# ON A VIBRATION OF AN ELASTIC CUSPED BARS

S. Kharibegashvili
A. Razmadze Mathematical Institute of the Georgian Academy of Sciences

G. Jaiani
I. Vekua Institute of Applied Mathematics
Tbilisi State University

Let us consider the following hyperbolic equation of the second order

$$Lu := \rho(x)h(x)\,\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(h(x)\frac{\partial u}{\partial x}\right) + a_1\,\frac{\partial u}{\partial t} + a_2\,\frac{\partial u}{\partial x} + a_3 u = F, \tag{1}$$

where the coefficients and the right hand part of equation (1) are defined in the rectangle

$$D := \left\{(x,t) \in R^2 \,:\, 0 < x < 1,\ 0 < t < T\right\},$$

and

$$\rho \in C[0,1], \quad h \in C[0,1] \cap C^1(0,1]; \quad a_1,\, a_2 \in C^1(\bar{D}), \quad a_3 \in C(\bar{D}),$$

$$\rho(x) > \rho_0 = const > 0, \quad h_1 x^\alpha \le h(x) \le h_2 x^\alpha, \quad 0 \le x \le 1, \tag{2}$$

$\alpha = const > 0$, $h_i = const > 0$, $i = 1, 2$. By virtue of (2) we have $h(0) = 0$, i.e., equation (1) has an order degeneration on the left side $\Gamma_0 : x = 0,\ 0 \le t \le T$ of the rectangle $D$. Let $\Gamma_1 : x = 1,\ 0 \le t \le T$, and $\Gamma_{*\alpha} := \begin{cases} \Gamma_0 \cup \Gamma_1 \ if\ \alpha < 1, \\ \Gamma_1 \qquad if\ \alpha \ge 1, \end{cases}$

**Problem 1.** Find a solution $u(x,t)$ of equation (1) satisfying the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad 0 \le x \le 1, \tag{3}$$

and the boundary condition

$$u|_{\Gamma_{*\alpha}} = 0. \tag{4}$$

Parallel with the problem (1), (3), (4) let us consider the problem of finding the solutions of

$$L^* v := \rho(x)h(x)\,\frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x}\left(h(x)\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial t}(a_1 v) - \frac{\partial}{\partial x}(a_2 v) + a_3 v = F \tag{5}$$

with the initial and boundary conditions as follows

$$v(x,T) = 0, \quad \frac{\partial v(x,T)}{\partial t} = 0, \quad 0 \le x \le 1, \tag{6}$$

$$v|_{\Gamma_{*\alpha}} = 0, \tag{7}$$

where $L^*$ is the operator formally adjoint to the operator $L$. Below it will be shown that the problems (1), (3), (4) and (5), (6), (7) are adjoint in corresponding function spaces.

**Definition 1.** Denote by $E$ and $E^*$ subspaces of the space $C^2(\bar{D})$ satisfying initial boundary value conditions (3), (4) and (6), (7) correspondingly. Let $W_+(W_+^*)$ be Hilbert spaces obtained from spaces $E(E^*)$ by closure with the norm

$$||u||^2_{W_+(W_+^*)} := \int_D h(x)\left[u^2 + \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right] dD.$$

Denote by $W_-(W_-^*)$ spaces with negative norms constructed by $L_2(D)$ and $W_+(W_+^*)$.

Let

$$||v||^2_{L_{2,h}} := \int_D h(x)u^2 dD,$$

**Definition 2.** Let $F \in L_{2,h^{-1}}(D)$. A function $u \in W_+$ will be called a strong generalized solution of the IBVP (1), (3), (4) if there exists a sequence $u_n \in E$ such that

$$\lim_{n\to+\infty} ||u_n - u||_{W_+} = 0, \quad \lim_{n\to+\infty} ||Lu_n - F||_{W_-^*} = 0.$$

**Theorem 1.** *Let*

$$M_i := \sup_D |h^{-1}(x)a_i(x,t)| < +\infty, \quad i = 1, 2; \tag{8}$$

$$M_4 := \sup_D |\sigma_1^{-1}(x)a_3(x,t)| < +\infty, \tag{9}$$

$$M_5 = \sup_D \left|\sigma_1^{-1}(x)\left(\frac{\partial a_1}{\partial t} + \frac{\partial a_2}{\partial x} - a_3\right)(x,t)\right| < +\infty, \tag{10}$$

$\sigma_1(x) := x^{\alpha-1}|\ln\frac{1}{2}x|^{-\frac{1}{2}-\varepsilon_0}$ *by* $\alpha \neq 1$ *and* $\sigma_1(x) := |\ln\frac{1}{2}x|^{-1-\varepsilon_0}$ *by* $\alpha = 1$; *let* $\varepsilon_0 =$*const*$> 0$ *be arbitrary small. Then for any* $F \in L_{2,h^{-1}}(D)$, *there exists a unique strong generalized solution* $u \in W_+$ *of IBVP (1), (3), (4), such that*

$$c||u||_{L_{2,h}} \le ||F||_{W_-^*} \tag{11}$$

*with positive constant* $c$ *independent of* $F$.

