ON THE SOME PROPORTIES OF ONE CLASS OF GEOMETRICAL FIGURS

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1. Let $ABA'B'$- be a rectangle (see fig. 1).

fig.1

Deffinition 1. The surface Mobius-Listing's type - M_n -is obtained by meains of identification of the edges such that: A) if number n -is odd, then the point A is identified with the point B', the edge AB to the edge $B'A'$. B) if number n -is even, then the point A is identified with the point A', the edge AB with the edge $A'B'$, where n is a number of rotation of the edge relativeli axis OO' until the identification.

In these parts of the article we give parametrical represantation of the surfaces Mobius-Listing's type in following restrictions: i) middle line OO' transforms in the circle. ii) Rotation the edge is even along the middle line. In particular, if $n = 1$ this deffinition is classical Mobius band (see for example $[1-3]$

Let a domain $Q \equiv Q^{-} \cup Q^{+} \in \mathbb{R}^2$ be a rectangle, where

$$Q^{-} = \left\{ (\tau, \theta) \in \Re^2; -\tau^* \le \tau \le \tau^*; -2\pi \le \theta < 0 \right\};$$

$$Q^{+} = \left\{ (\tau, \theta) \in \Re^2; -\tau^* \le \tau \le \tau^*; 0 \le \theta < 2\pi \right\};$$

and R, τ^*- are arbitrary positive constants; $\tau^* < R$.

Theorem 1. *The transformation* $F \colon Q^* \longrightarrow M_n$,

$$F = \begin{cases} x(\tau,\theta) = \left(R + \tau \cos \dfrac{\theta n}{2} \right) \cos\theta \\ y(\tau,\theta) = \left(R + \tau \cos \dfrac{\theta n}{2} \right) \sin\theta \\ z(\tau,\theta) = \tau \sin \dfrac{\theta n}{2} \end{cases} \tag{1}$$

where $(\tau, \theta) \in Q^*$; $Q^* = Q$ *if* $n = 2m+1\,(m \in \mathbb{Z})$ *and* $Q^* = Q^+$ *if* $n = 2m\,(m \in \mathbb{Z})$ *- are parametrical represantation of* M_n. i) If $n = 2m$, then M_n is a two-sided surface (in particular M_0 - is a ring, with two radius $R-\tau^*$ and $R+\tau^*$.) ii) If $n=2m+1$, then M_n is a one-sided surface (in particular M_1 - is Mobius band (see for example [1,2]).)

Remark 1. If $n=2m+1$, then every pair of points $(\tau_0,\theta_0)\in Q^-$ and $(-\tau_0,\theta+2\pi)\in Q^+$ yields one point of M_n

Remark 2. Tangential vectors of the surface M_n -

$$\vec{r}_\tau(\tau,\theta)=\frac{\partial x}{\partial\tau}\vec{i}+\frac{\partial y}{\partial\tau}\vec{j}+\frac{\partial z}{\partial\tau}\vec{k}=\cos\frac{\theta n}{2}\cos\theta\vec{i}+\cos\frac{\theta n}{2}\sin\theta\vec{j}+\sin\frac{\theta n}{2}\vec{k};$$

$$\vec{r}_\theta(\tau,\theta)=\frac{\partial x}{\partial\theta}\vec{I}+\frac{\partial y}{\partial\theta}\vec{j}+\frac{\partial z}{\partial\theta}\vec{k}=\left[-\left(R+\tau\cos\frac{\theta n}{2}\right)\sin\theta-\frac{n\tau}{2}\sin\frac{\theta n}{2}\cos\theta\right]\vec{i}+$$
$$+\left[\left(R+\tau\cos\frac{\theta n}{2}\right)\cos\theta-\frac{n\tau}{2}\sin\frac{\theta n}{2}\sin\theta\right]\vec{j}+\frac{n\tau}{2}\cos\frac{\theta n}{2}\vec{k}$$

are orthogonal (e.i. $(\vec{r}_\tau,\vec{r}_\theta)=0$).

Remark 3. The normal vectors of the surface M_n have the following form

$$\vec{\nu}(\tau,\theta)=\vec{r}_\tau\times\vec{r}_\theta=\left[\frac{n\tau}{2}\sin\theta-\frac{\tau}{2}\sin\theta ncos\theta-R\sin\frac{n\theta}{2}\cos\theta\right]\vec{i}+$$
$$\left[-\frac{n\tau}{2}\cos\theta-\frac{\tau}{2}\sin\theta nsin\theta-R\sin\frac{n\theta}{2}\sin\theta\right]\vec{j}+\left[R\cos\frac{n\theta}{2}+\tau\cos^2\frac{n\theta}{2}\right]\vec{k}.$$

If $n=2m+1$, for every pair of points $(\tau_0,\theta_0)\in Q^-$ and $(-\tau_0,\theta+2\pi)\in Q^+$

$$\vec{\nu}(\tau_0,\theta_0)=-\vec{\nu}(-\tau,\theta_0+2\pi).$$

Corollary 1. If $n=2m, m\in Z$ then the represantation (1) is a one-to-one correspondence $Q^+\longrightarrow M_n\in\mathbb{R}^3$;

If $n=2m+1, m\in Z$ then the represantation (1) is a one-to-one correspondence $Q\longrightarrow(M_n,\vec{\nu})\in\mathbb{R}^6$.

