APPLICATION OF VEKUA's DIMENSION REDUCTION METHOD TO CUSPED PLATES AND BARS

G. Jaiani

I. Vekua Institute of Applied Mathematics
of Tbilisi State University

Received June, 19, 2001

The part of the talk concerning cusped bars is published in [1,2]. A survey of results up to 1999 connected with cusped plates is given in [3,4]. The present paper contains a brief survey of mostly new results in the theory of cusped plates (i.e., of cusped prismatic shells) on the basis of I. Vekua's hierarchic models of prismatic shells.

1. Let us recall the following Vekua's relations in the so called $N-$th approximation (notation is slightly changed, see [5], pp. 202, 203):

$$\overset{N}{X}_{\alpha jr,\alpha} + \sum_{s=0}^{r} \overset{r}{a}_{is} \overset{N}{X}_{ijs} + \overset{N}{\Phi}{}^{0}_{jr} = \rho \, \frac{\partial^2 \overset{N}{u}_{jr}}{\partial t^2}, \quad r = \overline{0,N}, \quad j = 1,2,3; \tag{1}$$

$$\overset{N}{e}_{ijr} = \frac{1}{2}\left(\overset{N}{u}_{ir,j} + \overset{N}{u}_{jr,i}\right) + \frac{1}{2}\sum_{s=r}^{N}\left(\overset{r}{b}_{is}\overset{N}{u}_{js} + \overset{r}{b}_{js}\overset{N}{u}_{is}\right), \quad r = \overline{0,N}, \; i,j = 1,2,3; \tag{2}$$

$$\overset{N}{X}_{ijr} = \lambda\delta_{ij}\overset{N}{u}_{\gamma r,\gamma} + \mu\left(\overset{N}{u}_{ir,j} + \overset{N}{u}_{jr,i}\right) + \sum_{s=r}^{N}\overset{r}{B}_{ijks}\overset{N}{u}_{ks}, \quad r = \overline{0,N}, \quad i,j = 1,2,3, \tag{3}$$

where

$$\overset{r}{a}_{\alpha r} := r\frac{h_{,\alpha}}{h}; \quad \overset{r}{a}_{\alpha s} := (2s+1)\frac{\overset{(+)}{h}_{,\alpha} - (-1)^{r+s}\overset{(-)}{h}_{,\alpha}}{2h}, \quad s \neq r, \quad \alpha = 1,2; \tag{4}$$

$$\overset{r}{a}_{3s} := -(2s+1)\frac{1-(-1)^{s+r}}{2h}; \tag{5}$$

$$\overset{r}{B}_{ijks} := \lambda\delta_{ij}\overset{r}{b}_{ks} + \mu\delta_{kj}\overset{r}{b}_{is} + \mu\delta_{ik}\overset{r}{b}_{js},$$

$$\overset{r}{b}_{\alpha r} := -(r+1)\frac{h_{,\alpha}}{h}, \quad \overset{r}{b}_{3r} = 0; \tag{6}$$

$$\overset{r}{b}_{js} := \begin{cases} 0 & s < r, \\ -(2s+1)\dfrac{\overset{(+)}{h}_{,\alpha} - (-1)^{s+r}\overset{(-)}{h}_{,\alpha}}{2h}, & j = \alpha, \quad s > r, \\ (2s+1)\dfrac{1-(-1)^{s+r}}{2h}, & j = 3, \quad s > r; \end{cases}$$

$$(\alpha = 1,2; \qquad r,s = \overline{0,N}, \qquad j = 1,2,3),$$

δ_{ij} is Kroneker's symbol; λ and μ are Lamé constants; ρ is a density of the material; $x_3 = \overset{(+)}{h}(x_1, x_2)$ and $x_3 = \overset{(-)}{h}(x_1, x_2)$, $(x_1, x_2) \in \omega$, are upper and lower surfaces of the plate, ω is a plate projection, $2h(x_1, x_2) = \overset{(+)}{h}(x_1, x_2) - \overset{(-)}{h}(x_1, x_2) \geq 0$ is a thickness (a part of the projection boundary where $2h = 0$ is called a cusped part of the plate boundary); indices after comma mean differentiation with respect to the corresponding variables; the ranges of Latin and Creek indices are $\{1, 2, 3\}$ and $\{1, 2\}$ correspondingly; the repeated index convention for summation is systematically used, unless one of them is understressed; $\overset{N}{X}_{ijr}$, $\overset{N}{e}_{ijr}$, $\overset{N}{u}_{ir}$ are the so called $r-$th moments of the stress and strain tensors, and of the displacement vector components, which multiplied, by $a^{\frac{1}{2}}(r + \frac{1}{2})^{\frac{1}{2}}$ in the three-dimensional case $N = +\infty$ coincide with the $r-$th Fourier-Legendre coefficients of the stress (X_{ij}) and strain (e_{ij}) tensors, and of the displacement vector components (u_i);

