BOUNDARY VALUE PROBLEMS IN (1.0) APPROXIMATION OF A MATHEMATICAL MODEL OF BARS[1]

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Generalizing an idea of I. Vekua [1] of constructing a version of the theory of plates and shells, in the elastic bar model suggested in [2,3] fields of displacements, strains and stresses of the three-dimensional theory of linear elasticity have been expanded into orthogonal double Fourier-Legendre series with respect to the variables along thickness, and width of the bar with a variable rectangular cross-section.

Now let us consider symmetric ($2\bar{h}_i = 0,\ i = 2,3$) bar in (1.0) approximation of the above model. The corresponding full system of equations has the following form:

$$(\lambda + 2\mu)(h_2 h_3 \overset{0,0}{v}_{1,1})_{,1} + 3\lambda(h_2 h_3 \overset{1,0}{v}_{3})_{,1} + \overset{0,0}{X_1^0} = 0, \tag{1}$$

$$\mu(h_2 h_3^3 \overset{1,0}{v}_{3,1})_{,1} - 3(\lambda + 2\mu) h_2 h_3 \overset{1,0}{v}_{3} - \lambda h_2 h_3 \overset{0,0}{v}_{1,1} + h_3 \overset{1,0}{X_3^0} = 0, \tag{2}$$

$$\mu(h_2 h_3 \overset{0,0}{v}_{2,1})_{,1} + \overset{0,0}{X_2^0} = 0, \tag{3}$$

$$\mu(h_2 h_3^3 \overset{1,0}{v}_{2,1})_{,1} - 3\mu h_2 h_3 \overset{1,0}{v}_{2} + h_3 \overset{1,0}{X_2^0} = 0, \tag{4}$$

$$\mu(h_2 h_3 \overset{0,0}{v}_{3,1})_{,1} + 3\mu(h_2 h_3 \overset{1,0}{v}_{1})_{,1} + \overset{0,0}{X_3^0} = 0, \tag{5}$$

$$(\lambda + 2\mu)(h_2 h_3^3 \overset{1,0}{v}_{1,1})_{,1} - \mu h_2 h_3 \overset{0,0}{v}_{3,1} - 3\mu h_2 h_3 \overset{1,0}{v}_{1} + h_3 \overset{1,0}{X_1^0} = 0, \tag{6}$$

where

$$\overset{r,s}{v}_k(x_1) := \frac{\overset{r,s}{u}_k(x_1)}{h_2^{r+1}(x_1) h_3^{s+1}(x_1)}$$

λ and μ are Lamé's constants, index 1 after comma means differentiation with respect to x_1,

$$\overset{n_3,n_2}{X_j^0} := \int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2} \left[\sqrt{1 + \left(\overset{(+)}{h}_{3,1}\right)^2}\, \overset{(+)}{Q}{}_j^{h_3}(x_1, x_2, \overset{(+)}{h}_3) + (-1)^{n_3} \sqrt{1 + \left(\overset{(-)}{h}_{3,1}\right)^2} \times \right.$$

$$\left. \times \overset{(-)}{Q}{}_j^{h_3}(x_1, x_2, \overset{(-)}{h}_3) \right] P_{n_2}(a_2 x_2 - b_2) dx_2 + \int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3} \left[\sqrt{1 + \left(\overset{(+)}{h}_{2,1}\right)^2}\, \overset{(+)}{Q}{}_j^{h_2}(x_1, \overset{(+)}{h}_2, x_3) + \right.$$

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$$+(-1)^{n_3}\sqrt{1+\left(\overset{(-)}{h}_{2,1}\right)^2}\ \overset{(-)}{Q}{}_j^{h_3}(x_1,\overset{(-)}{h}_2,x_3)\Bigg]\ P_{n_3}(a_3x_3-b_3)dx_3+\overset{n_3,n_2}{X_j},\quad i=1,2,3,$$

$\overset{(+)}{Q}{}_j^{h_3}$, $\overset{(-)}{Q}{}_j^{h_3}$, $\overset{(+)}{Q}{}_j^{h_2}$, $\overset{(-)}{Q}{}_j^{h_2}$ are components of surface forces acting on surfaces $\overset{(+)}{h}_3$, $\overset{(-)}{h}_3$, $\overset{(+)}{h}_2$, $\overset{(-)}{h}_2$ respectively; the bar occupies the following domain

$$B:=\{(x_1,x_2,x_3):\ \ 0<x_1<l,\ \ \overset{(-)}{h}_i(x_1)<x_i<\overset{(+)}{h}_i(x_1),\ \ i=2,3,\ \ l=const\};$$

$$2h_i(x_1):=\overset{(+)}{h}_i(x_1)-\overset{(-)}{h}_i(x_1)\ge 0,\ \ 2\bar{h}_i(x_1):=\overset{(+)}{h}_i(x_1)+\overset{(-)}{h}_i(x_1)\ge 0,\ \ i=2,3;\ \ h_i\in C^1;$$

$$\overset{n_3,n_2}{X_j}(x_1):=\int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2}\int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3}X_j(x_1,x_2,x_3)P_{n_2}(a_2x_2-b_2)P_{n_3}(a_3x_3-b_3)dx_2dx_3,\ \ j=1,2,3,$$

