ON I.VEKUA'S METHOD IN THE THEORY OF ELASTIC MIXTURES

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Suppose we have static linear three-dimension equations of isotropic mixed material. Our aim is, from this to get two-dimension equations for the shell using I. Vekua's method of expansion by Legandre polynomials, also to consider finite approximations and to get general solutions for some cases.

Let us consider a curvilinear coordinate system. In this system static linear balance equations have the form

$$\begin{cases} \dfrac{1}{\sqrt{g}}\partial_i\left[\sqrt{g}\left(\boldsymbol{\sigma}^{'i}-(\Pi-\alpha_2)\,R^i\right)\right]+\rho_1 F^{'}=0,\\ \dfrac{1}{\sqrt{g}}\partial_i\left[\sqrt{g}\left(\boldsymbol{\sigma}^{''i}+(\Pi-\alpha_2)\,R^i\right)\right]+\rho_2 F^{''}=0,\end{cases} \tag{1}$$

where g is a metric tensor discriminate of space, $\boldsymbol{\sigma}^{'i},\boldsymbol{\sigma}^{''i}$ are contravariant stress tensors of mixture, ρ_1,ρ_2 are densities of mixture, R^i is contravariant basis vectors, α_2 are constant, $F^{'},F^{''}$ are volume forces,

$$\Pi=\frac{\alpha_2\rho_2}{\rho}\left(R^j\partial_j u^{'}\right)+\frac{\alpha_2\rho_1}{\rho}\left(R^j\partial_j u^{''}\right),$$

where $u^{'},u^{''}$ are displacement vectors, $\rho=\rho_1+\rho_2$.

Under repeating indexes we mean summation, the Latin letters taking the values 1,2,3.

If we introduce notation

$$\begin{aligned} P^{'i}&=P^{'ij}R_j:=\boldsymbol{\sigma}^{'i}-(\Pi-\alpha_2)\,R^i=\left(\sigma^{'ij}-g^{ij}(\Pi-\alpha_2)\right)R_j,\\ P^{''i}&=P^{''ij}R_j:=\boldsymbol{\sigma}^{''i}+(\Pi-\alpha_2)\,R^i=\left(\sigma^{''ij}+g^{ij}(\Pi-\alpha_2)\right)R_j,\end{aligned}$$

where g^{ij} are metric tensor's contravariant components, R_j are covariant basis vectors, then we have the following relations

$$\begin{cases} P^{'i}=\mathbb{C}_1^{'ij}\partial_j u^{'}+\mathbb{C}_2^{'ij}\partial_j u^{''},\\ P^{''i}=\mathbb{C}_1^{''ij}\partial_j u^{'}+\mathbb{C}_2^{''ij}\partial_j u^{''},\end{cases} \tag{2}$$

where $\mathbb{C}_1^{'ij},\mathbb{C}_2^{'ij},\mathbb{C}_1^{''ij},\mathbb{C}_2^{''ij}$ are operators, defined as follows

$$\begin{aligned} \mathbb{C}_1^{'ij}&=\left(\lambda_1-\frac{\alpha_2\rho_2}{\rho}\right)R^i\otimes R^j+(\mu_1+\lambda_5)\,R^j\otimes R^i+(\mu_1-\lambda_5)\left(R^iR^j\right)\mathbb{E},\\ \mathbb{C}_2^{'ij}&=\left(\lambda_3-\frac{\alpha_2\rho_1}{\rho}\right)R^i\otimes R^j+(\mu_3-\lambda_5)\,R^j\otimes R^i+(\mu_3+\lambda_5)\left(R^iR^j\right)\mathbb{E},\\ \mathbb{C}_1^{''ij}&=\left(\lambda_4+\frac{\alpha_2\rho_2}{\rho}\right)R^i\otimes R^j+(\mu_3-\lambda_5)\,R^j\otimes R^i+(\mu_3+\lambda_5)\left(R^iR^j\right)\mathbb{E},\\ \mathbb{C}_2^{''ij}&=\left(\lambda_2+\frac{\alpha_2\rho_1}{\rho}\right)R^i\otimes R^j+(\mu_2+\lambda_5)\,R^j\otimes R^i+(\mu_2-\lambda_5)\left(R^iR^j\right)\mathbb{E},\end{aligned}$$

where $\otimes$ is symbol of tensor multiplication, $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \mu_1, \mu_2, \mu_3$ are modulus of elasticity, $\alpha_2 = \lambda_3 - \lambda_4$, $\mathbb{E}$ is identify transformation's operator $\mathbb{E} = R_i \otimes R^i = R^i \otimes R_i$

Let us consider symetric shells, whose thickness are $2h$ and middle surface is S

We consider normal coordinate system on the middle surface. For any points radius-vector R has the form

$$R\left(x^1, x^2, x^3\right) = r\left(x^1, x^2\right) + x^3 n\left(x^1, x^2\right),$$

where x^1, x^2 are Gaus-parametrs of the surfaces $r\left(x^1, x^2\right)$ and $n\left(x^1, x^2\right)$ resfectivety radius-vector and unit vector of S in the point (x^1, x^2).