$$\Phi^0_{jr} := \sqrt{1 + \left(\overset{(+)}{h}_{,1}\right)^2 + \left(\overset{(+)}{h}_{,2}\right)^2}\, Q^+_{nj}$$

$$+(-1)^r \sqrt{1 + \left(\overset{(-)}{h}_{,1}\right)^2 + \left(\overset{(-)}{h}_{,2}\right)^2}\, Q^-_{nj} + \Phi_{jr},$$

Φ_{jr} are the $r-$th moments of the volume forces Φ_j :

$$\Phi_{jr}(x_1, x_2) := \int\limits_{\overset{(-)}{h}(x_1,x_2)}^{\overset{(+)}{h}(x_1,x_2)} \Phi_j(x_1, x_2) P_r(ax_3 - b) dx_3,$$

$$a = \frac{1}{h},; \quad b = \frac{\tilde{h}}{h}, \quad 2\tilde{h} := \overset{(+)}{h}(x_1, x_2) + \overset{(-)}{h}(x_1, x_2),$$

P_r are Legendre Polynomials, Q^+_{nj} and Q^-_{nj} are projections on the axis x_j, $j = 1, 2, 3$, of the surface forces Q^+_n and Q^-_n acting on the upper and lower surfaces of the prismatic shell, n is an outward normal to the prismatic shell face surfaces.

Remark 1. It is not difficult to show that relations (2), (3) can also be immediately obtained after substitution of the expressions

$$\{X_{ij},\ e_{ij},\ u_i,\} = \sum_{r=0}^{N} a\left(r + \frac{1}{2}\right)\{X_{ijr},\ e_{ijr},\ u_{ir}\} P_r(ax_3 - b), \ \ i, j = 1, 2, 3,$$

(with arbitrary $u_{ir}(x_1, x_2) \in C^2(\omega)$) of the stress and strain tensors, and of the displacement vector in Cauchy's relations and Hooke's law for an isotropic elastic material.

Taking into account (6) and that

$$\left(h^{r+1}\right)_{,\alpha} = -h^{r+1}\overset{r}{b}_{\alpha r}, \qquad \alpha = 1,2,$$

we can write relations (2) and (3) for the weighted moments

$$\overset{N}{v}_{jr} := h^{-r-1}\ \overset{N}{u}_{jr};$$

in the following forms

$$\overset{N}{e}_{ijr} = \frac{1}{2}h^{r+1}\left(\overset{N}{v}_{ir,j} + \overset{N}{v}_{jr,i}\right) + \frac{1}{2}\sum_{s=r+1}^{N} h^{s+1}\left(\overset{r}{b}_{is}\overset{N}{v}_{js} + \overset{r}{b}_{js}\overset{N}{v}_{is}\right), \qquad \sum_{s=N+1}^{N}(\cdot) = 0, \tag{7}$$

and

$$\overset{N}{X}_{ijr} = \lambda\delta_{ij}h^{r+1}\overset{N}{v}_{\gamma r,\gamma} + \mu h^{r+1}(\overset{N}{v}_{ir,j} + \overset{N}{v}_{jr,i}) + \sum_{s=r+1}^{N}\overset{r}{B}_{ijks}h^{s+1}\overset{N}{v}_{ks}, \qquad r = \overline{0,N}, \quad i,j = 1,2,3, \tag{8}$$

correspondingly.

