Part II : Ramsey's theorem computes through sparsity

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Ramsey's theorem

$[X]^n$ is the set of unordered n-tuples of elements of X

A k-coloring of $[X]^n$ is a map $f : [X]^n \to k$

A set $H \subseteq X$ is homogeneous for f if $|f([H]^n)| = 1$.

RT^n_k Every k-coloring of $[\mathbb{N}]^n$ admits an infinite homogeneous set.

Encodability vs Domination

Encodability

A set S is P-encodable if there is an instance of P such that every solution computes S

Domination

A function f is P-dominated if there is an instance of P such that every solution computes a function dominating f.

What sets are

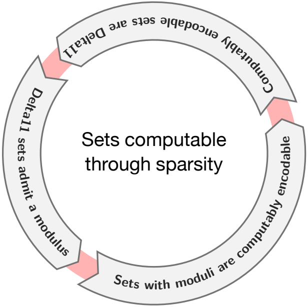
RT^n_k-encodable?

Thm (Jockusch)

Every function is RT^2_2-dominated.

Given $g : \omega \to \omega$, an interval $[x, y]$ is g-large if $y \geq g(x)$. Otherwise it is g-small.

$$f(x, y) = \begin{cases} 1 & \text{if } [x, y] \text{ is g-large} \\ 0 & \text{otherwise} \end{cases}$$

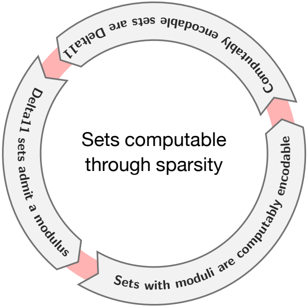


Every Δ^1_1 set is RT^2_2-encodable

Thm (Folklore)

Every RT^n_k-encodable set is computably encodable.

For every coloring $f : [\mathbb{N}]^n \to k$ and every infinite $X \subseteq \mathbb{N}$ there is an infinite f-homogeneous set $Y \subseteq X$.



Whenever $n \geq 2$ and $k \geq 2$,

$$\mathsf{RT}^n_k\text{-encodable} \equiv \Delta^1_1$$

The encodability power of RT^n_k comes from the

sparsity

of its homogeneous sets.

Thm (Dzhafarov and Jockusch)

The RT^1_2-encodable sets are the computable sets.

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

Sparsity of red implies non-sparsity of blue and conversely.

Cone avoidance 101

Strategy

Examples

Cohen forcing
Jockusch-Soare forcing

Pattern

Forcing question

Application

Pigeonhole forcing

Forcing in Computability Theory

Partial order
$(\mathbb{P}, \leq)$

Condition
$p \in \mathbb{P}$
approximation

Denotation
$[p] \subseteq 2^\omega$
class of candidates

Compatibility
If $q \leq p$ then $[q] \subseteq [p]$

Forcing in Computability Theory

Filter $\mathcal{F} \subseteq \mathbb{P}$

$\forall p \in \mathcal{F}\ \forall q \geq p\ q \in \mathcal{F}$
$\forall p, q \in \mathcal{F}, \exists r \in \mathcal{F}\ r \leq p, q$

Dense set $D \subseteq \mathbb{P}$

$\forall p \in \mathbb{P} \exists q \leq p\ q \in D$

Denotation

$[\mathcal{F}] = \bigcap_{p \in \mathcal{F}} [p]$

Forcing $p \Vdash \varphi(G)$

$\forall G \in [p]\ \varphi(G)$

Cohen forcing

$$(2^{<\omega}, \preceq)$$

$2^{<\omega}$ is the set of all finite binary strings

$\sigma \preceq \tau$ means σ is a prefix of τ

$[\sigma] = \{X \in 2^\omega : \sigma \prec X\}$

Thm (Folklore)

Let $C \not\leq_T \emptyset$. For every sufficiently Cohen generic G, $C \not\leq_T G$.

