A SOLID-FLUID MODEL WITH UNILATERAL CONTACT CONDITIONS

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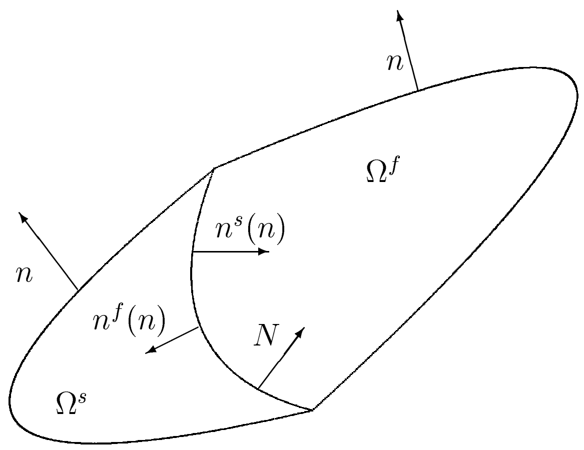
Let $\Omega = \Omega^s \cup \Omega^f \cup \Sigma''$ be a bounded domain in $\mathbb{R}^3$ with a Lipschitz boundary $\partial\overline{\Omega}$, where Ω^s and Ω^f are the subdomains occupied by the elastic solid and viscous fluid respectively, with the Lipschitz interface Σ (see Fig.1).



$$\Sigma' = \{x \in \Sigma | [u_i N_i] = 0\}, \quad u(x) := \{u_i\}_{i=1}^3 = \begin{cases} u_i^s & if \;\; x \in \Omega^s, \\ u_i^f & if \;\; x \in \Omega^f, \end{cases}$$

u_i^s and u_i^f denote components of the displacement vectors of the solid and fluid respectively, by the symbol $[\varphi]$ we denote the jump of a function $[\varphi] = = \varphi|_f - \varphi|_s$ on the interface Σ, $\Sigma'' \subset \Sigma'$.

In Ω^s we have the equations of the linear theory of elasticity:

$$\rho^s \frac{\partial^2 u_i^s}{\partial t^2} = \frac{\partial \sigma_{ij}^s}{\partial x_j} + f_i, \quad x \in \Omega^s \quad t \in [0,T], \tag{1}$$

where

$$\sigma_{ij}^s = a_{ijkh}^s e_{kh}(u^s),\ i,j = 1,2,3,\ e_{kh}(u^s) := \frac{1}{2}\left(\frac{\partial u_k^s}{\partial x_h} + \frac{\partial u_h^s}{\partial x_k}\right),\ k,h = 1,2,3, \tag{2}$$

σ_{ij}^s is the stress tensor, f is a given force and the Einstein summation convention has been used (the Latin indices run the values 1,2,3). The elastic coefficients a_{ijkh}^s are constants and they satisfy the conditions of symmetry and positive definiteness:

$$a_{ijkh}^s = a_{jikh}^s = a_{jihk}^s = a_{khij}^s, \tag{3}$$

$$a_{ijkh}^s e_{ij} e_{kh} \geq \alpha e_{ij} e_{ij},\ \ \alpha > 0,\ \ e_{ij} = e_{ji}. \tag{4}$$

In Ω^f we have the following equations

$$\rho^f \frac{\partial^2 u_i^f}{\partial t^2} = \frac{\partial \sigma_{ij}^f}{\partial x_j} + f_i, \quad x \in \Omega^s,\ \ t \in [0,T], \tag{5}$$

$$\sigma_{ij}^f = -\delta_{ij} p + (\eta \delta_{ij} \delta_{kh} + 2\mu \delta_{ik} \delta_{jh}) e_{kh}\left(\frac{\partial u^f}{\partial t}\right), \tag{6}$$

where σ_{ij}^f is the component of the stress tensor, η and μ are coefficients of viscosity satisfying the condition as follows:

$$\mu > 0,\ \ \eta/\mu > -(2/3)\alpha; \quad 0 < \alpha < 1.$$

δ_{ij} is Kronecker delta.

The symbols ρ^s and ρ^f in (1) and (5) are respectively the densities of solid and fluid in the position of a rest. Let us assume that they are constants.

p is a perturbation of pressure with respect to the position of a rest. It is connected with the perturbation of the density ρ by the following expression:

$$p = c^2 \rho, \tag{7}$$

where $c > 0$ is the velocity of sound spreading in the position of a rest of fluid.