Multiplying the both sides of (1) by h^r, in view of (4), (5), we can write it in the form as follows

$$h^r\overset{N}{X}_{\alpha jr,\alpha} + rh_{,\alpha}h^{r-1}\overset{N}{X}_{\alpha jr} + h^r\sum_{s=0}^{r-1}\overset{r}{a}_{is}\overset{N}{X}_{ijs} + h^r\Phi^0_{jr} = \rho h^r\ \frac{\partial^2 h^{r+1}\overset{N}{v}_{jr}}{\partial t^2},$$

$$\sum_{s=0}^{-1}(\cdot) = 0, \quad r = 0,N, \quad j = 1,2,3;$$

i.e.,

$$\left(h^r\overset{N}{X}_{\alpha jr}\right)_{,\alpha} + h^r\sum_{s=0}^{r-1}\overset{r}{a}_{is}\overset{N}{X}_{ijs} + h^r\Phi^0_{jr} = \rho h^r\ \frac{\partial^2 h^{r+1}\overset{N}{v}_{jr}}{\partial t^2}, \qquad r = 0,N, \quad j = 1,2,3. \tag{9}$$

Substituting (8) in (9), we get

$$\begin{aligned}
&\mu\left[\left(h^{2r+1}\overset{N}{v}_{\alpha r,j}\right)_{,\alpha} + \left(h^{2r+1}\overset{N}{v}_{jr,\alpha}\right)_{,\alpha}\right] \\
&+\lambda\delta_{\alpha j}\left(h^{2r+1}\overset{N}{v}_{\gamma r,\gamma}\right)_{,\alpha} + \sum_{s=r+1}^{N}\left(\overset{r}{B}_{\alpha jks}h^{r+s+1}\overset{N}{v}_{ks}\right)_{,\alpha} \\
&+\sum_{s=0}^{r-1}\overset{r}{a}_{is}\left[\lambda\delta_{ij}h^{r+s+1}\overset{N}{v}_{\gamma s,\gamma} + \mu h^{r+s+1}\left(\overset{N}{v}_{is,j} + \overset{N}{v}_{js,i}\right) + \sum_{l=s+1}^{N}\overset{s}{B}_{ijkl}h^{r+l+1}\overset{N}{v}_{kl}\right] \\
&+h^r\Phi^0_{jr} = \rho h^r\ \frac{\partial^2 h^{r+1}\overset{N}{v}_{jr}}{\partial t^2}, \quad r = \overline{0,N}, \quad j = 1,2,3.
\end{aligned} \tag{10}$$

Actually, system (10) is the explicit form of the general shell system (8.33) in [6] (where the lowest terms are not given explicitly) for the case of prismatic shells in the N-th approximation.

If $N = 0$, system (10) has the following form:

$$\mu\left[\left(h\overset{0}{v}_{\alpha 0,\beta}\right)_{,\alpha} + \left(h\overset{0}{v}_{\beta 0,\alpha}\right)_{,\alpha}\right] + \lambda\delta_{\alpha\beta}\left(h\overset{0}{v}_{\gamma 0,\gamma}\right)_{,\alpha} + \Phi^0_{\beta 0} = \rho h\,\frac{\partial^2 h\overset{0}{v}_{jr}}{\partial t^2}, \quad \beta = 1,2; \tag{11}$$

$$\mu\left(h\overset{0}{v}_{30,\alpha}\right)_{,\alpha} + \Phi^0_{30} = \rho h\,\frac{\partial^2 h\overset{0}{v}_{3r}}{\partial t^2}. \tag{12}$$

System (11) and equation (12) in the general case have been investigated under some restrictions in [7]. Considering Dirichlet boundary value problem, it has been shown that the plate projection boundary should be divided into two parts, $\Gamma_0 \cup \Gamma_1$, where

$$\Gamma_0 := \left\{x_1, x_2 \in \partial\omega : \;\; h = 0 \;\text{ and }\; \frac{\partial h}{\partial \nu} = 0\right\},$$

$$\Gamma_1 := \left\{x_1, x_2 \in \partial\omega : \;\; h = 0 \;\text{ and }\; \frac{\partial h}{\partial \nu} > 0\right\},$$

($\varkappa > 1$ and $\varkappa = 1$, respectively, for the case (20) (see below)) and $\Gamma_2 \cup \Gamma_3$, where

$$\Gamma_2 := \left\{x_1, x_2 \in \partial\omega : \;\; h = 0 \;\text{ and }\; \frac{\partial h}{\partial \nu} = +\infty\right\}$$