X_j are volume forces; $P_n(\cdot)$ are Legendre Polynomials;

$$a_i(x_1):=\frac{1}{h_i(x_1)},\quad b_i(x_1):=\frac{\bar{h}_i(x_1)}{h_i(x_1)},\quad i=2,3;$$

$$\overset{n_3,n_2}{u_j}(x_1):=\int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2}\int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3}u_j(x_1,x_2,x_3)P_{n_2}(a_2x_2-b_2)P_{n_3}(a_3x_3-b_3)dx_2dx_3,\ \ j=1,2,3,$$

$u_j(x_1,x_2,x_3)$ are components of the displacement vector.

Boundary value problems for equation (3) have been solved in [2]. Therefore for this equation e.g. the Dirichlet problem is correct if only $\overset{0,0}{I}_0<\infty$, $\overset{0,0}{I}_\ell<\infty$. For (4) the Dirichlet problem is correct if only $\overset{1,0}{I}_0<\infty$, $\overset{1,0}{I}_\ell<\infty$ (see also [4],[5]), where

$$\overset{N_3,N_2}{I_0}:=\int\limits_o^\varepsilon h_2^{-2N_2-1}(\tau)h_3^{-2N_3-1}(\tau)d\tau,\ \ \varepsilon=const>0,$$

$$\overset{N_3,N_2}{I_\ell}:=\int\limits_{\ell-\varepsilon}^{\ell}h_2^{-2N_2-1}(\tau)h_3^{-2N_3-1}(\tau)d\tau,\ \ \varepsilon=const>0.$$

Hence the approximate value of

$$u_2(x_1,x_2,x_3)\cong\frac{1}{4}\overset{0,0}{v}_2(x_1)+\frac{3x_3}{4}\overset{1,0}{v}_2(x_1)$$

can be preassigned on the bar ends if only $\overset{1,0}{I}_0 < +\infty$, $\overset{1,0}{I}_\ell < +\infty$.

Systems (5), (6) and (1), (2) can be correspondigly reduced to the following systems

$$3\mu\overset{1,0}{v}_1 = -\mu\overset{0,0}{v}_{3,1} - h_2^{-1}(x_1)h_3^{-1}(x_1)\left(\int_{x_1^0}^{x_1}\overset{0,0}{X_3^0}dx_1 - C_1\right),\quad C_1 = const,\ x_1^0 \in]0,\ell[,$$

$$(\lambda+2\mu)(h_2h_3^3\overset{0,0}{v}_{3,11})_{,11} = 3\overset{0,0}{X_3^0} + 3(h_3\overset{1,0}{X_1^0})_{,1} - \frac{\lambda+2\mu}{\mu}\left(h_2h_3^3\left(h_2^{-1}h_3^{-1}\int_{x_1^0}^{x_1}\overset{0,0}{X_3^0}dx_1 - \right.\right.$$

$$\left.\left.-h_2^{-1}h_3^{-1}C_1\right)_{,1}\right)_{,11};$$

and

$$3\lambda\overset{1,0}{v}_3 = -(\lambda+2\mu)\overset{0,0}{v}_{1,1} - h_2^{-1}(x_1)h_3^{-1}(x_1)\left(\int_{x_1^0}^{x_1}\overset{0,0}{X_1^0}(t)dt - C_2\right),\quad C_2 = const,$$

$$(\lambda+2\mu)\mu(h_2h_3^3\overset{0,0}{v}_{1,11})_{,11} - 12(\lambda+\mu)\mu(h_2h_3\overset{0,0}{v}_{1,1})_{,1} =$$

$$= -\mu\left(h_2h_3^3\left[\left(h_2^{-1}h_3^{-1}\int_{x_1^0}^{x_1}\overset{0,0}{X_1^0}(t)dt\right)_{,1} - C_2\left(h_2^{-1}h_3^{-1}\right)_{,1}\right]\right)_{,11} + 12\frac{(\lambda+\mu)\mu}{\lambda+2\mu}\overset{0,0}{X_1^0} +$$

$$+\frac{3\lambda^2}{\lambda+2\mu}\overset{0,0}{X_1^0} + 3\lambda(h_3\overset{1,0}{X_3^0})_{,1}.$$

The advantages of these systems consist in the following: the second equations are equations with respect to a only one unknown function and after their solution (see [6]) from the first equations correspondinly $\overset{1,0}{v}_1$ and $\overset{1,0}{v}_3$ can be readily calculated.