Is we use I. Vekuas method of expansion be Legendre polynomials from the system (1),

$$\begin{cases} \frac{1}{\sqrt{a}}\partial_\alpha(\sqrt{a}\overset{(k)}{P}{}'^{\alpha}) + \partial_\alpha \ln h \underline{\underline{\overset{(k)}{P}}}{}'^{\alpha} - \frac{1}{h}\underline{\overset{(k)}{P}}{}'^{3} + \overset{(k)}{\Phi}{}' = 0, \\ \frac{1}{\sqrt{a}}\partial_\alpha(\sqrt{a}\overset{(k)}{P}{}''^{\alpha}) + \partial_\alpha \ln h \underline{\underline{\overset{(k)}{P}}}{}''^{\alpha} - \frac{1}{h}\underline{\overset{(k)}{P}}{}''^{3} + \overset{(k)}{\Phi}{}'' = 0, \end{cases}$$

$$\left(k = 0, 1, ..., \overset{(-k)}{P}{}' = \overset{(-k)}{P}{}'' = 0, \ \ \textbf{when} \ \ k > 0\right)$$

where

$$\overset{(k)}{P}{}'^{i} = \left(k + \frac{1}{2}\right)\frac{1}{h}\int\limits_{-h}^{h}\sqrt{\frac{g}{a}}P'^{i}P_k\left(\frac{x^3}{h}\right)dx^3,$$

$$\overset{(k)}{P}{}''^{i} = \left(k + \frac{1}{2}\right)\frac{1}{h}\int\limits_{-h}^{h}\sqrt{\frac{g}{a}}P''^{i}P_k\left(\frac{x^3}{h}\right)dx^3,$$

$P_k\left(\frac{x^3}{h}\right)$ is a k-order polynomial,

$$\overset{(k)}{\Phi}{}' = \left(k + \tfrac{1}{2}\right)\tfrac{\rho_1}{h}\int\limits_{-h}^{h} F' P_k\left(\tfrac{x^3}{h}\right)dx^3 + \left(k + \tfrac{1}{2}\right)\tfrac{1}{h}\left\{\sqrt{\tfrac{g_+}{a}}\left[P'^{3}_{+} - \partial_\alpha h\, P'^{\alpha}_{+}\right] - \right.$$
$$\left. -(-1)^k\sqrt{\tfrac{g_-}{a}}\left[P'^{3}_{-} + \partial_\alpha h\, P'^{\alpha}_{-}\right]\right\},$$

Analogously we determine $\overset{(k)}{\Phi}{}''$,

$$P'^{j}_{\pm} = P'^{j}(x^1, x^2, \pm h), \ \ g_\pm = g(x^1, x^2, \pm h),$$

a is a metrix discriminant of the metric tenzor of S,

$$\underline{\underline{\overset{(k)}{P}}}{}'^{\alpha} = (k+1)\overset{(k)}{P}{}'^{\alpha} + (2k+1)\left(\overset{(k-2)}{P}{}'^{\alpha} + \overset{(k-4)}{P}{}'^{\alpha} + ...\right),$$
$$\underline{\overset{(k)}{P}}{}'^{3} = (2k+1)\left(\overset{(k-1)}{P}{}'^{3} + \overset{(k-3)}{P}{}'^{3} + ...\right),$$

Analogously we get definitions for $\overset{(k)}{\underline{\underline{P}}}{}''^{\alpha}$ and $\overset{(k)}{\underline{P}}{}''^{3}$.

Assume that

$$R_\alpha = \left(\delta_\alpha^\beta - x^3 b_\alpha^\beta\right) r_\beta \cong r_\alpha \quad - h \leq x_3 \leq h,$$

where b_α^β are coefficients of the second basic quadratic form,

$$u'(x^1, x^2, x^3) = \sum_{k=0}^{N} \overset{(k)}{u}{}'(x^1, x^2) P_k\left(\frac{x^3}{h}\right), \quad u''(x^1, x^2, x^3) = \sum_{k=0}^{N} \overset{(k)}{u}{}''(x^1, x^2) P_k\left(\frac{x^3}{h}\right),$$

where N is some fixed non-negative integer.

