On a Lattice of Degrees of Representations of Irrational Numbers

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What is a *representation* of the irrational numbers?

I will tell you.

Beware! I will not tell you what a representation of the real numbers is.

We identify an irrational number α with its Dedkind cut. The Dedekind cut of an irrational α is the function $\alpha : \mathbb{Q} \longrightarrow \{0, 1\}$ where

$$\alpha(q) = \begin{cases} 0 & \text{if } q < \alpha \\ 1 & \text{if } q > \alpha. \end{cases}$$

We will use the representation by Dedekind cuts to define what representation in general is. In principle, we could have use any other computable representation for this purpose, but it is

- convenient to use a representation that is unique
- a good idea to use a well-know representation.

What is a *representation*?

Definition

A class of functions R is a *representation (of the irrational numbers)* if there exist oracle Turing machines M and N such that

- for every irrational $\alpha \in (0,1)$ there exists $f \in R$ such that

$$\alpha = \Phi_M^f \text{ and } f = \Phi_N^\alpha$$

- for every $g \in R$ there exist an irrational $\alpha \in (0,1)$ such that

$$\alpha = \Phi_M^g = \Phi_M^f \text{ where } f = \Phi_N^\alpha$$

When $\alpha = \Phi_M^g$, we say that g *represents* α and that g is an *R-representation* of α.

Example

A function $C : \mathbb{N}^+ \longrightarrow \mathbb{Q}$ is a *Cauchy sequence* for α if

$$|\alpha - C(n)| < n^{-1} \ .$$

Let $\mathcal{C}$ be the class of all Cauchy sequences for irrational numbers in the interval $(0, 1)$.

Then $\mathcal{C}$ is a representation.

Example

A function $E : \mathbb{N}^+ \longrightarrow \{0, 1\}$ is the *base-2 expansion* of α if

$$\alpha = \sum_{i=1}^{\infty} E(i) 2^{-i} .$$

Let $2\mathcal{E}$ be the class of all base-2 expansions of irrational numbers in the interval $(0, 1)$.

Then $2\mathcal{E}$ is a representation.

Example

A function $T : \mathbb{Q} \cap [0,1] \longrightarrow (0,1)$ is the *trace function* for α if

$$|\alpha - T(q)| \; < \; |\alpha - q| \; .$$

Let $\mathcal{T}$ be the class of all trace functions for irrational numbers in the interval $(0,1)$.

Then $\mathcal{T}$ is a representation.

Example

Recall that the Dedekind cut of an irrational α is the function $\alpha : \mathbb{Q} \longrightarrow \{0, 1\}$ where

$$\alpha(q) = \begin{cases} 0 & \text{if } q < \alpha \\ 1 & \text{if } q > \alpha. \end{cases}$$

Let $\mathcal{D}$ be the class of all Dedekind cuts of irrational numbers in the interval $(0, 1)$.

Then $\mathcal{D}$ is a representation.

Example of something that is not a representation (but which come close).

A function $L : \mathbb{N} \longrightarrow (0, 1)$ is a *left cut* for α if the sequence

$$L(0), L(1), L(2), \ldots$$

contains (i) all the rationals less than α and (ii) only rational less that α.

A class of left cuts will *not* be a representation. We cannot compute the Dedekind cut of α from a left cut for α.

Example of something that is not a representation (but which come close).

Cauchy sequences without a modulus of convergence. Let $C : \mathbb{N}^+ \longrightarrow \mathbb{Q}$ be such that

$$\forall n \in \mathbb{N}^+ \; \exists N \left(i > N \quad \rightarrow \quad |\alpha - C(i)| < n^{-1} \right)$$

A class of such functions will *not* be a representation. We cannot compute the Dedekind cut of α from such a function.

Next we define an ordering relation $\preceq_S$ over the representations.

Intuitiviely, we want

- $R_1 \preceq_S R_2$ to be true if an R_2-representation of α can be subrecursively converted into an R_1-representation of α (subrecursively = "without unbounded search")
- $R_1 \not\preceq_S R_2$ to be true if unbounded search is required in order to convert R_2-representation of α into an R_1-representation of α.

More intuition ...

