallel decomposition method of the third degree precision for the solution of Cauchy abstract problem for the nonhomogenous evolution equation with an operator under a split form, is offered. Third degree precision is reached by introducing a complex coefficient with the positive real part. For the considered scheme the explicit a priori estimation is obtained.

Keywords and phrases: Decomposition Method, Semigroup, Trotter formula, Cauchy abstract problem.

1^0. **Introduction.** The study of the approximated schemes of a solution of evolution problems leads to the conclusion that to each approximated scheme there corresponds a definite operator (solving operator of a discrete problem), which approximates a solving operator (semigroup) of a source continuous problem. The opposite is also true: Constructing approximation of a continuous semigroup, we build an approximated scheme of a solution of an evolution problem.

For example, if we apply Rotte's method for a solution of an evolution problem, a solving operator of the obtained difference problem will be a discrete semigroup and we come to a problem of approximating a continuous semigroup with the help of discrete semigroups (in this case see T. Kato [1], Ch. IX).

In case of applying a decomposition method, the solving operator of the applicable decomposed problem generates the Ttrotter formula, (see Trotter H. [2]) or the Chernoff formula, (see Chernoff P. R. [3], [4]) or a formula, which is a combination of these formulas. Therefore, the error estimation of a decomposition method is equivalent to a problem of approximating of a continuous semigroup using Trotter type formulas. Papers [5], [6] (see also [7], Ch. II) are dedicated to the error estimations of Trotter type formulas.

The scheme of decomposition, associated with the Trotter formula, allows us to split Cauchy problem for an evolution equation with an operator $A = A_1+A_2+...+A_m$ to m problems correspondingly with operators $A_1, A_2, ..., A_m$, which are solved sequentially on each time interval with the length t/n.

The decomposition scheme, associated with the Chernoff formula, is known as a method of fractional steps (see N. N. Ianenko [8]).

As it is known, the decomposition method is sufficiently general for obtaining economical schemes for the solution of the multidimensional problems of mathematical physics. They can be divided into two groups: the schemes of sequential account (N. N. Ianenko [8], A.A. Samarskii [9], Marchuk G. I.

[10], A. A. Samarskii, P. N. Vabishchevich [11], I. V. Fryazinov [12], E. G. Diakonov [13], Temam R. [14], D.G. Gordeziani [15]) and the schemes of parallel account (D. G. Gordeziani and A. A. Samarskii [16], D. G. Gordeziani and H. V. Meladze [17], [18], A. M. Kuzyk and V. L. Makarov [19]). In [7] (see Ch. II) the explicit estimations for decomposition schemes of the parallel account are obtained, which were considered in [18]. At present, there are many works dedicated to the decomposition method (see references [10], [11]).

In the above-stated works the schemes considered are of the first or second degree precision. As far as we know, high degree precision decomposition formulas in case of two addends for the first time were obtained in [20].

In the present work, a symmetrized sequential-parallel decomposition method of the third degree precision for the solution of the Cauchy abstract problem with operator $A = A_1 + A_2 + ... + A_m$, is presented. For the considered scheme the explicit a priori estimation is obtained. Under explicit estimations we understand such a priori estimations for an error of solution, where the constants of a right member do not depend on a solution of an initial continuous problem, i.e. are absolute.

2^0. **Setting of the problem.** Let us consider Cauchy abstract problem in Banach space X :

$$\frac{du(t)}{dt} + Au(t) = f\left(t\right), \qquad t > 0, \qquad u(0) = \varphi. \tag{0.1}$$

Here A is a closed linear operator with the domain $D(A)$, which is everywhere dense in X, φ is a given element from $D\left(A\right)$, $f(t) \in C^1\left(X;[0;\infty)\right)$ and for every fixed t, $f\left(t\right) \in D\left(A\right)$.

Suppose that $(-A)$ operator generates a strongly continuous semigroup $\{\exp(-tA)\}_{t>0}$, then the solution of the problem (1) is given by the following formula (see [21],[22]):

$$u(t) = U(t, A)\varphi + \int_0^t U(t-s, A)f(s)ds, \tag{0.2}$$

where $U(t, A) = exp(-tA)$.

Let $A = A_1 + A_2 + ... + A_m$, where $A_i \quad (i = 1, 2, ..., m)$ are compactly defined, closed linear operators in X.

