ON A LINEAR SECOND ORDER DEGENERATE ELLIPTIC EQUATION

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Let Oxy be the Cartesian coordinate system and G be a bounded domain with sufficiently smooth boundary ∂G, where

$$\partial G = \Gamma_1 \bigcup \Gamma_2,$$

$$\Gamma_1 = \{(x, 0);\ a \leq x \leq b\},$$

$$\Gamma_2 = \{(x, y) \in \partial\Omega;\ y > 0\}.$$

Let us consider the following equation

$$L(u) \equiv div(h(y)\, grad\, u) + cu = 0, \tag{1}$$

on the domain G, where $h \in C^{1,\alpha}(]0, \ell[) \bigcap C([0, \ell])$; $\ell := \max\limits_{(x,y)\in\Gamma_2} \{y\}$; $h(y) > 0$, when $y > 0$; $h(0) = 0$; $c \in C^{\alpha}(\bar{G})$; $c \leq 0$.

We'll consider only regular solutions of the equation (1).

The following theorem is true.

Theorem 1. *If*

$$\int\limits_0^{\ell} h^{-1}(y)dy < \infty,$$

then the Dirichlet problem for the equation (1) is correct for any continuous boundary function.

Proof. To prove the theorem it suffices to show (see [1-4]), that at any point of the boundary ∂G there exists a barrier. At the point $(x_0, 0)$ barrier has the form

$$v(x, y) = \int\limits_0^{y} (1 - t)h^{-1}(t)dt + (x - x_0)^2. \tag{2}$$

Let us verify that the function (2) satisfies all the properties of a barrier function. For this let us compute $L(v)$.

$$L(v) = -1 + 2\, h(y) + cv.$$

Therefore, because of $c \leq 0$, for sufficiently small neighbourhood of the point $(x_0;\ 0)$, we have $L(v) < 0$. The other properties of a barrier for the function $v(x,\ y)$ can be easily verified.

Let us consider the problem.

Problem E. Find a bounded solution $u(x, y)$ of the equation (1) satisfying the boundary condition

$$u|_{\Gamma_2} = \varphi,$$

where φ is an arbitrary continuous function on ∂G.

The following theorem is true.

Theorem 2. *If*

$$\int_0^\ell h^{-1}(y)\,dy = \infty,$$

then the problem E for the equation (1) is uniquely solvable.

Proof. Let us consider the function

$$\omega = \int_y^\ell (1+t)h^{-1}(t)dt + k,$$

where $k > 0$ and such that $\omega > 0$ in G.

Hence we get

$$L(\omega) = -1 + c\omega < 0$$

in G.

Let $u(x,y)$ be an arbitrary bounded solution of the equation (1), which satisfies the following boundary condition

$$u|_{\Gamma_2} = 0.$$

Let $(x_1; y_1) \in G$, then fore any small $\varepsilon > 0$ we can write

$$-\varepsilon\,\omega(x_1,\,y_1) \le u(x_1,\,y_1) \le \varepsilon\,\omega(x_1,\,y_1).$$

Hence we'll have $u(x,\,y) \equiv 0$ in G.

Remark. If in (1) $c \equiv 0$ and u depends only on y, such critiria for corresponding boundary value problems for obtained ordinary differential equation are given in [5].

R e f e r e n c e s

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[5] Jaiani G.V., On a Model of a Bar with Variable Thickness. Bulletin of TICMI, 2, 1998, 36-40.