THE INVESTIGATION OF THE STABILITY OF THE MULTISTEP METHOD FOR THE OPERATOR VALUED ABSTRACT PARABOLIC EQUATION WITH THE USE OF ASSOCIATIVE POLYNOMIALS

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Let us consider the Cauchy problem in the Hilbert space H:

$$u'(t) + A(t)u(t) = f(t), \quad t > 0, \quad u(0) = \varphi, \tag{1}$$

where A is a self-adjoint positive definite operator with domain of definition $D(A)$ that is independent of t; $f(t)$ is an abstract function with values from the space H; φ is an initial vector; $u(t)$ is the sought for function.

Consider the multistep method for problem (1), which is described by the following scheme:

$$\sum_{i=0}^{q} \alpha_i u_{k-i} + \tau A(t_k) u_k = \tau f(t_k), \tag{2}$$

where

$$k = q, q+1, \cdots, (q \geq 2), \quad t_k = k \cdot \tau \quad (\tau > 0), \quad \alpha_0 > 0,$$
$$\sum_{l=1}^{q} \alpha_i = 0.$$

Theorem. *Assume the following :*
a) $A(t)$ is a self-adjoint positive definite operator for any t on $[0, +\infty[$;
b) for any t', t'' and s from $[0, +\infty[$, the following inequality holds

$$\|[A(t') - A(t'')]A^{-1}(s)\| \leq c_0 |t' - t''|, \quad c_o = const > 0;$$

c) for any x on $[0, 1]$ all the roots of the characteristic equation

$$\alpha_0 \lambda^q + \alpha_1 x^{q-1} \lambda^{q-1} + \cdots + \alpha_{q-1} x \lambda + \alpha_q = 0 \tag{3}$$

possibly, except one root, belong to the some circle, which is settled within the unit circle and the exceptional root belongs to the unit circle, or all the roots of the equation (3), possibly, with the exception of two of them, belong to the same circle, which is settled within the unit circle, and the residual roots $\lambda_1(x)$ and $\lambda_2(x)$ belong to the unit circle and satisfy the following conditions: 1)$\lambda_1(x) = \overline{\lambda_2(x)}$ or both are real; 2) there is a number $0 < \delta < 1$ that is independent of x, the distances from the point $(\lambda_1 + \lambda_2; \lambda_1\lambda_2)$ to the points $(2; 1)$ and $(-2; 1)$ is more or equal to δ. Then the true estimate for the scheme (2) is the following one:

$$\|u_{k+q-1}\| \leq c[e^{ct_{k-1}}(\|u_0\| + \|u_1\| + \cdots + \|u_{q-1}\|) + \tau \sum_{i=1}^{k} e^{ct_{k-i}} \|f(t_{i+q-1})\|] \tag{4}$$

where $k = q, q+1, \cdots, (q \geq 2), \quad c = const > 0.$

The scheme of the proof.

From (2) it follows:

$$u_k = \beta_1 L_k u_{k-1} + \beta_2 L_k u_{k-2} + \cdots + \beta_q L_k u_{k-q} + \tau \beta_0 L_k f(t_k), \tag{5}$$

where

$$\beta_0 = \frac{1}{\alpha_0}, \quad \beta_i = -\frac{\alpha_i}{\alpha_0}, \quad i = 1, 2, \cdots, q,$$
$$L_k = (I + \tau \beta_0 A(t_k))^{-1}, \quad k = q, q+1, \cdots$$

By the method of induction the following equation is obtained from (5):

$$\begin{aligned} u_{k+q-1} = {} & (\beta_1 \overset{q}{U}_{k-1} L_q + \beta_2 \overset{q+1}{U}_{k-2} L_{q+1} + \cdots + \beta_q \overset{2q-1}{U}_{k-q} L_{2q-1}) u_{q-1} + \\ & + (\beta_2 \overset{q}{U}_{k-1} L_q + \beta_3 \overset{q+1}{U}_{k-2} L_{q+1} + \cdots + \beta_q \overset{2q-2}{U}_{k-q+1} L_{2q-2}) u_{q-2} + \\ & + (\beta_3 \overset{q}{U}_{k-1} L_q + \beta_4 \overset{q+1}{U}_{k-2} L_{q+1} + \cdots + \beta_q \overset{2q-3}{U}_{k-q+2} L_{2q-3}) u_{q-3} + \\ & + \cdots + \beta_q \overset{q}{U}_{k-1} L_q u_0 + \tau \beta_0 \sum_{i=1}^{k} \overset{i+q-1}{U}_{k-i} L_{i+q-1} f(t_{i+q-1}), \end{aligned} \tag{6}$$

where $\overset{i}{U}_k$ operators are determined by the following recurrent dependence:

$$\overset{i}{U}_k = \beta_1 L_{k+i} \overset{i}{U}_{k-1} + \beta_2 L_{k+i} \overset{i}{U}_{k-2} + \cdots + \beta_q L_{k+i} \overset{i}{U}_{k-q}, \tag{7}$$

$$\overset{i}{U}_0 = I, \quad \overset{i}{U}_{-1} = \overset{i}{U}_{-2} = \cdots = \overset{i}{U}_{1-q} = 0.$$

