Is there a jump in the Weihrauch lattice?

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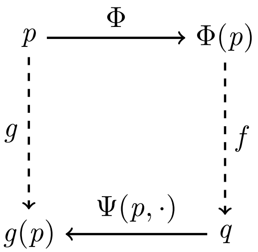
Weihrauch reducibility

Computational problem: partial multi-valued function $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$

$$\begin{aligned} \text{input} &: \text{any } x \in \operatorname{dom}(f) \\ \text{output} &: \text{any } y \in f(x) \end{aligned}$$

$g \le_{\mathrm{W}} f :\iff$ there are computable $\Phi, \Psi :\subseteq \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ s.t.

- Given $p \in \operatorname{dom}(g)$, $\Phi(p) \in \operatorname{dom}(f)$
- Given $q \in f(\Phi(p))$, $\Psi(p, q) \in g(p)$



If Ψ does not have access to p, we say that g is *strongly Weihrauch reducible to* f ($g \le_{\mathrm{sW}} f$).

More general spaces can be considered, but problems on $\mathbb{N}^\mathbb{N}$ are enough to study Weihrauch degrees.

The structure of Weihrauch degrees

Theorem (Pauly; Brattka, Gherardi)

The Weihrauch degrees are a distributive lattice with a bottom element but no top element.

Join: $(f_0 \sqcup f_1)(i, p) := f_i(p)$

Meet: $(f_0 \sqcap f_1)(p_0, p_1) := f_0(p_0) \sqcup f_1(p_1)$

Bottom: $\emptyset$

The non-existence of a "natural" top element is equivalent to a (relatively weak) form of choice.

Is there a jump in the Weihrauch lattice?

Definition (Brattka, Gherardi, Marcone)

For a problem f, we define f' as the problem that works as follows:

- input: sequence $(p_n)_{n\in\mathbb{N}}$ converging to $p \in \mathrm{dom}(f)$
- output: $f(p)$

We are just changing the representation of inputs for f!

Intuitively: to know the “true” input for f we first need to solve lim.

Historically, the name comes from:

- a basic analogy with the Turing jump for “simple” problems
- $\mathsf{lim} \equiv_{\mathrm{sW}} \mathsf{J}$, where $\mathsf{J} := p \mapsto p'$ is the Turing jump.

Is there a jump in the Weihrauch lattice?

A closer look reveals some "issues":

- the jump is not degree-theoretic: $f \equiv_W g$ does not imply $f' \equiv_W g'$
- the jump does not jump: there is f with $f \equiv_W f'$.

A counterexample for both is the constant function $c_0 := p \mapsto 0^{\mathbb{N}}$:

$$c' \equiv_W c \equiv_W \mathrm{id} <_W \mathrm{id}' \equiv_W \mathsf{lim}$$

At the higher levels, there are problems that are closed under composition with lim (e.g. $\mathsf{C}_{\mathbb{N}^{\mathbb{N}}}, \mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}$).

Is there a *better* jump in the Weihrauch lattice?

An abstract jump operator

Let $(P, \leq)$ be a partial order. A *jump operator* is a function $j\colon P \to P$ that is

1. *strictly increasing*: for every $p \in P$, $p < j(p)$
2. *weakly monotone*: for every $p, q \in P$, $p \leq q$ implies $j(p) \leq j(q)$

The structure $(P, \leq, j)$ is called *jump partial order*.

Countable JPO have been studied in the literature (Hinman and Slaman, Montalbán).

Using AC, every upper semilattice without maximum has a jump operator:
enumerate $(P, \leq, \oplus)$ as $(p_\alpha)_\alpha$ and let $j(p) := p \oplus p_\alpha$ where α is the least with $p_\alpha \not\leq p$.

In fact, every countable upper directed partial order has a jump (this does not extend to ω_1).

For countable partial orders, the existence of a jump operator is Σ^1_1-complete.

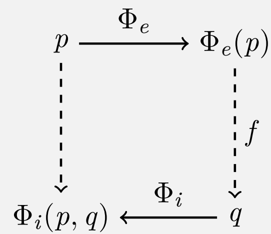
But of course we want a "natural" jump operation on the Weihrauch lattice (cardinality: $2^{\mathfrak{c}}$).

