ON SOME WEIGHED INEQUALITIES OF THE QUALITATIVE THEORY OF ELLIPTIC EQUATIONS

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The questions of regularity, Harnack's inequality and a Green's function estimations for the degenerated elliptic partial differential equations lead to consideration of the Sobolev's and Poincare's type inequalities with weights [1,2].

The study of the quantity properties of the spectrum of Shrodinger operator may lead to fractional analogous of such inequalities [3].

In this paper we have proved the weighted inegualities of Hardy-Sobolev's type for differences. Such a form of inequalities for one dimensional case was established in [4] under some additional condition (see also, [3]).

Introduce the notation. By c, c_1, c_2, c_3 and β, $1 \le \beta < \infty$, δ, $0 < \delta \le 1$ we denote positive constants, the values of which are not essentially one. A conjugate exponent to p will be denoted by p', so that $\frac{1}{p} + \frac{1}{p'} = 1$. Φ is the class of nondecreasing, positive and continuous functions on $[0, \infty)$. Q, $Q_\rho^x = \{y \,|y - x| \le \rho\}$ are a balls in R^n. For a local-summable function $v : R^n \to R^1_+$ and a subset $E \subset R^n$ we denote $v(E) = \int\limits_E v dx$. For a function of two variables ψ and subsets E, F we denote $\psi(E,F) = \int\int_{E\times F} \psi dxdy$. $|E|$ is the Lebesque measure of the set E.

We say, that function v belongs to A_∞ class, if there are positive constants c_1, c_2, δ, β such that for any subset E of the arbitrary ball Q holds inequality

$$c_1 \left(\frac{|E|}{|Q|}\right)^\beta \le \frac{v(E)}{v(Q)} \le c_2 \left(\frac{|E|}{|Q|}\right)^\delta .$$

The function of two variables ψ we apply to $\widetilde{A}_\infty$ class if, there exist positive constants c, β such that for any ball $Q \subset R^n$ and its subsets E, F with $E \cap F = \emptyset$, $|E| > \gamma\,|Q|\,,\ |F| > \gamma\,|Q|\,,\ \gamma \in (0, \frac{1}{2^n})$ the inequality $\psi(E,F) \ge c\gamma^\beta \psi(Q,Q)$ takes place. $C_0(R^n)$ is a class of continuous functions vanishing at infinity.

The main result of this note is as follows.

Theorem. *Let* $1 \le p \le q \le r < \infty$, $\lambda \in \Phi$, $v \in A_\infty$, $\psi \in \widetilde{A}_\infty$ *and,* $\omega : R^n \times R^n \to [0,\infty)$ *be a local-summable function.*

If there exists $A > 0$ *such that, for any ball* Q *the following condition satisfies*

$$\frac{v(Q)^{1/r}\lambda(c_0|Q|^{1/n})}{\psi(Q,Q)} \left[\int\limits_Q dx \left(\int\limits_Q (\frac{\psi^p(x,y)}{\omega(x,y)})^{p'-1} dy\right)^{q'/p'}\right]^{1/q'} \le A. \qquad (1)$$

Then for any function $u \in C_0^1(R^n)$ the following inequality holds

$$\left(\int\limits_{R^n} |u|^r \, v dx\right)^{1/r} \leq cA \left[\int\limits_{R^n} dx \left(\int\limits_{R^n} \frac{||u(x)| - |u(y)||^p}{\lambda(|x-y|)^p} \omega(x,y) dy\right)^{q/p}\right]^{1/q}, \quad (2)$$

here c_0, c are positive constants, c depends on n, p, q, r and on constants in the A_∞, $\widetilde{A}_\infty$ of functions v, ψ respectively, c_0-only on the dimension n, $n \geq 1$.

Proof. Let $\alpha > 0$ be fixed. Denote $e_\alpha = \{x \in R^n : |u(x)| > \alpha\}$. Then e_α is an open and bounded set; $d_\alpha = diam e_\alpha < \infty$.

There exists a number $\rho(x,\alpha)$, $0 < \rho(x,\alpha) < 2d_\alpha$ such that, for any point $x \in e_{2\alpha}$ it is hold

$$\left|Q^x_{\rho(x,\alpha)} \backslash e_\alpha\right| = \gamma \left|Q^x_{\rho(x,\alpha)}\right|, \quad (3)$$

where $0 < \gamma < \frac{1}{2^n}$ will be chosen later.

Indeed, the function $F(t) = |Q^x_t \backslash e_\alpha| \,/\, |Q^x_t| - \gamma$ is continuous and less than zero when $t = \varepsilon$ and, $F(2d_\alpha) \geq 0$.

Then with help of Cauchy's theorem on continuous functions we can find some constant c, $0 < c < 2d_\alpha$ that, $F(c) = 0$. Now choosing $\rho(x,\alpha) = c$ we obtain (3).

The family of balls $\{Q^x_{\rho(x,\alpha)}\}$, $x \in e_{2\alpha}$ makes a cover of the set e_α. Then with the help of the Besicoviche's lemma [5] we can choose a subcover $\{Q^i\}$, $i \in N$ with finite multiplicity-θ_n.

Fix some ball Q^i. Taking into account $|Q^i \backslash e_\alpha| = \gamma |Q^i|$ and $v \in A_\infty$, it follows that $v(Q^i \backslash e_\alpha) \leq c\gamma^\delta v(Q^i)$. Then from the identity $v(Q^i) = v(Q^i \backslash e_\alpha) + v(Q^i \cap e_{2\alpha})$ we obtain $v(Q^i)(1 - c\gamma^\delta) \leq v(Q^i \cap e_\alpha)$, where $c\gamma^\delta < 1$.

