ON SOME NON-LOCAL PROBLEMS OF THE THEORY OF ELASTICITY

N. Gordeziani

I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University

The present work studies non-local in time generalized problem for the second order evolutionary equation. Here we suggest an iteration method of finding approximate solution of such a problem. The results obtained herein make it possible to investigate non-local in time problems of the theory of elasticity. Non-local in space as well as in time problems were stated and explored by many scientists (see [1]-[8]).

We consider Hilbert spaces V and H satisfying the following three conditions (see e.g. [9]):

1) $V \subset H$;
2) V is dense in H;
3) V and H are separable spaces.

If we identify H with its dual space and denote the dual space of V by V' , then we will have one more condition

4) $V \subset H \subset V'$.

Further on we will use an operator $A \in L(V, V')$, where $L(\cdot, \cdot)$ means the set of linear continuous operators acting from V to V'; $(A\varphi, \psi) = a(\varphi, \psi)$, where $(\cdot, \cdot)$ is a scalar product in H ($|\cdot|$- norm in H).

We put some restriction on $a(\varphi, \psi)$:

a) the function $t \longrightarrow a(\varphi, \psi)$ is continuously differentiable on $[0, T]$ for $\forall \varphi, \psi \in V$;

b) $a(\varphi, \psi) = a(\psi, \varphi)$;

c) $\varphi \geq \alpha \left\|\varphi\right\|^2$, where $\alpha = const > 0, \quad \forall \varphi \in V, \forall t \in [0, T]$.

Here $\|\cdot\|$ and $((\cdot, \cdot))$ are respectively the norm and scalar product in V.

We would like to point out that the whole investigation brought herein is conducted for real spaces V and H .

Now we get to the statement of the problem itself.

Problem. We have to find the function $u(t)$ satisfying the condition $u(t) \in L^2(0, T; H)$ (see the definition in [9]), second order evolutionary equation

$$\frac{d^2 u}{dt^2} + Au(t) = f(t), \quad 0 < t < T, \tag{1}$$

and the following non-local initial conditions:

$$u(0) = \sum_{i=1}^{m} \gamma_i u(t_i) + \int\limits_0^{\bar{t}} \rho_1(\tau) u(\tau) d\tau + u_0, \tag{2}$$

$$u_t'(0)=\sum_{i=1}^{m}\gamma_i u_t'(t_i)+\int_0^{\bar{t}}\rho_2(\tau)u(\tau)d\tau+u_1, \qquad (3)$$

where $\gamma_i=const\quad(i=1,...,m)$ are prescribed constants, ρ_1 and ρ_2 are given continuous scalar functions defined on $[0,T]$; $u_0\in V,\quad u_1\in H$ are also prescribed vectors, $\{t_i\}$ are given points of $(0,T)$, $0<t_1<\cdots<t_m<T$.

According to the theorem of J.-L. Lions (see e.g. [9],[10]) Cauchy problem for the equation (1) has a unique solution.

On the basis of the latter statement we can prove the following theorem:

Theorem 1. *Let us assume that*

$$\lambda_i\geq 0,\quad \rho_1\equiv\rho_2\equiv 0\quad(i=1,...,m),\quad \sum_{i=1}^{m}\gamma_i=q=const<1, \qquad (4)$$

and

$$a\left(\sum_{i=1}^{m}\gamma_i u(t_i),\sum_{i=1}^{m}\gamma_i u(t_i)\right)\leq\sum_{i=1}^{m}\gamma_i a\left(u(t_i),u(t_i)\right); \qquad (5)$$

If there exists the solution of the problem (1)-(2)-(3), then it is unique.

As was mentioned above, to prove this theorem we use the theorem of J.-L. Lions ([9]) as well as the coercitiveness of the form $a(u,v)$.

Now we suggest iteration procedures to resolve the problem (1)-(2)-(3). The iteration procedure formulated hereby reduces resolution of the initial problem to resolving the sequence of Cauchy problems formulated for classical second order abstract equations. The above mentioned iteration procedure has the following form:

$$\frac{d^2u^{(k+1)}(t)}{dt^2}+Au^{(k+1)}(t)=f(t),\quad 0<t<T, \qquad (6)$$

$$u^{(k+1)}(0)=\sum_{i=1}^{m}\gamma_i u^{(k)}(t_i)+\int_0^{\bar{t}}\rho_1(\tau)u^{(k)}(\tau)d\tau+u_0, \qquad (7)$$

$$u_t'^{(k+1)}(0)=\sum_{i=1}^{m}\gamma_i u'^{(k)}(t_i)+\int_0^{\bar{t}}\rho_2(\tau)u_t^{(k)}(\tau)d\tau+u_1, \qquad (8)$$

$$(k=0,1,...),\quad u^{(0)}=0,\quad u_t'^{(0)}(0)=0. \qquad (9)$$

To make the judgment simple we consider discrete and integral variants of the conditions (2) and (3) separately.