The totalizing jump

Definition

The *totalizing jump* (*tot-jump* for short) of f is defined as follows: for $e, i \in \mathbb{N}$ and $p \in \mathbb{N}^{\mathbb{N}}$,

$$\mathsf{tJ}(f)(e,i,p) := \begin{cases} \{\Phi_i(p,q) : q \in f\Phi_e(p)\} & \text{if } \Phi_e(p) \in \mathrm{dom}(f) \\ & \text{and } (\forall q \in f\Phi_e(p))(\Phi_i(p,q)\downarrow) \\ \mathbb{N}^{\mathbb{N}} & \text{otherwise} \end{cases}$$



The totalizing jump

Theorem

The map $\mathsf{tJ}(\cdot)$ is a jump operator.

Proof (Sketch)

$f \le_{\mathrm{W}} \mathsf{tJ}(f)$ and weak monotonicity are easy.

To show that $\mathsf{tJ}(f) \not\le_{\mathrm{W}} f$, define $d \colon \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ as

$$d(p)(0) := p(0) + 1 \quad \text{and} \quad d(p)(n+1) := p(n+1)$$

Observe that d is computable and, for every $X \subseteq \mathbb{N}^{\mathbb{N}}$, $X \not\subseteq d(X)$.
We define a new function $f_d \equiv_{\mathrm{W}} \mathsf{tJ}(f)$ as follows: for every $x = (e,i)^\frown p \in \mathbb{N}^{\mathbb{N}}$,

$$f_d(x) := \begin{cases} \{d\Phi_i(x,q) \,:\, q \in f\Phi_e(x)\} & \text{if } \Phi_e(x) \in \mathrm{dom}(f) \text{ and } (\forall q \in f\Phi_e(x))(\Phi_i(x,q)\downarrow), \\ \mathbb{N}^{\mathbb{N}} & \text{otherwise.} \end{cases}$$

If $f_d \le_{\mathrm{W}} f$ via Φ_e and Φ_i then consider the input $z := (e,i)^\frown 0^{\mathbb{N}}$ for f_d.
Let $X = \{\Phi_i(z,y) \,:\, y \in f\Phi_e(x)\}$. There is $q \in f\Phi_e(x)$ s.t.

$$\Phi_i(z,q) \notin d(X) = \{d\Phi_i(z,y) \,:\, y \in f\Phi_e(x)\} = f_d$$

□

The totalization of a problem

Definition

The *totalization* or *total continuation* of $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ is the total problem $\mathsf{T}f\colon \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ defined as

$$\mathsf{T}f(x) := \begin{cases} f(x) & \text{if } x \in \operatorname{dom}(f) \\ \mathbb{N}^\mathbb{N} & \text{otherwise.} \end{cases}$$

The totalization is not a degree-theoretic operator: there is a computable function with no total computable (multi-valued) extension.

However, the totalizations of known problems can be useful benchmarks ($\mathsf{TC}_\mathbb{N}$, $\mathsf{TC}_{\mathbb{N}^\mathbb{N}}$).

Observe that both the definition of the totalization and of the tot-jump are split in a "main" case and in the "otherwise" case (where the output is $\mathbb{N}^\mathbb{N}$).

The tot-jump and the totalization

For every f, define

$$f_W(e,i,p) := \{\Phi_i(p,q) \,:\, q \in f\Phi_e(p)\}$$

with $\operatorname{dom}(f_W) := \{(e,i,p) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \,:\, \Phi_e(p) \in \operatorname{dom}(f) \text{ and } (\forall q \in f\Phi_e(p))(\Phi_i(p,q)\downarrow)\}$.

In other words, f_W is just the "main" case of $\mathsf{tJ}(f)$.

Theorem

For every problem f,

$$\mathsf{tJ}(f) \equiv_{\mathrm{W}} \max_{\leq_{\mathrm{W}}}\{\mathsf{T}g \,:\, g \equiv_{\mathrm{W}} f\}$$

Proof

$\leq_{\mathrm{W}}$: $\mathsf{tJ}(f) = \mathsf{T}(f_W)$ and $f_W \equiv_{\mathrm{W}} f$.