Besides, ρ and $v^f := \dfrac{\partial u^f}{\partial t}$ are connected with each other by the continuity equation, which in the linear case has the following form:

$$\frac{\partial \rho}{\partial t} + \rho^f div v^f = 0,$$

Hence taking into account, that $\rho = 0$ when $u^f = 0$, after integration we obtain:

$$\rho + \rho^f divu^f = 0.$$

Finally, the expresion (7) takes the following form:

$$p = -c^2 \rho^f divu^f. \tag{8}$$

From expression (6) using (8) we get

$$\sigma_{ij}^f = \delta_{ij} c^2 \rho^f divu^f + (\eta \delta_{ij} \delta_{kh} + 2\mu \delta_{ik} \delta_{jh}) e_{kh} \left(\frac{\partial u^f}{\partial t} \right), \tag{9}$$

Therefore σ_{ij}^f is a function of the vector u^f as well as of the velocity vector $\dfrac{\partial u^f}{\partial t}$.

Let us consider the following boundary and initial conditions:

$$u = 0 \qquad on \;\; \partial\Omega \tag{10}$$

$$u = \frac{\partial u}{\partial t} = 0 \quad in \;\; \Omega \qquad when \;\; t = 0. \tag{11}$$

On the interface Σ we set conditions as follows:

$$[u_i N_i] \geq 0 \qquad on \;\; \Sigma; \tag{12}$$

$$\sigma_{ij}^s n_j^s = \sigma_{NN}^s N_i; \quad \sigma_{ij}^f n_j^f = -\sigma_{NN}^s N_i, \quad i = 1, 2, 3; \tag{13}$$

$$\sigma_{NN}^s \leq 0 \tag{14}$$

$$and \quad \sigma_{NN}^s = 0 \quad if \quad [u_i N_i] > 0, \tag{15}$$

where (see Fig.1) N is a unit normal vector to the interface Σ with an external direction with respect to Ω^s, n is a unit outer normal vector to $\partial(\bar{\Omega} \backslash \Sigma)$, φ, σ_{NN} is the projection on the direction N of the stress vector acting on the surface Σ.

The relation (12) means that on the interface Σ the quantity of the projection u_N^f of the vector u^f on the direction N can not be less than u_N^s; the relations (13) denote that the inner force acting on Σ is normal to Σ and the action is opposite to the reaction, (14) means that there is compression (but not traction!) on Ω^s through Σ; (15) means that if the crack is open at a point, the acting force is zero at this point.

Let us introduce the following spaces:

$$V := \{u | u_i \in H^1(\Omega^s \cup \Omega^f \cup \Sigma''); u_i|_{\partial\overline{\Omega}} = 0\} \tag{16}$$

and

$$K := \{u | u_i \in V; \; [u_i N_i] \geq 0 \; a.e. \; on \; \Sigma\}, \tag{17}$$

where $H^1(\Omega^s \cup \Omega^f \cup \Sigma'')$ is the usual Sobolev space.

Theorem 1. *K is the closed convex set.*

Proof. Let us first prove that K is closed. Let $v^m \in K,\ m = 1, 2, ...,$ and $v^m \to v$, if $m \to \infty$, by the norm of the space V. Let us prove that $v \in K$. Obviously,

$$v_N^m|_f - v_N^m|_s \geq 0 \tag{18}$$

According to the trace theorem, we have

$$v_N^m|_f \to v_N|_f, \quad and \quad v_N^m|_s \to v_N|_s,$$

so $0 \leq v_N^m|_f - v_N^m|_s \to v_N|_f - v_N|_s$, that means, that $v_N|_f - v_N|_s \geq 0$, hence $v \in K$.

Now let us prove the convexity of the set K. Let $u, v \in K$. Now, we shall prove that

$$\alpha u + \beta v \in K,\ \forall \alpha, \beta = const > 0, \ \ \alpha + \beta = 1.$$

We have

$$u_N|_f - u_N|_s \geq 0, \quad and \quad v_N|_f - v_N|_s \geq 0$$

i.e.

$$(\alpha u_N)|_f - (\alpha u_N)|_s \geq 0 \quad and \quad (\beta v_N)|_f - (\beta v_N)|_s \geq 0,$$

from which we obtain that

$$(\alpha u_N + \beta v_N)|_f - (\alpha u_N + \beta v_N)|_s \geq 0.$$

Hence

$$\alpha u + \beta v \in K.$$

Let $H := L^2(\Omega)$.