Remark 4. For any $n\in Z$ an every point of the surface M_n is a regular point, i.e.

$$\begin{aligned}&E(\tau,\theta)=(\vec{r}_\tau(\tau,\theta),\vec{r}_\tau(\tau,\theta))=1>0;\\&F(\tau,\theta)=(\vec{r}_\tau(\tau,\theta),\vec{r}_\theta(\tau,\theta))=0;\\&G(\tau,\theta)=(\vec{r}_\theta(\tau,\theta),\vec{r}_\theta(\tau,\theta))=\left(R+\tau cos\frac{\theta n}{2}\right)^2+\frac{\tau^2n^2}{4}>0.\end{aligned}\tag{2}$$

and therefore the first quadratic form of the surface M_n is positive.

Remark 5. For any $n\in\mathbb{Z}$ every point of the surface M_n is a saddle point, i.e. - Gauss's curvature of the M_n is negative.

proof. It is known that Gauss's curvature of the M_n is

$$K=\frac{LN-M^2}{EG-F^2},$$

where E, F, G are deffined by (2) and L, M, N have the following form

$$
\begin{aligned}
&L(\tau,\theta) = \frac{[\vec{r}_{\tau\tau}(\tau,\theta)\,\vec{r}_{\tau}(\tau,\theta)\,\vec{r}_{\theta}(\tau,\theta)]}{\sqrt{EG-F^2}} = 0;\\
&M(\tau,\theta) = \frac{[\vec{r}_{\tau\theta}(\tau,\theta)\,\vec{r}_{\tau}(\tau,\theta)\,\vec{r}_{\theta}(\tau,\theta)]}{\sqrt{EG-F^2}} = \frac{Rn}{2\sqrt{\left(R+\tau cos\frac{\theta n}{2}\right)^2+\frac{\tau^2 n^2}{4}}};\\
&N(\tau,\theta) = \frac{[\vec{r}_{\theta\theta}(\tau,\theta)\,\vec{r}_{\tau}(\tau,\theta)\,\vec{r}_{\theta}(\tau,\theta)]}{\sqrt{EG-F^2}} = \sin\frac{\theta n}{2}\left[\left(R+\tau cos\frac{\theta n}{2}\right)^2 + \right.\\
&\left.\frac{\tau^2 n^2}{2}\sqrt{\left(R+\tau cos\frac{\theta n}{2}\right)^2+\frac{\tau^2 n^2}{4}}\right]
\end{aligned}
\tag{3}
$$

and therefore $K < 0$ i.e. every point of M_n is a saddle point.

2. Let $ABCD$ ɔgonal parallelepiped (see fig 2).

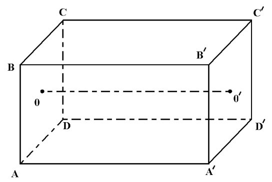

fig.2

Definition 2. Mobius-Listing's body ML_n -is obtained by means of identify of the opposite end-walls, such that: A) if number n -is odd, the point A identifies to the point C', the edge AB to the edge $C'D'$ and edge AD to the edge $C'B'$ correspondingly. B) if number n -is even, the point A identifies to the point A', the edge AB to the edge $A'B'$ and edge AD to the edge $A'D'$ correspondingly, where n is a number of rotations of the end-wall with respect to the axis OO' until the identification.

In this parts of the articl we give parametrical reprezantation of the M(bius-Listing's body in following restrictions: i) middle line OO' transforms in the circle. ii) Rotatin the end-walls are evenly along the middle line . Let a domain $\Omega \equiv \Omega^- \cup \Omega^+$ be a parallelepiped, where

$$
\begin{gathered}
\Omega^- = \left\{(\tau,t,\theta)\in\Re^3; -\tau^*\le\tau\le\tau^*; -t^*\le t\le t^*; -2\pi\le\theta<0\right\};\\
\Omega^+ = \left\{(\tau,t,\theta)\in\Re^3; -\tau^*\le\tau\le\tau^*; -t^*\le t\le t^*; 0\le\theta<2\pi\right\};
\end{gathered}
$$

and R, τ^*, t- are arbitrary positive constants; $\tau^* < R$,$t^* < R$.