$\geq_{\mathrm{W}}$: if $g \leq_{\mathrm{W}} f$ via Φ_e, Φ_i, the reduction $\mathsf{T}g \leq_{\mathrm{W}} \mathsf{tJ}(f)$ is witnessed by $p \mapsto (e,i,p)$ and id.

The tot-jump and the totalization

Here is another characterization of the tot-jump:

Definition

We define $\mathsf{W}_{\Pi^0_2\to\Pi^0_1} :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$ as

$$\mathsf{W}_{\Pi^0_2\to\Pi^0_1}(p) := \{q \in \mathbb{N}^\mathbb{N} \,:\, (\forall i)(q(i+1) > q(i) \text{ and } p(q(i)) = 0)\}$$

In other words, the domain of $\mathsf{W}_{\Pi^0_2\to\Pi^0_1}$ is the set of p with infinitely many zeroes, and solutions are lists of infinitely many zeroes.

Theorem

For every f, $\mathsf{tJ}(f) \equiv_\mathrm{W} \mathsf{T}(\mathsf{W}_{\Pi^0_2\to\Pi^0_1} * f * \mathsf{W}_{\Pi^0_2\to\Pi^0_1})$.

In general, the $\mathsf{W}_{\Pi^0_2\to\Pi^0_1}$ cannot be dropped on either side of f!

The range of the tot-jump

Theorem

The map $\mathsf{tJ}(\cdot)$ is injective. Moreover, for every f, g

$$f \leq_{\mathrm{W}} g \iff \mathsf{tJ}(f) \leq_{\mathrm{W}} \mathsf{tJ}(g).$$

Thus it induces an injective endomorphism on the Weihrauch degrees.

As a corollary, this gives two new (and, by iterating, infinitely many) embeddings of the Medvedev degrees in the Weihrauch degrees.

Some examples

- $\mathsf{tJ}(\emptyset) \equiv_W \mathrm{id}$
- $\mathsf{tJ}(\mathrm{id}) \equiv_W \mathsf{T}(\mathsf{W}_{\Pi^0_2 \to \Pi^0_1}) \equiv_W \chi_{\Pi^0_2 \to \Pi^0_1}$

  Moreover, $\widehat{\mathsf{tJ}(\mathrm{id})} \equiv_W \mathsf{lim}$
- $\mathsf{tJ}^2(\mathrm{id}) \equiv_W g_2$, where g_2 is defined as

$$g_2(p) := \begin{cases} 0^{\mathbb{N}} & \text{if } p \text{ has infinitely many } 0 \\ 0^{<\mathbb{N}}1^{\mathbb{N}} & \text{if } p \text{ has finitely many } 0 \text{ and infinitely many } 1 \\ 0^{<\mathbb{N}}1^{<\mathbb{N}}2^{\mathbb{N}} & \text{if } p \text{ has finitely many } 0 \text{ and } 1 \end{cases}$$

Intuitively, you are allowed to ask 2 Σ^0_2 questions in series.
If the answer to the first one is “yes” you will see it in finite time, and can move to the second.
If the answer is “no”, you hang on the first answer and are not able to check the second.

Some examples

For many problems, we in fact have $\mathsf{tJ}(f) \equiv_{\mathrm{W}} \mathsf{T}f$.

Theorem

Fix a problem f. Assume that there are two total computable functions $\varphi, \psi \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ s.t.

- for every e, i, $\varphi(e,i)$ and $\psi(e,i)$ are indexes of total functionals;
- whenever $g \le_{\mathrm{W}} f$ via Φ_e, Φ_i, then $g \le_{\mathrm{W}} f$ via $\Phi_{\varphi(e,i)}$ and $\Phi_{\psi(e,i)}$

Then $\mathsf{tJ}(f) \equiv_{\mathrm{W}} \mathsf{T}f$.

Intuitively, this is slightly stronger than:

any reduction $g \le_{\mathrm{W}} f$ can be uniformly witnessed by total functionals.