Let us introduce a difference net domain:

$$\overline{\omega}_\tau = \{t_k = k\tau, k = 1, 2, ..., \tau > 0\}.$$

Suppose that $f\left(t\right) \in C^2\left(X;[0,\infty)\right)$. Along with problem (1) we consider two sequences of the following problems on each interval $[t_{k-1}, t_k]$:

$$\begin{aligned} \frac{dv_k^1(t)}{dt} + \alpha A_1 v_k^1(t) &= \frac{\alpha}{m} f(t_k) - 2\sigma_1(t_k - t)f'(t_k), \\ v_k^1(t_{k-1}) &= u_{k-1}(t_{k-1}), \end{aligned}$$

$$\begin{aligned} \frac{dv_k^{i+1}(t)}{dt} + \alpha A_{i+1} v_k^{i+1}(t) &= \frac{\alpha}{m} f(t_k) - 2\sigma_{i+1}(t_k - t) f'(t_k), \\ v_k^{i+1}(t_{k-1}) &= v_k^i(t_k), \qquad i = 1, ..., m-2, \end{aligned}$$

$$\begin{aligned} \frac{dv_k^m(t)}{dt} + A_m v_k^m(t) &= \frac{1}{m} f(t_k) - 2\sigma_m(t_k - t) f'(t_k), \\ v_k^m(t_{k-1}) &= v_k^{m-1}(t_k), \end{aligned}$$

$$\begin{aligned} \frac{dv_k^{m+i}(t)}{dt} + \overline{\alpha} A_{m-i} v_k^{m+i}(t) &= \frac{\overline{\alpha}}{m} f(t_k) - 2\sigma_{m+i}(t_k - t) f'(t_k), \\ v_k^{m+i}(t_{k-1}) &= v_k^{m+i-1}(t_k), \qquad i = 1, ..., m-2, \end{aligned}$$

$$\begin{aligned} \frac{dv_k^{2m-1}(t)}{dt} + \overline{\alpha} A_1 v_k^{2m-1}(t) &= \frac{\overline{\alpha}}{m} f(t_k) - 2\sigma_{2m-1}(t_k - t) f'(t_k) + \frac{(t_k - t)^2}{2} f''(t_k), \\ v_k^{2m-1}(t_{k-1}) &= v_k^{2m-2}(t_k); \end{aligned}$$

$$\begin{aligned} \frac{dw_k^1(t)}{dt} + \alpha A_m w_k^1(t) &= \frac{\alpha}{m} f(t_k) - 2\sigma_1(t_k - t) f'(t_k), \\ w_k^1(t_{k-1}) &= u_{k-1}(t_{k-1}), \end{aligned}$$

$$\begin{aligned} \frac{dw_k^{i+1}(t)}{dt} + \alpha A_{m-i} w_k^{i+1}(t) &= \frac{\alpha}{m} f(t_k) - 2\sigma_{i+1}(t_k - t) f'(t_k), \\ w_k^{i+1}(t_{k-1}) &= w_k^i(t_k), \qquad i = 1, ..., m-2, \end{aligned}$$

$$\begin{aligned} \frac{dw_k^m(t)}{dt} + A_1 v_k^m(t) &= \frac{1}{m} f(t_k) - 2\sigma_m(t_k - t) f'(t_k), \\ w_k^m(t_{k-1}) &= w_k^{m-1}(t_k), \end{aligned}$$

$$\begin{aligned} \frac{dw_k^{m+i}(t)}{dt} + \overline{\alpha} A_{i+1} w_k^{m+i}(t) &= \frac{\overline{\alpha}}{m} f(t_k) - 2\sigma_{m+i}(t_k - t) f'(t_k), \\ v_k^{m+i}(t_{k-1}) &= v_k^{m+i-1}(t_k), \qquad i = 1, ..., m-2, \end{aligned}$$

$$\begin{aligned} \frac{dw_k^{2m-1}(t)}{dt} + \overline{\alpha} A_m w_k^{2m-1}(t) &= \frac{\overline{\alpha}}{m} f(t_k) - 2\sigma_{2m-1}(t_k - t) f'(t_k) + \frac{(t_k - t)^2}{2} f''(t_k), \\ w_k^{2m-1}(t_{k-1}) &= w_k^{2m-2}(t_k). \end{aligned}$$

Here $\sigma_1, \sigma_2, ..., \sigma_{2m-1}$ and α are numerical complex parameters, $Re(\alpha) > 0$, $u_0(0) = \varphi$. Suppose that $(-A_j)$, $(-\alpha A_j)$ and $(-\overline{\alpha} A_j)$ $(j = 1, 2, ..., m)$ operators generate strongly continuous semigroups.

On each $[t_{k-1},t_k]$ $(k=1,2,...)$ interval $u_k(t)$ are defined as follows:

$$u_k(t)=\frac{1}{2}[v_k^{2m-1}(t)+w_k^{2m-1}(t)].$$

We consider the function $u_k(t)$ as an approximate solution of the problem (1) on the interval $[t_{k-1},t_k]$.

The above-stated scheme in case of $m=2$ addends for the homogenous equation is considered in [23] and for the non-homogenous equation in [24].