Lem

For every non-computable set C and Turing functional Φ_e, the following set is dense in $(2^{<\omega}, \preceq)$.

$$D = \{\sigma \in 2^{<\omega} : \sigma \Vdash \Phi_e^G \neq C\}$$

Given $\sigma \in 2^{<\omega}$, define the Σ^0_1 set

$$W = \{(x, v) : \exists \tau \succeq \sigma \; \Phi_e^\tau(x) \downarrow = v\}$$

- Case 1: $(x, 1 - C(x)) \in W$ for some x
  Then τ is an extension forcing $\Phi_e^G \neq C$
- Case 2: $(x, C(x)) \notin W$ for some x
  Then σ forces $\Phi_e^G \neq C$
- Case 3: W is a Σ^0_1 graph of C
  Impossible, since $C \not\leq_T \emptyset$

Weak König's lemma

$2^{<\omega}$ is the set of all finite binary strings

A binary tree is a set $T \subseteq 2^{<\omega}$ closed under prefixes

A path through T is an infinite sequence P such that every initial segment is in T

WKL Every infinite binary tree admits an infinite path.

Jockusch-Soare forcing

$$(\mathcal{T}, \subseteq)$$

$\mathcal{T}$ is the collection of infinite computable binary trees

$[T] = \{X \in 2^\omega : \forall \sigma \prec X \; \sigma \in T\}$

Thm (Jockusch-Soare)

Let $C \not\leq_T \emptyset$. For every infinite computable binary tree $T \subseteq 2^{<\omega}$, there is a path $P \in [T]$ such that $C \not\leq_T P$.

Lem

For every non-computable set C and Turing functional Φ_e, the following set is dense in $(\mathcal{T}, \subseteq)$.

$$D = \{T \in \mathcal{T} : T \Vdash \Phi_e^G \neq C\}$$

Given $T \in \mathcal{T}$, define the Σ^0_1 set

$$W = \{(x, v) : \exists \ell \in \mathbb{N} \forall \sigma \in 2^\ell \cap T \; \Phi^\sigma_e(x) \downarrow = v\}$$

- Case 1: $(x, 1 - C(x)) \in W$ for some x
  Then T forces $\Phi^G_e \neq C$
- Case 2: $(x, C(x)) \notin W$ for some x
  Then $\{\sigma \in T : \neg(\Phi^\sigma_e(x) \downarrow = v)\}$ forces $\Phi^G_e \neq C$
- Case 3: W is a Σ^0_1 graph of C
  Impossible, since $C \not\leq_T \emptyset$

Forcing question

$$p \mathrel{?}\!\vdash \varphi(G)$$

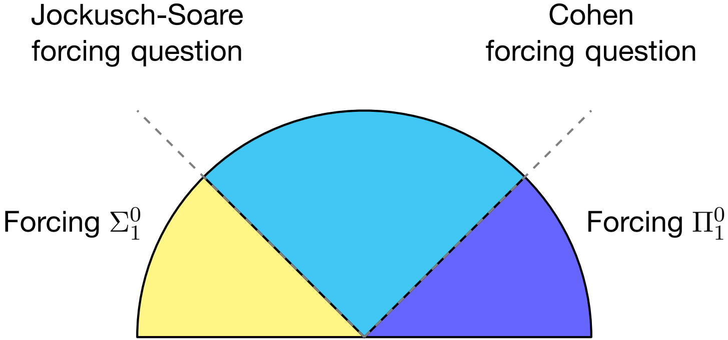
where $p \in \mathbb{P}$ and $\varphi(G)$ is Σ^0_1

Lem

Let $p \in \mathbb{P}$ and $\varphi(G)$ be a Σ^0_1 formula.

(a) If $p \mathrel{?}\!\vdash \varphi(G)$, then $q \Vdash \varphi(G)$ for some $q \leq p$;

(b) If $p \mathrel{?}\!\nvdash \varphi(G)$, then $q \Vdash \neg\varphi(G)$ for some $q \leq p$.


Jockusch-Soare
forcing question
Cohen
forcing question
Forcing Σ^0_1
Forcing Π^0_1

Suppose $p \mathrel{?}\!\vdash \varphi(G)$ is uniformly Σ^0_1 whenever $\varphi(G)$ is Σ^0_1

Lem

For every non-computable set C and Turing functional Φ_e, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{p \in \mathbb{P} : p \Vdash \Phi_e^G \neq C\}$$

Given $p \in \mathbb{P}$, define the Σ^0_1 set

$$W = \{(x, v) : p \mathrel{?\!\vdash} \Phi^G_e(x) \downarrow= v\}$$

- Case 1: $(x, 1 - C(x)) \in W$ for some x
  Then there is an extension forcing $\Phi^G_e \neq C$
- Case 2: $(x, C(x)) \notin W$ for some x
  Then there is an extension forcing $\Phi^G_e \neq C$
- Case 3: W is a Σ^0_1 graph of C
  Impossible, since $C \not\leq_T \emptyset$

Pigeonhole principle

RT^1_k Every k-partition of $\mathbb{N}$ admits an infinite subset of a part.