Some examples

In particular:

- $\mathsf{tJ}(\mathsf{lim}) \equiv_{\mathrm{W}} \mathsf{Tlim}$
- $\mathsf{tJ}(\mathsf{lim}^{[n]}) \equiv_{\mathrm{W}} \mathsf{T}(\mathsf{lim}^{[n]})$

Let $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ be the problem: given an ill-founded tree in $\mathbb{N}^{<\mathbb{N}}$, produce a path.
Let $\mathsf{UC}_{\mathbb{N}^\mathbb{N}}$ be the restriction of $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ to trees with a unique path.

- $\mathsf{tJ}(\mathsf{C}_{\mathbb{N}^\mathbb{N}}) \equiv_{\mathrm{W}} \mathsf{T}(\mathsf{C}_{\mathbb{N}^\mathbb{N}})$
- $\mathsf{tJ}(\mathsf{UC}_{\mathbb{N}^\mathbb{N}}) \equiv_{\mathrm{W}} \mathsf{T}(\mathsf{UC}_{\mathbb{N}^\mathbb{N}})$

This is not the case for
$\mathsf{C}_{2^\mathbb{N}}$: given an ill-founded tree in $2^{<\mathbb{N}}$, produce a path.

$$\mathsf{C}_{2^\mathbb{N}} \equiv_{\mathrm{W}} \mathsf{TC}_{2^\mathbb{N}} <_{\mathrm{W}} \mathsf{tJ}(\mathsf{C}_{2^\mathbb{N}}) \equiv_{\mathrm{W}} \mathsf{T}(\mathsf{C}_{2^\mathbb{N}} \times \mathsf{W}_{\Pi^0_2 \to \Pi^0_1})$$

$\mathsf{C}_\mathbb{N}$: given $p \in \mathbb{N}^\mathbb{N}$, find $n \notin \operatorname{ran}(p)$

$$\mathsf{C}_\mathbb{N} <_{\mathrm{W}} \mathsf{TC}_\mathbb{N} <_{\mathrm{W}} \mathsf{tJ}(\mathsf{C}_\mathbb{N}) \equiv_{\mathrm{W}} \mathsf{T}(\mathsf{C}_\mathbb{N} \times \mathsf{W}_{\Pi^0_2 \to \Pi^0_1})$$

More on the range of tJ

Theorem

$\mathsf{TC}_{\mathbb{N}} \notin \operatorname{ran}(\mathsf{tJ})$

More generally, we know many problems that are not in $\operatorname{ran}(\mathsf{tJ})$.

Definition (Brattka, Gherardi)

A problem f is called *co-total* if, for all problems g, $\quad f \le_{\mathrm{W}} \mathsf{T}g \iff f \le_{\mathrm{W}} g$.

Theorem

A problem f is co-total iff for all problems g, $\quad f \le_{\mathrm{W}} \mathsf{tJ}(g) \iff f \le_{\mathrm{W}} g$.

This implies that $\mathsf{C}_{\mathbb{N}}, \mathsf{C}_{2^{\mathbb{N}}}$ are not in the range of tJ (they are known to be co-total).

However, there are non-cototal problems that are not in $\operatorname{ran}(\mathsf{tJ})$ (e.g. $\mathsf{TC}_{\mathbb{N}}$).

More on the range of tJ

Let $\mathsf{DIS} := p \mapsto \{q : \mathsf{U}(p) \neq q\}$. This is "one of the weakest" discontinuous problems.

Theorem (Joint with Pauly)

If $\mathsf{DIS} \times g \le_{\mathrm{W}} \mathsf{tJ}(f)$ then $g \le_{\mathrm{W}} f$.

Proof (Sketch)

Assume that $\mathsf{DIS} \times g \le_{\mathrm{W}} \mathsf{tJ}(f)$ via Φ, Ψ. In particular, the backward functional Ψ has to produce a pair.
Using the recursion theorem, we can find a computable $p \in \mathbb{N}^{\mathbb{N}}$ such that

$$\mathsf{U}(\langle p, x\rangle) = \pi_1 \Psi((\langle p, x\rangle, x), 0^{\mathbb{N}}).$$

When applied to $\Phi(\langle p, x\rangle, x)$, $\mathsf{tJ}(f)$ is forced to not go in the "otherwise" case, hence inducing a reduction $g \le_{\mathrm{W}} f$.

More on the range of tJ

Corollary

If $\mathsf{DIS} \times g \le_{\mathrm{W}} g$ then $g \notin \mathrm{ran}(\mathsf{tJ})$.