- If $R_1 \preceq_S R_2$ holds, then the representation R_2 gives more information than the representation R_1.

More intuition ...

Let C be a Cauchy sequence for the irrational number α.

More intuition . . .

Let C be a Cauchy sequence for the irrational number α.

How can we decide if α lies above or below $1/3$?

Consider C as an oracle. (We assume that α is irrational, so α lies strictly above or strictly below $1/3$.)

We may ask C ...

- $C(0) = ?$
- $C(1) = ?$
- $C(2) = ?$
- $C(3) = ?$
- $\vdots$

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- $C(3) = 1/3$
- $\vdots$
- $C(16) = 1/3$
- $C(17) = 1/3$

We may ask C . . .

- $\vdots$
- $C(16) = 1/3$
- $C(17) = 1/3$

Now, we know that α is close to $1/3$, that is

$$\left|\alpha - \frac{1}{3}\right| < \frac{1}{17}$$

but we still don't know if α lies above or below $1/3$.

Now

$$C(\text{a sufficiently large number}) = \begin{array}{l}\text{a rational that allows}\\ \text{me to conclude if } \alpha\\ \text{lies above or below } 1/3\end{array}$$

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but we need unbounded search to find that number.

I cannot find the number by a subrecursive computation.

I need full Turing computability.

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Just one question is needed. No unbounded search is required.

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Just one question is needed. No unbounded search is required.

A subrecursive computation is sufficient to answer the question.

This example shows that we cannot compute the Dedekind cut of α subrecursively in a Cauchy sequence for α.

We want

$$\mathcal{D} \not\preceq_S \mathcal{C}$$

where

- $\mathcal{D}$ is the representation by Dedekind cuts
- $\mathcal{C}$ is the representation by Cauchy sequences.

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- $\mathcal{D}$ is the representation by Dedekind cuts
- $\mathcal{C}$ is the representation by Cauchy sequences.

In contrast, we can compute a Cauchy sequence for α subrecursively in the Dedekind cut of α.

Let α be an irrational number between 0 and 1.

We can compute a Cauchy sequence C for α subrecursively in the Dedekind cut of α: Let $C(1) = 2^{-1}$ and

$$C(n+1) \;=\; \begin{cases} C(n) - 2^{-n-1} & \text{if } D(C(n)) = 0 \\ C(n) + 2^{-n-1} & \text{otherwise.} \end{cases}$$

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We want that

$$\mathcal{C} \;\preceq_S\; \mathcal{D}\;.$$

Now, ... the formal definition of $\preceq_S$.

We need some auxiliary definitions.

We need the *time bounds*.

Definition

A function $t : \mathbb{N} \longrightarrow \mathbb{N}$ is a *time bound* if (i) $n \leq t(n)$, (ii) t is increasing and (iii) t is time-constructible: there is a single-tape Turing machine that, on input 1^n, computes $t(n)$ in $O(t(n))$ steps.

We need the notation $O(t)_R$.

Definition

Let t be a time-bound and let R be a representation. Then, $O(t)_R$ denotes the class of all irrational α in the interval $(0, 1)$ such that at least one R-representation of α is computable by a Turing machine running in time $O(t(n))$ (where n is the length of the input).

Example

Let $\mathcal{C}$ be the representation by Cauchy sequences. Let $\alpha \in (0, 1)$ be irrational.

Then the following two statements are equivalent (by definition).

1. $\alpha \in O(n^2)_{\mathcal{C}}$
2. at least one Cauchy sequence for α can be computed by a Turing machine running in time $O(n^2)$ (where n is the length of the input).

Example

Let $2\mathcal{E}$ be the representation by base-2 expansions. Let $\alpha \in (0,1)$ be irrational.

Then the following two statements are equivalent.

1. $\alpha \in O(2^{4n^2})_{2\mathcal{E}}$
2. the base-2 expansion of α can be computed by a Turing machine running in time $O(2^{4n^2})$ (where n is the length of the input).

Now we are ready for the definition of $\preceq_S$.

Definition

Let t be a time-bound. Let R_1 and R_2 be representations. The relation $R_1 \preceq_S R_2$ holds if there for any time-bound t exists a time-bound s such that

$$O(t)_{R_2} \subseteq O(s)_{R_1} .$$

If the relation $R_1 \preceq_S R_2$ holds, we will say that the representation R_1 is *subrecursive* in the representation R_2.