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

$\longmapsto$

0	1	2	3	4
5	6	7	8	9
10	11	12	13	14
15	16	17	18	19
20	21	22	23	24
25	26	27	28	....

Thm (Dzhafarov and Jockusch)

A set is RT^1_2-encodable iff it is computable.

Thm (Dzhafarov and Jockusch)

A set is RT^1_2-encodable iff it is computable.

Input : a set $C \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $C \not\leq_T G$

$$(F_0, F_1, X)$$

Initial segment

Reservoir

- F_i is finite, X is infinite, $\max F_i < \min X$ (Mathias condition)
- $C \not\leq_T X$ (Weakness property)
- $F_i \subseteq A_i$ (Combinatorics)

Extension

$(E_0, E_1, Y) \leq (F_0, F_1, X)$

- $F_i \subseteq E_i$
- $Y \subseteq X$
- $E_i \setminus F_i \subseteq X$

Denotation

$\langle G_0, G_1 \rangle \in [F_0, F_1, X]$

- $F_i \subseteq G_i$
- $G_i \setminus F_i \subseteq X$

$$[E_0, E_1, Y] \subseteq [F_0, F_1, X]$$

$$(F_0, F_1, X) \Vdash \varphi(G_0, G_1)$$

Condition

Formula

$\varphi(G_0, G_1)$ holds for every $\langle G_0, G_1\rangle \in [F_0, F_1, X]$

Input : a set $C \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $C \not\leq_T G$

Input : a set $C \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $C \not\leq_T G$

$$\Phi_{e_0}^{G_0} \neq C \vee \Phi_{e_1}^{G_1} \neq C$$

Input : a set $C \not\leq_T \emptyset$ and a 2-partition $A_0 \sqcup A_1 = \mathbb{N}$

Output : an infinite set $G \subseteq A_i$ such that $C \not\leq_T G$

$$\Phi_{e_0}^{G_0} \neq C \vee \Phi_{e_1}^{G_1} \neq C$$

The set $\{p \in \mathbb{P} : p \Vdash \Phi_{e_0}^{G_0} \neq C \vee \Phi_{e_1}^{G_1} \neq C\}$ is dense

Disjunctive forcing question

$$p \mathrel{?{\vdash}} \varphi_0(G_0) \vee \varphi_1(G_1)$$

where $p \in \mathbb{P}$ and $\varphi_0(G_0), \varphi_1(G_1)$ are Σ^0_1

Lem

Let $p \in \mathbb{P}$ and $\varphi_0(G_0), \varphi_1(G_1)$ be Σ^0_1 formulas.

(a) If $p \mathrel{?{\vdash}} \varphi_0(G_0) \vee \varphi_1(G_1)$, then $q \Vdash \varphi_0(G_0) \vee \varphi_1(G_1)$ for some $q \leq p$;

(b) If $p \mathrel{?{\nvdash}} \varphi_0(G_0) \vee \varphi_1(G_1)$, then $q \Vdash \neg\varphi_0(G_0) \vee \neg\varphi_1(G_1)$ for some $q \leq p$.

Suppose the following relation is uniformly $\Sigma^0_1(X)$ whenever $\varphi_0(G_0), \varphi_1(G_1)$ are Σ^0_1

$$(F_0, F_1, X) \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$$

Lem

For every non-computable set C and Turing functionals Φ_{e_0}, Φ_{e_1}, the following set is dense in $(\mathbb{P}, \leq)$.

$$D = \{p \in \mathbb{P} : p \Vdash \Phi^{G_0}_{e_0} \neq C \vee \Phi^{G_1}_{e_1} \neq C\}$$

Consider the $\Sigma^0_1(X)$ set

$$W = \{(x, v) : p \mathrel{?\vdash} \Phi^{G_0}_{e_0}(x) \downarrow = v \vee \Phi^{G_0}_{e_0}(x) \downarrow = v\}$$

Problem: complexity of the instance

"Can we find an extension for this instance of RT^1_2?"