The following theorem is true:

Theorem 2.*If $\gamma_i\geq 0,\quad \sum\limits_{i=1}^{m}\gamma_i=q<1,\quad \rho_1(\tau)=\rho_2(\tau)\equiv 0$, then $u^{(k)}(t)$ sequence converges to $u(t)$ solution of the problem (1)-(2)-(3) and the following estimation is in force:*

$$\left\| u^{(k)}(t) - u(t) \right\|^* \le cq^k$$

where c does not depend on k and $\|v\|^* = \left(\|v\|^2 + \|v_t'\|^2\right)^{1/2}$.

Proof. We bring in the difference $w^{(k)}(t) = u^{(k)}(t) - u(t)$ which satisfies conditions:

$$\frac{d^2 w^{(k+1)}(t)}{dt^2} + Aw^{(k+1)}(t) = 0, \quad 0 < t < T, \tag{10}$$

$$w^{(k+1)}(0) = \sum_{i=1}^{m} \gamma_i w^{(k)}(t_i), \tag{11}$$

$$w_t'^{(k+1)}(0) = \sum_{i=1}^{m} \gamma_i w_t'^{(k+1)}(t_i), \tag{12}$$

$$(k = 0, 1, ...)\,, \quad w^{(0)}(t) = -u(t), \quad w_t'^{(0)}(t) = u_t'(t). \tag{13}$$

Applying the theorem of J.-L. Lions (see [9]) and particularly the energetic equality for Cauchy problem, we get:

$$a\left(\sum_{i=1}^{m} \gamma_i w^{(k+1)}(t_i), \sum_{i=1}^{m} \gamma_i w^{(k+1)}(t_i)\right) + \left\| \sum_{i=1}^{m} \gamma_i w_t'^{(k+1)}(t_i) \right\|^2 \le \tag{14}$$

$$q\left[a\left(w^{(k+1)}(0), w^{(k+1)}(0)\right) + \left\| w_t'^{(k+1)}(0) \right\|^2\right].$$

Using the initial conditions (11)-(12) and letting k run through all its meanings, we get:

$$a\left(w^{(k+1)}(0), w^{(k+1)}(0)\right) + \left\| w_t'^{(k+1)}(0) \right\|^2 \le cq^k, \tag{15}$$

where c does not depend on k.

The inequality (15) means that the iteration process converges at $t = 0$ moment and

$$a\left(w^{(k+1)}(t), w^{(k+1)}(t)\right) + \left\| w_t'^{(k+1)}(t) \right\|^2 \le \tag{16}$$

$$a\left(w^{(k+1)}(0), w^{(k+1)}(0)\right) + \left\| w_t'^{(k+1)}(0) \right\|^2.$$

Taking into account inequality (16) and coercitiveness of the bilinear form $a(\cdot, \cdot)$ we easily get convinced in the correctness of the Theorem 2.

Considering the other version of the problem with an integral non-local condition we can state the following theorem:

Theorem 3. *If* $\gamma_i \equiv 0, \quad (i = 1, ..., m), \quad \left| \int\limits_0^{\bar{t}} \rho(\tau)u(\tau)d\tau \right| = q < 1$ *and*

$$\int\limits_0^{\bar{t}} \rho(\tau)a\left(u(\tau), u(\tau)\right)d\tau \geq a\left(\int\limits_0^{\bar{t}} \rho(\tau)u(\tau)d\tau, \int\limits_0^{\bar{t}} \rho(\tau)u(\tau)d\tau\right), \tag{17}$$

where $\rho(\tau) = \rho_1(\tau) = \rho_2(\tau)$,

then the iteration process (6)-(9) converges and the following estimation is true:

Theorem 4.

$$\left\| u^{(k)}(t) - u(t) \right\|^* \leq cq^k.$$

We would like to emphasize that the results obtained herein let us state and investigate non-local problems for non-stationary equations of membrane oscillation (see [13]), the theories of elasticity (see [10]) and elastic mixtures (see [11],[12]) in the case of classical (by space) boundary conditions.

R e f e r e n c e s

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