We will need natural degrees of the operator $A=A_1+A_2+...+A_m$ $(A^s$, $s=2,3,4)$. In case of two addends $(m=2)$ they are defined as follows:

$$A^2=\left(A_1^2+A_2^2\right)+\left(A_1A_2+A_2A_1\right),$$

$$A^3=\left(A_1^3+A_2^3\right)+\left(A_1^2A_2+...+A_2^2A_1\right)+\left(A_1A_2A_1+A_2A_1A_2\right),$$

$$A^4=\left(A_1^4+A_2^4\right)+\left(A_1^3A_2+...+A_2^3A_1\right)+\left(A_1^2A_2A_1+...+A_2^2A_1A_2\right)+$$
$$+\left(A_1A_2A_1A_2+A_2A_1A_2A_1\right).$$

Analogously are defined A^s $(s=2,3,4)$ when $m>2$. Obviously, the domain $D\left(A^s\right)$ of the operator A^s is the intersection of the domains of its addends.

Let us introduce the following definitions:

$$\|\varphi\|_A=\|A_1\varphi\|+...+\|A_m\varphi\|, \quad \varphi\in D\left(A\right),$$

$$\|\varphi\|_{A^2}=\sum_{i,j=1}^{m}\|A_iA_j\varphi\|, \quad \varphi\in D\left(A^2\right),$$

where $\|\cdot\|$ is a norm in X, similarly are defined $\|\varphi\|_{A^s}$ $(s=3,4)$.

Theorem. *Let the following conditions be satisfied:*

a) $\alpha=\frac{1}{2}\pm i\frac{1}{2\sqrt{3}}$ $\left(i=\sqrt{-1}\right)$;

b) $(-\gamma A_j)$, $\gamma=1,\alpha,\overline{\alpha}$ $(j=1,2,...,m)$ *and* $(-A)$ *operators generate strongly continuous semigroups, for which the following estimations hold correspondingly:*

$$\|U(t,\gamma A_j)\|\le e^{\omega t},$$

$$\|U(t,A)\|\le Me^{\omega t}, \quad M,\omega=const>0;$$

c) $U\left(s,A\right)\varphi\in D\left(A^4\right)$ *for every fixed* $s\ge 0$;

d) $f(t)\in C^3(X;[0,\infty))$; $f\left(t\right)\in D\left(A^3\right)$, $f^{(k)}(t)\in D\left(A^{3-k}\right)$, $k=1,2$ *and* $U\left(s,A\right)f\left(t\right)\in D\left(A^4\right)$ *for every fixed* t *and* s $(t,s\ge 0)$;

e) $m>2$, $\sigma_j=\widetilde{\sigma}_1$, $\sigma_m=\widetilde{\sigma}_2$, $\sigma_{m+j}=\widetilde{\sigma}_3$ $(j=1,2,...,m-1)$,

$$\widetilde{\sigma}_3=\frac{1}{2\left[\left(9m+1\right)\alpha-3m\right]}, \quad \widetilde{\sigma}_1=\left(9\alpha-4\right)\widetilde{\sigma}_3, \quad \widetilde{\sigma}_2=\frac{1}{2}-\widetilde{\sigma}_1-\widetilde{\sigma}_3.$$

Then the following estimation holds:

$$\|u(t_k) - u_k(t_k)\| \le ce^{\omega_0 t_k} t_k \tau^3 \left(\sup_{s\in[0,t_k]} \|U(s,A)\varphi\|_{A^4} + t_k \sup_{s,t\in[0,t_k]} \|U(s,A)f(t)\|_{A^4} + \right.$$

$$\left. + \sup_{t\in[0,t_k]} \|f(t)\|_{A^3} + \sup_{t\in[0,t_k]} \|f'(t)\|_{A^2} + \sup_{t\in[0,t_k]} \|f''(t)\|_{A} + \sup_{t\in[0,t_k]} \|f'''(t)\| \right),$$

where c, ω_0 are positive constants.

3^0. **Conclusion.** In case operators A_1 A_2 ... A_m are matrixes, then it is obvious, that conditions of the theorem are automatically met. Also conditions of the theorem are met, if A_1, A_2 ... A_m and A are self-adjoint, positive definite operators.

Third degree precision is reached by introducing complex parameter. Because of this, each equation of the given decomposed system is changed by a pair of real equations, unlike lower order precision schemes. To solve the specific problem, for example the matrix factorization may be used, where the coefficients are the matrixes of the second order, unlike lower order precision schemes, where common factorization may be used.

It must be noted, that the sum of modules of coefficients of $V(\tau)$ transition operator equals to one, unlike the high degree precision decomposition schemes considered in the work [20]. Because of this, the considered schema is stable for any A_1 A_2 ... A_m bounded operators.

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