Defi

$$(F_0, F_1, X) \mathrel{?}\vdash \varphi_0(G_0) \vee \varphi_1(G_1)$$

$$\equiv$$

$$(\exists i < 2)(\exists E_i \subseteq X \cap A_i)\varphi_i(F_i \cup E_i)$$

The formula is $\Sigma^0_1(X \oplus A_i)$

Idea: make an overapproximation

"Can we find an extension for every instance of RT^1_2?"

Defi

$$(F_0, F_1, X) \mathrel{?\vdash} \varphi_0(G_0) \vee \varphi_1(G_1)$$

$$\equiv$$

$$(\forall B_0 \sqcup B_1 = \mathbb{N})(\exists i < 2)(\exists E_i \subseteq X \cap B_i)\varphi_i(F_i \cup E_i)$$

The formula is $\Sigma^0_1(X)$

Case 1: $p \mathrel{?}\vdash \varphi_0(G_0) \vee \varphi_1(G_1)$

Letting $B_i = A_i$, there is an extension $q \leq p$ forcing

$$\varphi_0(G_0) \vee \varphi_1(G_1)$$

Case 2: $p \mathrel{?}\nvdash \varphi_0(G_0) \vee \varphi_1(G_1)$

$$(\exists B_0 \sqcup B_1 = \mathbb{N})(\forall i < 2)(\forall E_i \subseteq X \cap B_i)\neg\varphi_i(F_i \cup E_i)$$

The condition $(F_0, F_1, X \cap B_i) \leq p$ forces

$$\neg\varphi_0(G_0) \vee \neg\varphi_1(G_1)$$

Ramsey's theorem

Over n-tuples

$$\mathrm{RT}^{n}_{k}$$

Using k colors

Ramsey's theorem

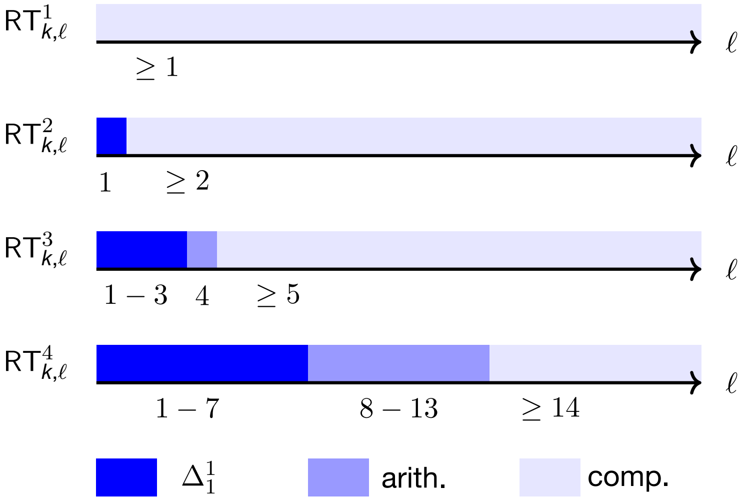
Over n-tuples

$$\mathrm{RT}^{n}_{k,r}$$

Allows r colors

Using k colors

$\mathsf{RT}^n_{k,\ell}$-encodable sets

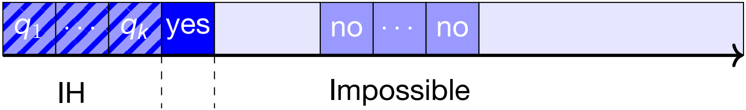


Thm (Cholak, P.)

Every function is $\mathsf{RT}^n_{k,\ell}$-dominated for $\ell < 2^{n-1}$.

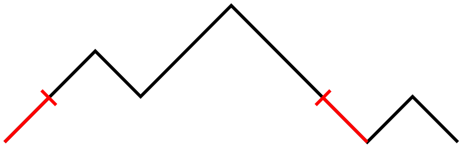
$$f(x_1, x_2, \ldots, x_n) = \langle [x_1, x_2] \text{ g-large?}, \ldots, [x_{n-1}, x_n] \text{ g-large?} \rangle$$

- Case 1: the color $\langle no, \ldots, no \rangle$ is avoided
- Case 2: the color $\langle q_1, \ldots, q_k, yes, no, \ldots, no \rangle$ is avoided



Catalan numbers

C_n is the number of trails of length $2n$.



$$C_0 = 1 \qquad \text{and} \qquad C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786,...