($0 < \varkappa < 1$ for the case (20)),

$$\Gamma_3 := \{x_1, x_2 \in \partial\omega : \;\; h > 0\}$$

(ν is an inward normal to $\partial\omega$). On the part $\Gamma_2 \cup \Gamma_3$ the Dirichlet conditions for the desired functions $\overset{0}{v}_{i0}$, $i = 1, 2, 3$, should be given while the part $\Gamma_0 \cup \Gamma_1$ should be set free of boundary conditions. The criterion

$$h(P) = 0, \quad \frac{\partial h(P)}{\partial \nu} \geq 0, \qquad P \in \partial\omega, \tag{13}$$

is equivalent to the divergence of the following integral along the normal ν

$$\int_P^Q \frac{d\nu}{h(Q)} = +\infty, \quad h(P) = 0, \quad Q \in \omega,$$

Q is an arbitrary point of the projection in a neighbourhood of P.

The criterion

$$h(P) = 0, \quad \frac{\partial h(P)}{\partial \nu} = +\infty, \quad P \in \partial\omega,$$

is equivalent to the convergence of the integral

$$\int\limits_{P}^{Q} \frac{d\nu}{h(Q)} < +\infty, \quad h(P) = 0. \tag{14}$$

In [7] (see also [3]) a more general system than (11). Actually the system which contains system (10) is considered. Let us recall it for $n = 2$:

$$Lw := (A^{\alpha\beta} w_{,\alpha})_{,\beta} + E^{\alpha} w_{,\alpha} + Cw = F, \quad (x_1, x_2) \in \omega, \tag{15}$$

where

$$w := (w_1, \dots, w_m)^T;\ F := (F_1, \dots, F_m)^T;\ A^{\alpha\beta} := \|a_{kl}^{\alpha\beta}\|;\ E^{\alpha} := \|e_{kl}^{\alpha}\| \in C^1(\overline{\omega});$$

$$C := \|c_{kl}\| \in C(\overline{\omega}), \quad i, j = 1, 2, \quad 1 \leq k, l \leq m.$$

It is easy to rewrite system (10) in the above vector form.

In fact, in this case

$$w := \left(\overset{N}{v}_{10}, \overset{N}{v}_{20}, \overset{N}{v}_{30}, \cdots, \overset{N}{v}_{1N}, \overset{N}{v}_{2N}, \overset{N}{v}_{3N} \right)^T,$$

$$A^{\alpha\beta} := \left\| \alpha_{kl}^{\alpha\beta} \right\|_1^{N+1}, \quad E^{\alpha} := \|\varepsilon_{kl}^{\alpha}\|_1^{N+1}, \quad C := \|\gamma_{kl}\|_1^{N+1}, \quad \alpha, \beta = 1, 2, \tag{16}$$

with the 3×3 matrices

$$\alpha_{kl}^{\alpha\beta} := \|0\|_1^3, \quad k \neq l, \quad \varepsilon_{k\underline{k}}^{\alpha} := \|0\|_1^3, \quad \alpha, \beta = 1, 2, \quad k, l = \overline{1, N+1};$$

$$\alpha_{k\underline{k}}^{11} := \begin{Vmatrix} \lambda + 2\mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{Vmatrix} h^{2k-1}, \quad \alpha_{k\underline{k}}^{22} := \begin{Vmatrix} \mu & 0 & 0 \\ 0 & \lambda + 2\mu & 0 \\ 0 & 0 & \mu \end{Vmatrix} h^{2k-1},$$

$$\alpha_{k\underline{k}}^{12} := \begin{Vmatrix} 0 & \lambda & 0 \\ \mu & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} h^{2k-1}, \quad \alpha_{k\underline{k}}^{21} := \begin{Vmatrix} 0 & \mu & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} h^{2k-1}, \quad k = \overline{1, N+1}; \tag{17}$$

$$\varepsilon_{kl}^{\alpha} := \left\| \overset{k-1}{B}_{\alpha jml-1} \right\|_1^3 h^{l+k-1}, \quad k < l, \quad \alpha = 1, 2,$$

$$\varepsilon_{kl}^{1} := \begin{Vmatrix} (\lambda + 2\mu) \overset{k-1}{a}_{1l-1} & \mu \overset{k-1}{a}_{2l-1} & \mu \overset{k-1}{a}_{3l-1} \\ \lambda \overset{k-1}{a}_{2l-1} & \mu \overset{k-1}{a}_{1l-1} & 0 \\ \lambda \overset{k-1}{a}_{3l-1} & 0 & \mu \overset{k-1}{a}_{1l-1} \end{Vmatrix} h^{l+k-1},$$