**Theorem 2.** *Let (8)-(10) be satisfied. Then for any* $F \in W_-^*$, *there exists a unique strong generalized solution* $u \in L_{2,h}$ *of the IBVP (1), (3), (4) such that (11) is fulfilled.*

**Remark 1.** In view of (2) norms

$$||\cdot||_{W_+(W_+^*)} \quad \textit{and} \quad ||u||^2 := \int_D x^\alpha \left[u^2 + \left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right] dD, \tag{12}$$

$$||u||^2 := \int\limits_D x^\alpha \left[ \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial u}{\partial x} \right)^2 \right] dD,$$

and

$$||u||^2 := \int\limits_D h(x) \left[ \left( \frac{\partial u}{\partial t} \right)^2 + \left( \frac{\partial u}{\partial x} \right)^2 \right] dD$$

are equivalent.

**Remark 2.** In virtue of remark 1, the metric (12) according to [1] ensures in the sense of traces conservation of the zero boundary data (4). When $\alpha \geq 1$, the zero boundary data on $\Gamma_0 : x = 0,\ 0 \leq t \leq T$ are not preserved. They are preserved only on the lateral side $\Gamma_1 : x = 1,\ 0 \leq t \leq T$ of the domain $D$.

Let us now consider the (0,0) approximation of a bar model constructed in [2] (see also [3]). Then the solving system will have the form

$$(h_2 h_3 v_{j,1}(x_1,t))_{,1} + \overset{0,0}{Y_j} = \Lambda_j^{-1} \rho h_2 h_3 \frac{\partial^2 v_j(x_1,t)}{\partial t^2},\ j = 1,2,3, \tag{13}$$

$$v_j(x_1,t) := \frac{\overset{0,0}{w_j}(x_1,t)}{h_2(x_1) h_3(x_1)}, \qquad \overset{0,0}{Y_1} := \frac{\overset{0,0}{X_1^0}}{\lambda + 2\mu}, \qquad \overset{0,0}{Y_i} := \frac{\overset{0,0}{X_i^0}}{\mu}, \qquad i = 2,3.$$

Let us consider the static case of (13). Obviously,

$$v_j(x_1) = -\int\limits_{x_1^0}^{x_1} \frac{d\tau}{h_2(\tau)h_3(\tau)} \int\limits_{x_1^0}^{\tau} \overset{0,0}{Y_j}(t)dt + c_1^j \int\limits_{x_1^0}^{x_1} \frac{d\tau}{h_2(\tau)h_3(\tau)} + c_2^j, \tag{14}$$

$$x_1^0 = const \in ]0, L[, \quad c_\alpha^j = const, \quad \alpha = 1,2, \quad j = 1,2,3.$$

Let further

$$v_j(0) = \varphi_j^0 \quad if \quad I_0 < +\infty, \quad j = 1,2,3, \tag{15}$$

$$v_j(L) = \varphi_j^L \quad if \quad I_L < +\infty, \ j = 1,2,3, \tag{16}$$

$$v_j(x_1) = O(1), \quad x_1 \to 0+ \quad if \ I_0 = +\infty, \ j = 1,2,3, \tag{17}$$

$$v_j(x_1) = O(1), \quad x_1 \to L- \quad if \ I_L = +\infty, \ j = 1,2,3, \tag{18}$$

$$\overset{0,0}{X}_{1j}(0) = \Lambda_j h_2 h_3 v_{j,1}|_{x_1=0} = \psi_j^0, \quad if \ I_0 \leq +\infty, \quad j = 1,2,3, \tag{19}$$

$$\overset{0,0}{X}_{1j}(L) = \Lambda_j h_2 h_3 v_{j,1}|_{x_1=L} = \psi_j^L, \quad if \ I_L \leq +\infty, \quad j = 1,2,3, \tag{20}$$

$$I_0 := \int\limits_0^\varepsilon h_2^{-1}(\tau) h_3^{-1}(\tau) d\tau, \ \ I_L := \int\limits_{L-\varepsilon}^L h_2^{-1}(\tau) h_3^{-1}(\tau) d\tau, \ \ \varepsilon = const > 0,$$

$\varphi_j^0,\ \psi_j^0,\ \varphi_j^L,\ \psi_j^L,\ j = 1,2,3$, are given constants.