Defi

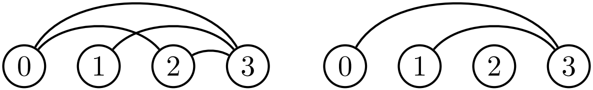
A largeness graph is a pair $(\{0, \ldots, n-1\}, E)$ such that

(a) If $\{i, i+1\} \in E$, then for every $j > i+1$, $\{i, j\} \notin E$

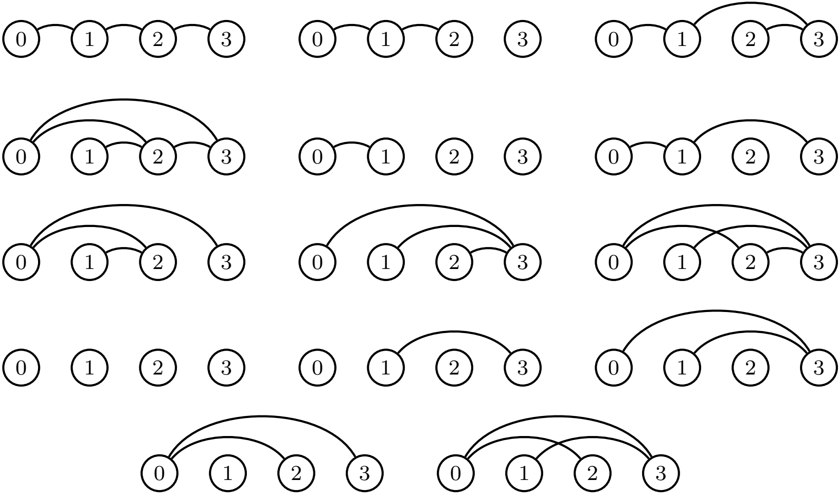
(b) If $i < j < n$, $\{i, i+1\} \notin E$ and $\{j, j+1\} \in E$, then $\{i, j+1\} \in E$

(c) If $i+1 < j < n-1$ and $\{i, j\} \in E$, then $\{i, j+1\} \in E$

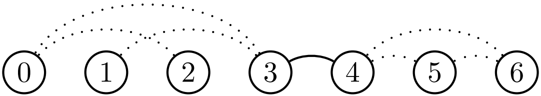
(d) If $i+1 < j < k < n$ and $\{i, j\} \notin E$ but $\{i, k\} \in E$, then $\{j-1, k\} \in E$



Largeness graphs of size 4



Counting largeness graphs



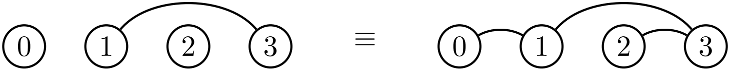
A largeness graph $\mathcal{G} = (\{0, \ldots, n-1\}, E)$ is packed if for every $i < n - 2$, $\{i, i+1\} \notin E$.

- L_n = number of largeness graphs of size n
- P_n = number of packed largeness graphs of size n

$$L_0 = 1 \qquad \text{and} \qquad L_{n+1} = \sum_{i=0}^{n} P_{i+1} L_{n-i}$$

Counting packed largeness graphs

A largeness graph $\mathcal{G} = (\{0, \ldots, n-1\}, E)$ of size $n \geq 2$ is normal if $\{n-2, n-1\} \in E$.



Thm (Cholak, P.)

The following are in one-to-one correspondance:

(a) packed largeness graphs of size n

(b) normal largeness graphs of size n

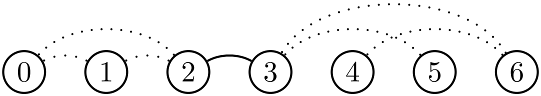
(c) largeness graphs of size $n-1$

Thm (Cholak, P.)

Every left-c.e. function is $\mathrm{RT}^n_{k,\ell}$-dominated for $\ell < C_n$.

$f(x_1, x_2, \ldots, x_n) =$ the largeness graph of g

- Case 1: a packed graph is avoided
- Case 2: a graph of the following form is avoided



Conclusion

RT^n_k for $n \geq 2$ has instances having only sparse solutions, hence encodes all the Δ^1_1 sets

RT^1_k cannot force having sparse solutions, so encodes only the computable sets

A trichotomy appears when we allow more colors in the solutions

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