$$\varepsilon_{kl}^{2} := \begin{Vmatrix} \mu \overset{k-1}{a}_{2l-1} & \lambda \overset{k-1}{a}_{1l-1} & 0 \\ \mu \overset{k-1}{a}_{1l-1} & (\lambda + 2\mu) \overset{k-1}{a}_{2l-1} & \mu \overset{k-1}{a}_{3l-1} \\ 0 & \lambda \overset{k-1}{a}_{3l-1} & \mu \overset{k-1}{a}_{2l-1} \end{Vmatrix}, \quad k > l, \quad k, l = \overline{1, N+1};$$

$$\gamma_{kl} := \left\| \left(\overset{k-1}{B^*}_{\alpha jml-1} h^{k+l}\right)_{,\alpha} + \sum_{s=0}^{k-2} \overset{k-1}{a}_{is} \overset{s}{B^*}_{ijml-1} h^{k+l} \right\|_1^3, \quad k,l=\overline{1,N+1};$$

where

$$\overset{s}{B^*}_{ijml-1} := \begin{cases} \overset{s}{B}_{ijml-1}, & l-1>s, \\ 0, & l-1\le s, \end{cases} \quad i,j,m=1,2,3, \quad l=\overline{1,N+1}; \quad s=\overline{0,N}.$$

Assuming $\overset{(\pm)}{h}_{,\alpha} \in C(\overline{\omega})$, on the one hand, obviously,

$$\left.\frac{\partial h}{\partial \nu}\right|_{h=0} \ge 0 \tag{18}$$

(in the case (20) (see below) it means that $\varkappa \ge 1$); on the other hand, according to (16), (17), we have

$$A^{\alpha\beta} \in C^1(\overline{\omega}), \quad E^\alpha \in C(\overline{\omega}),$$

and

$$E^\alpha\,|_{h=0} = 0$$

since

$$\overset{k-1}{a}_{jl-1} h^{l+k-1}\,|_{h=0} = 0, \quad k>l; \quad \overset{k-1}{B}_{\alpha jml-1} h^{l+k-1}\,|_{h=0} = 0,$$
$$k<l; \quad \alpha=1,2, \quad j,m=1,2,3,; \quad k,l=\overline{1,N+1},$$

hence

$$E^\alpha \nu_\alpha\,|_{h=0} = 0. \tag{19}$$

But (19) means (see [3,7]) that the part of the plate projection boundary where $h=0$ should be set free of boundary conditions. In the case under consideration (18) and (19) are fulfilled at the same time. Thus, criterion (13), i.e., (18) remains valid also in the $N-$th approximation.

By virtue of (17), in general,

$$A^{12} \ne A^{21}.$$

But it can be avoided. Indeed,

$$Lw = (D^{\alpha\beta} w_{,\alpha})_{,\beta} + F^\alpha w_{,\alpha} + Cw = F,$$

where

$$D^{\alpha\beta} := \begin{cases} A^{\alpha\beta}, & \alpha=\beta, \\ \frac{1}{2}\left(A^{12}+A^{21}\right), & \alpha\ne\beta, \end{cases} \quad \alpha=1,2,$$

i.e., $D^{12}=D^{21}$, and

$$F^\alpha := E^\alpha + \frac{1}{2}\left(A^{\alpha\beta} - A^{\beta\alpha}\right)_{,\underline{\beta}}, \quad \alpha\ne\beta, \ \alpha,\beta=1,2.$$

In the approximation $N = 1$ for the case of cylindrical deformation of a plate, which actually coincides with the bar deformation in the approximation (1.0), more refined results are obtained in [1,8]. Namely, $\overset{1}{v}_{10}, \overset{1}{v}_{20}, \overset{1}{v}_{30}$ should be given if (14) is fulfilled while $\overset{1}{v}_{11}, \overset{1}{v}_{21}, \overset{1}{v}_{31}$ should be given if

$$\int\limits_P^Q \frac{d\nu}{h^3(Q)} < +\infty$$

(in the case (20) it means that $\varkappa < \frac{1}{3}$).

Otherwise, i.e., if the above integrals are divergent, the cusped part of the boundary should be set free of boundary conditions for corresponding unknown functions.