Remark 1. Let us consider (N_3, N_2) approximation of a model of bars given in [2,3]. Further the notation of above works will be used.

Let the width and the thickness of the bar be given by $2h_i(x_1) = h_0^i x_1^{\alpha_i}$, $i = 2,3$ ($h_0^i = const > 0$, $\alpha_i = const \geq 0$), respectively.

Problem 1. *Find bounded weighted double moments* $\overset{n_3,n_2}{v_j} \in C^3(]0,\ell[) \cap C(]0,\ell])$, $j = 1,2,3$, $n_i = \overline{0,N_i}$, $i = 2,3$, *of displacement vector components* $u_j(x_1,x_2,x_3)$ *under boundary conditions:*

$$\overset{n_3,n_2}{v_j}(0) = \overset{n_3,n_2}{\varphi_j^0},\quad j=1,2,3,\ n_i = \overline{0,N_i},\ i=2,3,\quad if\ (2N_2+1)\alpha_2 + (2N_3+1)\alpha_3 < 1; \tag{7}$$

$$\overset{n_3,n_2}{v_j}(\ell) = \overset{n_3,n_2}{\varphi_j^\ell},\quad j=1,2,3,\ n_i = \overline{0,N_i},\ i=2,3,$$

provided that the lateral boundaries $(x_i = \overset{(\pm)}{h}_i(x_1),\ i=2,3,\ 0<x_1<\ell)$ *of the bar are arbitrarily loaded.* $\overset{n_3,n_2}{\varphi_j^0}$, $\overset{n_3,n_2}{\varphi_j^\ell}$, *and loadings* $\overset{(\pm)}{f_j^i}$ *are given.*

Statement. Problem 1 is correct in the sense of Hadamard.

If we consider the limit $N_i \to +\infty$, $i=2,3$, then from (7) we have

$$\alpha_i < \frac{1}{2N_i+1}, \quad and \quad \lim_{N_i\to+\infty} \alpha_i = 0, \quad i=2,3.$$

Thus, Problem 1 tends to the following three-dimensional model

Problem 2. *Find*

$$u_i(x_1,x_2,x_3) \in C^3(V) \cap \begin{cases} C(\overline{V})\ if\ \alpha_i = 0,\ i=2,3, \\ C(\overline{V}\backslash\{0,x_2,x_3\})\ if\ \alpha_2+\alpha_3>0, \end{cases}$$

bounded in the latter case, under boundary conditions

$$u_j(0,x_2,x_3) = \varphi_j^0(x_2,x_3), \quad j=1,2,3, \quad if \quad \alpha_i=0, \quad i=2,3;$$

$$u_j(\ell,x_2,x_3) = \varphi_j^\ell(x_2,x_3), \quad j=1,2,3,$$

$$X_{ij}(x_1,x_2,x_3)\Big|_{x_i=\overset{(\pm)}{h}_i(x_1)} = \overset{(\pm)}{f_j^i}(x_1,x_{5-i}), \quad j=1,2,3,\ i=2,3.$$

Remark 2. In the forthcoming paper the similar mathematical model of bars with the variable cross-section of general form will be constructed.

Remark 3. In [7] to deduce the system of one-dimensional equations of ceramic bar of uniform cross-section from the three-dimensional equations of thermopiezoelasticity, the method of power series expansion is used for the variables along the bar thickness and width. In contrast to this, by using the method of the present paper, one-dimensional equations of a ceramic bar of a variable cross-section can be deduced.

Remark 4. If we waive the assumption that ℓ is much greater than width and thickness which is necessary in case of bars then B should be understood as an essentially three-dimensional elastic body. Generalizing the idea of this paper we could expand the displacement vector components into triple Fourier-Legendre series. This approach leads in the general case to the infinite system of algebrical equations. In case of a prismatic body ($a_j = const,\ j=1,2,3$), $a_1 = \ell^{-1}$ (N_3,N_2,N_1) approximation is equivalent to the polynomial approximation of solution of three-dimensional boundary value problems of the linear theory of elasticity. In dynamical case $(0,0,0)$ approximation characterizes rigid transfer of the body under action of volume forces; (1.1.1) approximation coincides with the homogeneous deformation of the body. This approach is convinient especially in point of view of numerical solution of the three-dimensional problems.

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