If $|e_{2\alpha} \cap Q^i| < \gamma |Q^i|$, then taking into account $v \in A_\infty$, we find $v(Q^i \cap e_{2\alpha}) \leq c\gamma^\delta v(Q^i)$, therefore

$$v(Q^i \cap e_{2\alpha}) \leq \frac{c\gamma^\delta}{1 - c\gamma^\delta} v(Q^i \cap e_\alpha). \quad (4)$$

If $|Q^i \cap e_{2\alpha}| \geq \gamma |Q^i|$, then according to $\psi \in \widetilde{A}_\infty$ we find

$$\int\limits_{Q^i \backslash e_\alpha} dx \left(\int\limits_{Q^i \cap e_{2\alpha}} \psi(x,y) dy\right) \geq (c\gamma^\beta)^2 \psi(Q^i, Q^i)$$

Using the Holder's inequality we obtain

$$1 \leq \frac{1}{c_1 \gamma^{2\beta} \psi(Q^i, Q^i)} \left[\int\limits_{Q^i \backslash e_\alpha} \left(\int\limits_{Q^i \cap e_{2\alpha}} \omega dy\right)^{1/p} \left(\int\limits_{Q^i \cap e_{2\alpha}} (\frac{\psi^p}{\omega})^{p'-1} dy\right)^{1/p'} dx\right].$$

Since $\lambda \in \Phi$, and $\lambda(|x-y|) \leq \lambda(c_0 |Q^i|^{1/n})$ for $x \in Q^i \backslash e_\alpha$, $y \in Q^i \cap e_{2\alpha}$ it follows that

$$v(Q^i \cap e_{2\alpha}) \leq v(Q^i) \left(\frac{\lambda(c_0 |Q^i|^{1/n})}{c_1 \gamma^{2\beta} \psi(Q^i, Q^i)}\right)^r \left[\int\limits_{Q^i \backslash e_\alpha} dx \left(\int\limits_{Q^i \cap e_{2\alpha}} (\frac{\psi^p}{\omega})^{p'-1} dy\right)^{q'/p'}\right]^{r/q'} \times$$

$$\times\left[\int\limits_{Q^i\setminus e_\alpha} dx\left(\int\limits_{Q^i\cap e_{2\alpha}}\frac{\omega}{\lambda^p}dy\right)^{q/p}\right]^{r/q}\leq\frac{cA^r}{\gamma^{2\beta r}}\left[\int\limits_{Q^i\setminus e_\alpha} dx\left(\int\limits_{Q^i\cap e_{2\alpha}}\frac{\omega}{\lambda^p}dy\right)^{q/p}\right]^{r/q}. \quad (5)$$

From the (4) and (5) we conclude that

$$v(Q^i\cap e_{2\alpha})\leq\frac{c\gamma^\delta}{1-c\gamma^\delta}v(Q^i\cap e_\alpha)+$$

$$+\frac{cA^r}{\gamma^{2\beta r}}\left[\int\limits_{Q^i\setminus e_\alpha} dx\left(\int\limits_{Q^i\cap e_{2\alpha}}\frac{\omega}{\lambda^p}dy\right)^{q/p}\right]^{r/q}.$$

After summing by i and taking into account the finite multiplicity of cover $\{Q^i\}$, $i\in N$ and also that $1\leq p\leq q\leq r<\infty$ we have

$$v(e_{2\alpha})\leq\frac{c\gamma^\delta}{1-c\gamma^\delta}v(e_{2\alpha})+\frac{cA^r}{\gamma^{2\beta r}}\left[\int\limits_{R^n\setminus e_\alpha} dx\left(\int\limits_{e_{2\alpha}}\frac{\omega}{\lambda^p}dy\right)^{q/p}\right]^{r/q}.$$

By integrating the previous inequality on interval $(0,\infty)$ we find

$$\int\limits_0^\infty v(e_{2\alpha})d\alpha^r\leq\frac{c\gamma^\delta}{1-c\gamma^\delta}\int\limits_0^\infty v(e_\alpha)d\alpha^r+$$

$$+\frac{cA^r}{\gamma^{2\beta r}}\left[\int\limits_{R^n} dx\left(\int\limits_{R^n}\frac{||u(x)|-|u(y)||^p}{\lambda(|x-y|)^p}\omega(x,y)dy\right)^{q/p}\right]^{r/q}.$$

Now choosing the γ such that $\frac{1}{2^r}-\frac{c\gamma^\delta}{1-c\gamma^\delta}>0$, and by $\int\limits_0^\infty v(\alpha)d\alpha^r=\int\limits_{R^n}|u|^r\,vdx$ we complete the proof.

Corollary. Let $1\leq p\leq q\leq r<\infty$, functions $\frac{1}{v}$, λ belong to class Φ, moreover v be locally integrable on interval $(0,\infty)$.

Suppose that

$$\sup_{h>0}\lambda(h)\left(\int\limits_0^h v(s)ds\right)^{1/r}h^{-1/p-1/q}=\mu<\infty,$$

then for any function $u\in C_0(0,\infty)$ there holds the estimate

$$\left(\int\limits_0^\infty v|u|^r dt\right)^{1/r}\leq c_0\mu\left[\int\limits_0^\infty\left(\int\limits_0^\infty\left(\frac{|u(x)-u(t)|}{\lambda(|x-t|)}\right)^p dy\right)^{q/p}\right]^{1/q}.$$

The proof of corollary follows from the just proved theorem by setting $\psi\equiv\omega\equiv1$ in the case $n=1$.

R e f e r e n c e s

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