If the operators $\overset{i}{U}_k$ are uniformly bounded, then according to equation (6) a priori estimate (4) holds,thereby the scheme (2) is stable, and on the contrary, if the scheme (2) is stable, then the operators $\overset{i}{U}_k$ are uniformly bounded (in this case we assume that t is varying in a finite interval).

Let's see if the $\overset{i}{U}_k$ operators are uniformly bounded when the conditions of the theorem hold. To prove this fact the operator-valued polynomials are necessary, which are determined by the following recurrent dependence:

$$\begin{gathered} U_k(\beta_1 L, \beta_2 L, \cdots, \beta_q L) = \beta_1 L U_{k-1}(\beta_1 L, \beta_2 L, \cdots, \beta_q L) + \\ + \beta_2 L U_{k-2}(\beta_1 L, \beta_2 L, \cdots, \beta_q L) + \cdots + \beta_q L U_{k-q}(\beta_1 L, \beta_2 L, \cdots, \beta_q L), \\ U_0 = I, \quad U_{-1} = U_{-2} = \cdots = U_{1-q} = 0, \\ L = (I + \tau \beta A(t_j))^{-1}. \end{gathered} \tag{8}$$

These polynomials are called polynomials associated to the scheme (2).

The scheme (2) for an operator A which does not depend on t with the use of associated polynomials was investigated in [6].

The operators $\overset{i}{U}_k$ are related to the operator-valued polynomials $U_k(\beta_1 L, \beta_2 L, \cdots, \beta_q L)$ by the following equation:

$$\overset{q+j-1}{U}_k = U_k - \tau\beta_0 \sum_{i=1}^{k} \overset{i+q+j-1}{U}_{k-i} L_{i+q+j-1}[A(t_{i+j}) - A(t_j)]U_i. \tag{9}$$

The following estimates hold:

$$\|U_k(\beta_1 L, \beta_2 L, \cdots, \beta_q L)\| \leq c; \tag{10}$$

$$\|U_k(\beta_1 S^{q-1}, \beta_2 S^{q-2}, \cdots, \beta_q I)\| \leq c; \tag{11}$$

$$\|\tau A(t_j) S^k\| \leq \frac{q}{\beta_0 k}, \tag{12}$$

where $S = L^{\frac{1}{q}}$, $\quad c = const > 0$.

Taking into account

$$U_k\left(\beta_1 L, \beta_2 L, ..., \beta_q L\right) = L^{\frac{k}{q}} U_k\left(\beta_1 S^{q-1}, \beta_2 S^{q-2}, ..., \beta_q I\right),$$

according to inequalities (11), (12) and the conditions of the theorem the following is true:

$$\|(A(t_{k+j}) - A(t_j))U_k(\beta_1 L, \beta_2 L, \cdots, \beta_q L)\| \leq$$

$$\leq \|(A(t_{k+j}) - A(t_j))A^{-1}(t_j)\| \|A(t_j) L^{\frac{k}{q}}\| \|U_k(\beta_1 S^{q-1}, \beta_2 S^{q-2}, \cdots, \beta_q I)\| \leq$$

$$\leq c_0 \frac{t_{k+j} - t_j}{\tau} \cdot \frac{q}{\beta_0 k} \cdot c = c, \quad c = const > 0. \tag{13}$$

With the help of estimates (10) and (13) from (9), we obtain

$$\| \overset{q+j-1}{U}_k \| \leq c + c\tau \sum_{i=1}^{k} \| \overset{i+q+j-1}{U}_{k-i} \|. \tag{14}$$

Applying the discrete analog of the Gronwall lemma the following estimate can be obtained:

$$\| \overset{i+q+j-1}{U}_{k-i} \| \leq c e^{c t_{k-i}}. \tag{15}$$

Considering this estimate a priori estimate (4) follows from equation (6).

The proof is complete.

To the investigation of multistep method for an abstract parabolic equation are dedicated the papers [1] – [5].

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