Theorem 2. *The transformation* $F\colon \Omega^* \longrightarrow ML_n$,

$$F=\begin{cases} x\left(\tau,t,\theta\right)=\left(R+\tau cos\dfrac{\theta n}{2}-t\sin\dfrac{\theta n}{2}\right)cos\theta,; \\ y\left(\tau,t,\theta\right)=\left(R+\tau cos\dfrac{\theta n}{2}-t\sin\dfrac{\theta n}{2}\right)\sin\theta,; \\ z\left(\tau,t,\theta\right)=\tau\sin\dfrac{\theta n}{2}+tcos\dfrac{\theta n}{2}. \end{cases} \qquad (4)$$

where $(\tau,t,\theta)\in\Omega^*$; $\Omega^*=\Omega$ *if* $n=2m+1\,(m\in\mathbb{Z})$ *and* $\Omega^*=\Omega^+$ *if* $n=2m\,(m\in\mathbb{Z})$ - *are parametrical represantation of* ML_n. In particular ML_0 is a cylinder.

Remark 6. If $n=2m+1$, then every pair of points $(\tau_0,t_0,\theta_0)\in\Omega^-$ and $(-\tau_0,-t_0,\theta+2\pi)\in Q^+$ yields one point of ML_n

Remark 7. If $n=2m, m\in Z$ then the represantation (2) is a one-to-one correspondence $\Omega^+\longrightarrow ML_n\in\Re^3$;

Remark 8. The principal vectors of the body ML_n have the following form:

$$\vec{r}_\tau\left(\tau,t,\theta\right)=\frac{\partial x}{\partial\tau}\vec{i}+\frac{\partial y}{\partial\tau}\vec{j}+\frac{\partial z}{\partial\tau}\vec{k}=cos\frac{\theta n}{2}cos\theta\vec{i}+cos\frac{\theta n}{2}\sin\theta\vec{j}+\sin\frac{\theta n}{2}\vec{k};$$

$$\vec{r}_t\left(\tau,t,\theta\right)=\frac{\partial x}{\partial t}\vec{i}+\frac{\partial y}{\partial t}\vec{j}+\frac{\partial z}{\partial t}\vec{k}=\sin\frac{\theta n}{2}cos\theta\vec{i}-\sin\frac{\theta n}{2}\sin\theta\vec{j}+cos\frac{\theta n}{2}\vec{k};$$

$$\begin{aligned}\vec{r}_\theta\left(\tau,t,\theta\right) &= \frac{\partial x}{\partial\theta}\vec{i}+\frac{\partial y}{\partial\theta}\vec{j}+\frac{\partial z}{\partial\theta}\vec{k}\\ &= \left[-\left(R+\tau cos\frac{\theta n}{2}-t\sin\frac{\theta n}{2}\right)\sin\theta-\frac{n\tau}{2}\sin\frac{\theta n}{2}cos\theta-\frac{nt}{2}cos\frac{\theta n}{2}cos\theta\right]\vec{i}\\ &+ \left[\left(R+\tau cos\frac{\theta n}{2}-t\sin\frac{\theta n}{2}\right)cos\theta-\frac{n\tau}{2}\sin\frac{\theta n}{2}\sin\theta-\frac{nt}{2}cos\frac{\theta n}{2}\sin\theta\right]\vec{j}\\ &+ \left[\frac{n\tau}{2}cos\frac{\theta n}{2}-\frac{nt}{2}\sin\frac{\theta n}{2}\right]\vec{k}.\end{aligned}$$

These three vectors are relatively orthogonal, i.e.

$$(\vec{r}_\tau,\vec{r}_t)=0;\quad(\vec{r}_\theta,\vec{r}_t)=0;\quad(\vec{r}_\theta,\vec{r}_\tau)=0.$$

Remark 9. For every number n the surface M_n and body ML_n are refined in the torus $T\left(R,\tau^*\right)$ and $T\left(R,\sqrt{\tau^{*2}+t^{*2}}\right)$ respectively.

References

[1] Courant.R., Robbins G. What is Mathematics? Oxford University Press, New York, 1948.

[2] Korn.G.A, Korn T.M. Mathematical Handbook. McGraw-Hill Book Company, New York-San Francisco-Toronto-London-Sydney, 1968.

[3] Scwartz Laurent. Analyse Mathematique. Cours Prof. a L'ecole Politechnique, v.II. Paris, Hermann 1967.

[4] Kosnevsky I. A First Course in Algebraic Topology. Cambridge. 1980.