Generalizing this fact for the $N-$th approximation, the expected criterion looks as follows:

$$\overset{N}{v}_{is}, \; i = 1, 2, 3, \; s = \overline{0, N},$$

should be given as Dirichlet data on the part of the projection boundary, where $h = 0$, if

$$\int\limits_P^Q \frac{d\nu}{h^{2s+1}(Q)} < +\infty,$$

while the above part should be set free of Direchlet data if

$$\int\limits_P^Q \frac{d\nu}{h^{2s+1}(Q)} = +\infty.$$

The above-cited result for the first approximation in the general case of deformation has been recently obtained by G. Devdariani, when the thickness of the plate has the form

$$2h(x_1, x_2) = h_0 x_2^{\varkappa}, \qquad h_0, \varkappa = const > 0, \tag{20}$$

and the plate projection is a domain lying in the half-plane $x_2 > 0$ and adjacent to the axis x_1.

Remark 2. By the variational formulation the natural working space is a Hilbert space with the following (or equivalent) norm:

$$\|w\|^2 := \sum_{\beta=1}^{2} \sum_{i=1}^{3} \sum_{r=1}^{N} \| \left(\frac{r}{2} + 1\right)^{\frac{1}{2}} \; h^{\frac{2r+1}{2}} \; v_{ir,\beta} \|^2_{L_2(\omega)}$$

$$+\sum_{i=1}^{3}\sum_{r=0}^{N}\| \left(\frac{r}{2}+1\right)^{\frac{1}{2}}\ h^{\frac{2r+1}{2}}\ v_{ir}\|^2_{L_2(\omega)}.$$[1]

Acknowledgements.
Research supported by the Collaborative Linkage Grant NATO PST.CLG. 976426/5437.

References

[1] Jaiani, G.V. On a Mathematical Model of Bars with Variable Rectangular Cross-sections. ZAMM-Zeitschrift fuer Angewandte Mathematik und Mechanik, 81, 3, 2001, 147-173

[2] Kharibegashvili, S., Jaiani, G.V. On a Vibration of an Elastic Cusped Bars. Bulletin of TICMI, 4, 1999, 24-28 (web-site: http//www.viam.hepi.edu.ge/others/ticmi)

[3] Jaiani, G.V. Initial and Boundary Value Problems for Singular Differential Equations and Applications to the Theory of Cusped Bars and Plates, Complex methods for partial differential equations (ISAAC Serials, Vol. 6), Eds.: H. Begehr, O. Celebi, W. Tutschke. Kluwer, Dordrecht, 1999, 113-149

[4] Jaiani, G.V. Elastic Bodies with Non-smooth Boundaries-Cusped Plates and Shells. ZAMM, 76, Suppl. 2, 1996, 117-120

[5] Vekua, I.N. On a Way of Calculating of Prismatic Shells. Proceedings of A. Razmadze Institute of Mathematics of Georgian Academy of Sciences, 21, 1955, 191-259 (Russian)

[6] Vekua, I.N. The Theory of Thin Shallow Shells of Variable Thickness. Proceedings of A. Razmadze Institute of Mathematics of Georgian Academy of Sciences, 30, 1965, 5-103 (Russian)

[7] Jaiani, G.V. The First Boundary Value Problem of Cusped Prismatic Shell Theory in Zero Approximation of Vekua Theory. Proceedings of I.Vekua Institute of Applied Mathematics, 29, 1988, 5-38 (Russian, Georgian and English summaries)

[8] Kharibegashvili, S., Jaiani, G.V. Dynamical Problems in the (0.0) and (1.0) Approximations of a Mathematical Model of Cusped Bars (to appear)

[1] If we introduce such a norm, then, according to Parseval's equality, for $u \in L_2(\Omega)$

$$\|u\|^2_{L_2(\Omega)}=\sum_{i=1}^{3}\sum_{r=0}^{\infty}\| \left(\frac{r}{2}+1\right)^{\frac{1}{2}}\ h^{\frac{2r+1}{2}}\ v_{ir}\|^2_{L_2(\omega)},$$

and when moreover $u \in H^1(\Omega)$ and $h=const$

$$\|u_{,\beta}\|^2_{L_2(\Omega)}=\sum_{i=1}^{3}\sum_{r=0}^{\infty}\| \left(\frac{r}{2}+1\right)^{\frac{1}{2}}\ h^{\frac{2r+1}{2}}\ v_{ir,\beta}\|^2_{L_2(\omega)}.$$