Definition. A vector-function u is called a generalized solution of the problem (1)-(6), (10)-(15) , if u satisfies the following conditions

$$\begin{aligned}
&u, u' \in L^\infty(0, T; V), \quad u'' \in L^\infty(0, T; H), \quad u' \in K,\\
&(\rho u''(t), v - u'(t)) + a(u(t), v - u'(t)) + b(u'(t), v - u'(t)) \geq (f(t), v - u'(t)),\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall v \in K,\ \forall t \in [0, T],\\
&u = 0 \quad and \quad u' = 0 \quad when \quad t = 0,
\end{aligned}$$

where $L^\infty(0, T; V) := \{f(t) | f(t) : [0, T] \to V$, is measurable and $\underset{t\in[0,T]}{ess\ sup} \|f(t)\|_v = \|f\|_{L^\infty(0,T,V)} < \infty\}$, by the symbol $(.,.)$ we denote the scalar product in H,

$$\rho = \rho(x) = \begin{cases} \rho^s & if \ \ x \in \Omega^s, \\ \rho^f & if \ \ x \in \Omega^f; \end{cases}$$

$$\begin{aligned}
a(u, w) &= \int_\Omega a_{ijkh}(x) e_{kh}(u) e_{ij}(w) dx,\\
b(v, w) &= \int_\Omega b_{ijkh}(x) e_{kh}(v) e_{ij}(w) dx;
\end{aligned}$$

$$a_{ijkh}(x) = \begin{cases} a^s_{ijkh} & if \ \ x \in \Omega^s, \\ a^f_{ijkh} \equiv \rho^f c^2 \delta_{ij}\delta_{kh} & if \ \ x \in \Omega^f; \end{cases}$$

$$b_{ijkh}(x) = \begin{cases} b^s_{ijkh} = 0 & if \ \ x \in \Omega^s, \\ b^f_{ijkh} \equiv 2\mu\delta_{ik}\delta_{jh} + \eta\delta_{ij}\delta_{kh} & if \ \ x \in \Omega^f. \end{cases}$$

Theorem 2. *The classical solution is also the generalized solution of the problem* (1)-(6), (10)-(15).

Proof. Let u be the classical solution of problem (1)-(6), (10)-(15) and let $v := \{v_i\} \in K$. If we multiply both sides of equations (1) and (5) by

$$v_i = \begin{cases} v^s_i & if \quad x \in \Omega^s, \quad i = 1,2,3, \\ v^f_i & if \quad x \in \Omega^f, \end{cases}$$

and use the formula of integration by parts respectively on Ω^s and Ω^f, we get the relations as follows

$$\int\limits_{\Omega^s} \rho^s \frac{\partial u^s_i}{\partial t^2} v^s_i dx - \int\limits_{\Sigma} n_j \sigma^s_{ij} v^s_i d\Sigma = \int\limits_{\Omega^s} f_i v^s_i dx - \int\limits_{\Omega^s} \sigma^s_{ij} \frac{\partial v^s_i}{\partial x_j} dx \tag{19}$$

and

$$\int\limits_{\Omega^f} \rho^f \frac{\partial u^f_i}{\partial t^2} v^f_i dx - \int\limits_{\Sigma} n_j \sigma^f_{ij} v^f_i d\Sigma = \int\limits_{\Omega^f} f_i v^f_i dx - \int\limits_{\Omega^f} \sigma^f_{ij} \frac{\partial v^f_i}{\partial x_j} dx \tag{20}$$

Let us sum the relations (19) and (20), and take into account the interface conditions (13), then we obtain:

$$\int\limits_{\Omega} \rho \frac{\partial^2 u_i}{\partial t^2} v_i dx + \int\limits_{\Sigma} \sigma^s_{NN}[v_i N_i] d\Sigma = \int\limits_{\Omega} f_i v_i dx - \int\limits_{\Omega} \sigma_{ij} \frac{\partial v_i}{\partial x_j} dx, \tag{21}$$

where $\rho = \begin{cases} \rho^s & if\ x \in \Omega^s, \\ \rho^f & if\ x \in \Omega^f, \end{cases}$ $\sigma_{ij}(x) = \begin{cases} \sigma^f_{ij} & if\ x \in \Omega^s, \\ \sigma^s_{ij} & if\ x \in \Omega^f, \end{cases}$ $u_i = \begin{cases} u^s_i & if\ x \in \Omega^s, \\ u^f_i & if\ x \in \Omega^f, \end{cases}$ $i = 1, 2.3.$

From (21) and (17) we get

$$\int\limits_{\Omega} \rho \frac{\partial^2 u_i}{\partial t^2} v_i dx + \int\limits_{\Omega} \sigma_{ij} \frac{\partial v_i}{\partial x_j} dx \geq \int\limits_{\Omega} f_i v_i dx.$$

Substituting here $v - u'$ instead of v the proof is fulfilled.

The following theorems can also be proved:

Theorem 3. *If the generalized solution is smooth enough then it will be also a classical solution of the problem* (1)-(6), (10)-(15).

Theorem 4. *There exists the unuque generalized solution of the problem* (1)-(6), (10)-(15).

References

[1] Sanches-Palencia E., Non-homogeneous Media and Vibration Theory. Lecture Notes in Physics 127. Springer-Verlag, New-York, 1980, (Russian).

[2] Lions J.L., Quelqes Methodes de Resolution de Problems aux Limites Non- lineares. Dunod, Gauthier-Villars, Paris, 1969, (Russian).