Since $\mathsf{DIS} \le_{\mathrm{W}} \mathsf{C}_2$ we have

$$\mathsf{C}_{2^{\mathbb{N}}}, \mathsf{lim}, \mathsf{UC}_{\mathbb{N}^{\mathbb{N}}}, \mathsf{C}_{\mathbb{N}^{\mathbb{N}}} \notin \mathrm{ran}(\mathsf{tJ})$$

This does not characterize the range: DIS is not co-total and $\mathsf{DIS} \times \mathsf{DIS} \not\le_{\mathrm{W}} \mathsf{DIS}$ but $\mathsf{DIS} \notin \mathrm{ran}(\mathsf{tJ})$.

Open question: Does closure under product with DIS imply co-totality?

More on the range of tJ

Recall: $(f_0 \sqcup f_1)(i, p) := f_i(p)$

Theorem

For every f, $\mathsf{tJ}(f)$ is join-irreducible.

Proof

There are $\overline{e}, \overline{i} \in \mathbb{N}$ s.t. for every f and every $k \in \mathbb{N}$, $\mathsf{tJ}(f)$ is Weihrauch equivalent to its restriction to $X_k := \{(\overline{e}, \overline{i})^\frown 0^{k\frown} x : x \in \mathbb{N}^\mathbb{N}\}$.

If $\mathsf{tJ}(f) \le_{\mathrm{W}} g_0 \sqcup g_1$ via Φ, Ψ, by the continuity of the forward functional, there is k such that $\Phi(\overline{e}, \overline{i}, 0^k)(0) \downarrow= b$. In particular, this yields a reduction $\mathsf{tJ}(f) \le_{\mathrm{W}} g_b$.

Corollary

For every f, g, $\mathsf{tJ}(f) \sqcup \mathsf{tJ}(g) \le_{\mathrm{W}} \mathsf{tJ}(f \sqcup g)$. Moreover, the reduction is strict iff $f \mid_{\mathrm{W}} g$.

More on the range of tJ

Open problem: find a (better) description for ran(tJ).

Theorem

For every $g \neq \emptyset$ there is $f <_{\mathrm{W}} g$ s.t. $\mathsf{tJ}(f) \not\leq_{\mathrm{W}} g$.

In other words, lower cones are not closed under tot-jump.

Theorem (Lempp, J. Miller, Pauly, M. Soskova, V.)

The Weihrauch lattice above id is dense. Morever, if id $\not\leq_{\mathrm{W}} f$ there are g, h such that $f \leq_{\mathrm{W}} g <_{\mathrm{W}} h$ and the interval (g, h) is empty.

Theorem

For every f there is h with $f <_{\mathrm{W}} h <_{\mathrm{W}} \mathsf{tJ}(f)$.

Moreover, there are f, g s.t. $f <_{\mathrm{W}} g$ and, for every h with $\mathsf{tJ}(f) <_{\mathrm{W}} h <_{\mathrm{W}} \mathsf{tJ}(g)$, h is not in the range of $\mathsf{tJ}(\cdot)$.

Final comments

The definition of $\mathsf{tJ}(f)$ is $\Delta^{1,f}_2$ in the language of third-order arithmetic, i.e. there is a Δ^1_2 formula with parameter f that says "$q \in \mathsf{tJ}(f)(e,i,p)$".

No jump operator can be defined in a Σ^1_1 way.

Proof (Sketch)

The counterexample would be $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

Indeed, "$x \in \mathsf{C}_{\mathbb{N}^\mathbb{N}}(A)$" is Π^0_1 relatively to A. Any arithmetic formula using $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ is still arithmetic.

Using the properties of the Baire space: $\mathsf{C}_{\mathbb{N}^\mathbb{N}} \equiv_{\mathrm{W}} \boldsymbol{\Sigma}^1_1\text{-}\mathsf{C}_{\mathbb{N}^\mathbb{N}}$.

In other words, $\mathsf{C}_{\mathbb{N}^\mathbb{N}}$ would compute its jump.

Open question: Can we define the tot-jump (or any jump) in a $\Pi^{1,f}_1$ way?

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