Let $\overset{0,0}{Y_j} \in L([0,L])$, and if $I_0 = +\infty$ ($I_L = +\infty$), let it be such that the iterated integral in (14) is bounded by $x_1 \to 0+$ ($L-$).

If $\dfrac{1}{h_2 \cdot h_3}$ is locally summable in $]0, L[$, then from (14) it follows that for regular solutions ($v_j \in C^2(]0, L[)$) only the following problems are well-posed: (13), (15), (16) ($v_j \in C([0, L])$); (13), (17), (16) ($v_j \in C(]0, L])$); (13), (15), (18) ($v_j \in C([0, L[)$); (13), (17), (18) ($v_j$ is bounded); (13), (15), (20) ($v_j \in C([0, L[)$, $h_2h_3v_{j,1} \in C(]0, L])$; (13), (19), (16) ($v_j \in C(]0, L])$, $h_2h_3v_{j,1} \in C([0, L[)$; the mixed problems when on the one end of the bar for some displacement components the Dirichlet and for others either the Neumann or (17), (18) type conditions are given but on the other end for first components either the Neumann or the Dirichlet or (17), (18) type conditions and for the second components the Dirichlet conditions (the Neumann and (17), (18) type conditions are not admissible) are given, are well-posed.

In the case (13), (17), (18) we have got the solution up to the rigid-body translation ($c_1^j = 0$, $c_2^j$, $j = 1, 2, 3$, are arbitrary). It should be so since in this case the bar under consideration as a three-dimensional body is loaded only by surface forces (which are included in $\overset{0,0}{Y_j}$) acting on the lateral surfaces of the bar. All other BVPs have unique solutions and constants $c_\alpha^j$, $\alpha = 1, 2$, $j = 1, 2, 3$, can be explicitly calculated.

**Remark 3.** Let

$$2h_i(x_1) = h_0^i x_1^{\varkappa_i}(L - x_1)^{\delta_i}, \quad h_0^i = const > 0, \quad \varkappa_i, \ \delta_i = const \geq 0, \quad i = 2, 3.$$

Then we can give transparent geometric interpretation of setting of above BVPs. To $I_0(I_L) < +\infty$ there corresponds $\varkappa_1 + \varkappa_2 < 1$ ($\delta_1 + \delta_2 < 1$), and to $I_0(I_L) = +\infty$ there corresponds $\varkappa_1 + \varkappa_2 \geq 1$ ($\delta_1 + \delta_2 \geq 1$). If at least one of the bar profiles and projections has a sharp cusp (see the definition in [4]), i.e., $\varkappa_1 + \varkappa_2 \geq 1$ ($\delta_1 + \delta_2 \geq 1$), then at this end the components of the displacement vector can not be given. If both the profile and projection have blunt cusps (see the definition in [4]) and $\varkappa_1 + \varkappa_2 \geq 1$ ($\delta_1 + \delta_2 \geq 1$), then the same is true but for $\varkappa_1 + \varkappa_2 < 1$ ($\delta_1 + \delta_2 < 1$) the above components can be given.

If $\dfrac{1}{h_2 \cdot h_3}$ has nonsummable singularities in $]0, L[$ then only arbitrary three (but one for any fixed $j$) out of six conditions of (15), (16) can be given (in particular, it contains cases when either only (19) or only (16) can be given).

If

$$2h_i := h_i^0 x_1^{\alpha_i}, \ h_i^0, \alpha_i = const > 0, \ i = 2, 3, \tag{21}$$

equations (13) are of the type of (1) for which the conditions (8), (9), (10) are fulfilled since in case of (13) the coefficients $\alpha_i = 0, i = 1, 2, 3$, in (13). Hence, Theorems 1 and 2 are valid for (13), where $h := h_2h_3$ with (21), $F := \overset{0,0}{Y_j}$. Therefore, the edge $x_1 = 0$ of the bar can be fixed only if $\alpha = \alpha_1 + \alpha_2 < 1$.

More precisely, without loss of generality we can assume

$$L = 1, \ \ \varphi_j^0 = 0, \ \ \psi_j^L = 0, \ j = 1, 2, 3.$$

Then, if

$$\overset{0,0}{Y}_j \in W_-^* \;\; or \;\; L_{2,h^{-1}}(D),$$

there exist unique strong generalized solutions

$$v_j(x_1, t) \in W_+ \;\; or \;\; L_{2,h}, \quad j = 1, 2, 3,$$

of the initial boundary value problems (13), (21),

$$v_j \,|_{t=0} = 0, \qquad \left.\frac{\partial v_j}{\partial t}\right|_{t=0} = 0, \quad x_1 \in ]0, L[, \quad j = 1, 2, 3, \tag{22}$$

(15), (16) and (13), (21), (22), (16), (17).

**R e f e r e n c e s**

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