$$\overset{(k)}{P}{}'^{i} \cong \left(k + \frac{1}{2}\right) \frac{1}{h} \int_{-h}^{h} P'^{i} P_k\left(\frac{x^3}{h}\right) dx^3, \quad \overset{(k)}{P}{}''^{i} \cong \left(k + \frac{1}{2}\right) \frac{1}{h} \int_{-h}^{h} P''^{i} P_k\left(\frac{x^3}{h}\right) dx^3,$$

$$\overset{(k)}{u}{}' = \overset{(k)}{u}{}'' = 0, \quad \overset{(k)}{P}{}'^{i} = \overset{(k)}{P}{}''^{i} = 0, \quad \textbf{when} \ \ k > N.$$

If we consider equations for displacement then we get $6N + 6$ equations, i.e, well have second order partial differential equations, for $6N + 6$ unknown functions. In the isometric coordinate system there first basic square form can be written as follows

$$ds^2 = \Lambda(d\xi_1^2 + d\xi_2^2), \quad \Lambda > 0,$$

we have

$$\begin{cases} a_1 \Delta \overset{(k)}{u}{}'_\alpha + b_1 \partial_\alpha (\partial_1 \overset{(k)}{u}{}'_1 + \partial_2 \overset{(k)}{u}{}'_2) + c \Delta \overset{(k)}{u}{}''_\alpha + \\ + d \partial_\alpha (\partial_1 \overset{(k)}{u}{}''_1 + \partial_2 \overset{(k)}{u}{}''_2) + \overset{(k)}{M}{}'_\alpha + \overset{(k)}{\Phi}{}'_\alpha = 0, \\ a_1 \Delta \overset{(k)}{u}{}'_3 + c \Delta \overset{(k)}{u}{}''_3 + \overset{(k)}{M}{}'_3 + \overset{(k)}{\Phi}{}'_3 = 0, \\ c \Delta \overset{(k)}{u}{}'_\alpha + d \partial_\alpha (\partial_1 \overset{(k)}{u}{}'_1 + \partial_2 \overset{(k)}{u}{}'_2) + a_2 \Delta \overset{(k)}{u}{}''_\alpha + \\ + b_2 \partial_\alpha (\partial_1 \overset{(k)}{u}{}''_1 + \partial_2 \overset{(k)}{u}{}''_2) + \overset{(k)}{M}{}''_\alpha + \overset{(k)}{\Phi}{}''_\alpha = 0, \\ c \Delta \overset{(k)}{u}{}'_3 + a_2 \Delta \overset{(k)}{u}{}''_3 + \overset{(k)}{M}{}''_3 + \overset{(k)}{\Phi}{}''_3 = 0, \end{cases} \tag{3}$$

$$\left(k = 0, 1, ..., N, \ \ \partial_\alpha = \frac{\partial}{\partial \xi_\alpha}, \ \ \alpha = 1, 2\right), \quad \Delta = \frac{\partial^2}{\partial \xi_1^2} + \frac{\partial^2}{\partial \xi_2^2} \quad - \ \text{is Laplacian},$$

$$\begin{aligned} &a_1 = \mu_1 - \lambda_5, \ \ c = \mu_3 + \lambda_5, \ \ b_1 = \mu_1 + \lambda_5 + \lambda_1 - \frac{\alpha_2 \rho_2}{\rho}, \\ &d = \mu_3 + \lambda_3 - \lambda_5 - \frac{\alpha_2 \rho_1}{\rho} = \mu_3 + \lambda_4 - \lambda_5 + \frac{\alpha_2 \rho_2}{\rho}, \\ &a_2 = \mu_2 - \lambda_5, \ \ b_2 = \mu_2 + \lambda_2 + \lambda_5 + \frac{\alpha_2 \rho_1}{\rho}. \end{aligned}$$

$\overset{(k)}{M}{}'_j$ and $\overset{(k)}{M}{}''_j$ are homogeneous linear differential expressiens, which contain unknown functions $\overset{(k)}{u}{}'_j$ and $\overset{(k)}{u}{}''_j$ and the first differentials.

Let us consider approximation $N=1$ for a plate of constant thickness. In case (3) system is split into two independend systems of stretch-press and bend by quantity $\overset{(0)}{u}{}'_{\alpha}$, $\overset{(0)}{u}{}''_{\alpha}$, $\overset{(1)}{u}{}'_{3}$, $\overset{(1)}{u}{}''_{3}$, and $\overset{(1)}{u}{}'_{\alpha}$, $\overset{(1)}{u}{}''_{\alpha}$, $\overset{(0)}{u}{}'_{3}$, $\overset{(0)}{u}{}''_{3}$.

The equations of stretch-press in Cartesian coordinate system have the following form

$$\left\{\begin{array}{l} a_1\Delta \overset{(0)}{u}{}'_{+} + 2b_1\partial_{\bar z}\theta' + \dfrac{2}{h}(\lambda_1 - \dfrac{\alpha_2\rho_2}{\rho})\partial_{\bar z}\overset{(1)}{u}{}'_{3} + c\Delta\overset{(0)}{u}{}''_{+} + \\ +2d\partial_{\bar z}\theta'' + \dfrac{2}{h}(\lambda_3 - \dfrac{\alpha_2\rho_1}{\rho})\partial_{\bar z}\overset{(1)}{u}{}''_{3} = 0, \\ c\Delta\overset{(0)}{u}{}'_{+} + 2d\partial_{\bar z}\theta' + \dfrac{2}{h}(\lambda_4 + \dfrac{\alpha_2\rho_2}{\rho})\partial_{\bar z}\overset{(1)}{u}{}'_{3} + a_2\Delta\overset{(0)}{u}{}''_{+} + \\ +2b_2\partial_{\bar z}\theta'' + \dfrac{2}{h}(\lambda_2 + \dfrac{\alpha_2\rho_1}{\rho})\partial_{\bar z}\overset{(1)}{u}{}''_{3} = 0, \\ a_1\Delta\overset{(1)}{u}{}'_{3} - \dfrac{3}{h}(\lambda_1 - \dfrac{\alpha_2\rho_2}{\rho})\theta' - \dfrac{3(a_1+b_1)}{h^2}\overset{(1)}{u}{}'_{3} + c\Delta\overset{(1)}{u}{}''_{3} - \\ -\dfrac{3}{h}(\lambda_3 - \dfrac{\alpha_2\rho_1}{\rho})\theta'' - \dfrac{3(d+c)}{h^2}\overset{(1)}{u}{}''_{3} = 0, \\ c\Delta\overset{(1)}{u}{}'_{3} - \dfrac{3}{h}(\lambda_4 + \dfrac{\alpha_2\rho_2}{\rho})\theta' - \dfrac{3(c+d)}{h^2}\overset{(1)}{u}{}'_{3} + a_2\Delta\overset{(1)}{u}{}''_{3} - \\ -\dfrac{3}{h}(\lambda_2 + \dfrac{\alpha_2\rho_1}{\rho})\theta'' - \dfrac{3(b_2+a_2)}{h^2}\overset{(1)}{u}{}''_{3} = 0, \end{array}\right. \tag{4}$$

where

$$\overset{(0)}{u}{}'_{+} = \overset{(0)}{u}{}'_{1} + i\overset{(0)}{u}{}'_{2}, \quad \overset{(0)}{u}{}''_{+} = \overset{(0)}{u}{}''_{1} + i\overset{(0)}{u}{}''_{2},$$

$$\theta' = \partial_z\overset{(0)}{u}{}'_{+} + \partial_{\bar z}\overline{\overset{(0)}{u}}{}'_{+}, \quad \theta'' = \partial_z\overset{(0)}{u}{}''_{+} + \partial_{\bar z}\overline{\overset{(0)}{u}}{}''_{+},$$

General solution of (4) systems has the form

$$\begin{aligned} \overset{(0)}{u}{}'_{+} &= A_1'\varphi_1(z) + B_1' z\overline{\varphi_1'(z)} + A_2'\varphi_2(z) + B_2' z\overline{\varphi_2'(z)} + C_1'\partial_{\bar z}\chi_1(z,\bar z) \\ &+ C_2'\partial_{\bar z}\chi_2(z,\bar z) - \overline{\psi_1(z)}, \\ \overset{(0)}{u}{}''_{+} &= A_1''\varphi_1(z) + B_1'' z\overline{\varphi_1'(z)} + A_2''\varphi_2(z) + B_2'' z\overline{\varphi_2'(z)} \\ &+ C_1''\partial_{\bar z}\chi_1(z,\bar z) + C_2''\partial_{\bar z}\chi_2(z,\bar z) - \overline{\psi_2(z)}, \\ \overset{(1)}{u}{}'_{3} &= K_1'(\varphi_1'(z) + \overline{\varphi_1'(z)}) + K_2'(\varphi_2'(z) + \overline{\varphi_2'(z)}) + \chi_1(z,\bar z) + \chi_2(z,\bar z) \\ \overset{(1)}{u}{}''_{3} &= K_1''(\varphi_1'(z) + \overline{\varphi_1'(z)}) + K_2''(\varphi_2'(z) + \overline{\varphi_2'(z)}) + K_1\chi_1(z,\bar z) + K_2\chi_2(z,\bar z), \end{aligned}$$

where $\varphi_1(z)$, $\varphi_2(z), \psi_1(z)$, $\psi_2(z)$ are analytic functions, $\chi_1(z,\bar z), \chi_2(z,\bar z)$ are general solutions of Helmholtz's equations

$$\Delta\chi - æ_\alpha\chi = 0, \quad \alpha = 1,2,$$

where $æ_1$ and $æ_2$ are characteristic square equation roots, $A_1', ..., K_2$ are constants, which are expressed by modulus of Elastisity.

References

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