Let us see why we have $\mathcal{C} \preceq_S 2\mathcal{E}$.

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Analyse that algorithm and conclude: *If a Turing machine can compute the base-2 expansion of α in time $O(t(n))$, then a Turing machine can compute a Cauchy sequence for α in time $O(2^{5t(n)})$.*

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Thus

$$O(t(n))_{2\mathcal{E}} \subseteq O(2^{5t(n)})_{\mathcal{C}} .$$

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(this holds when $s(n) = 2^{5t(n)}$).

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(this holds when $s(n) = 2^{5t(n)}$).

Thus, we have $\mathcal{C} \preceq_S 2\mathcal{E}$ (by the definition of $\preceq_S$).

This generalises. In general we can prove $R_1 \preceq_S R_2$ by the following recipe.

Find a subrecursive algorithm for converting an R_2-representation of α into an R_1-representation of α (no unbounded search).

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Analyse that algorithm and conclude: *If a Turing machine can compute an R_2-representation of α in time $O(t(n))$, then a Turing machine can compute an R_1-representation α in time $O(s_t(n))$* where s_t is a time-bound depending on t.

This generalises. In general we can prove $R_1 \preceq_S R_2$ by the following recipe.

Find a subrecursive algorithm for converting an R_2-representation of α into an R_1-representation of α (no unbounded search).

Analyse that algorithm and conclude: *If a Turing machine can compute an R_2-representation of α in time $O(t(n))$, then a Turing machine can compute an R_1-representation α in time $O(s_t(n))$* where s_t is a time-bound depending on t.

Thus, for any time-bound t there exists a time-bound s such that

$$O(t(n))_{R_2} \subseteq O(s(n))_{R_1} .$$

Thus, we have $R_1 \preceq_S R_2$ (by the definition of $\preceq_S$).

To prove $R_1 \not\preceq_S R_2$ migh not be all that easy.

Then we have to prove that

there exists a time-bound t such that
for any time-bound s

$$O(t(n))_{R_2} \not\subseteq O(s(n))_{R_1}$$

To prove $R_1 \not\preceq_S R_2$ might not be all that easy ...

... which again can be proved by proving

there exists a time-bound t such that
for any time-bound s
there exists an irrational $\beta \in (0,1)$ such that
$\beta \in O(t(n))_{R_2} \setminus O(s(n))_{R_1}$

That there for any time-bond s exists such a β will typically be proved by a diagonalisation argument. Such arguments may be tedious and involved.

The relation $\preceq_S$ is a preorder. Thus $\preceq_S$ induce a degree structure on the representations (standard stuff will follow).

Let R and Q be representations.

$$R \equiv_S Q \quad \Leftrightarrow_{\text{def}} \quad R \preceq_S Q \text{ and } Q \preceq_S R \; .$$

$$R \prec_S Q \quad \Leftrightarrow_{\text{def}} \quad R \preceq_S Q \text{ and } Q \not\preceq_S R \; .$$

We define the *degree* of the representation R, denoted $\deg(R)$, as the equivalence class given by

$$\deg(R) \; = \; \{ \; Q \mid Q \equiv_S R \; \} \; .$$

The set of all degrees, denoted $\mathcal{S}$, is given by

$$\mathcal{S} \; = \; \{ \; \deg(R) \mid R \text{ is a representation } \} \; .$$

We will use $\mathbf{a}, \mathbf{b}, \mathbf{c}$ (possible decorated) to denote degrees. We will use $\leq$ and $<$ to denote the ordering relations induced on the degrees by $\preceq_S$ and $\prec_S$, respectively.

It turns out that this degree structure is a lattice. That is, there are operators $\cup$ and $\cap$ on the degrees such that

- $\mathbf{a} \cup \mathbf{b}$ is the least upper bound of $\mathbf{a}$ and $\mathbf{b}$
- $\mathbf{a} \cap \mathbf{b}$ is the greatest lower bound of $\mathbf{a}$ and $\mathbf{b}$.

for any $\mathbf{a}, \mathbf{b} \in \mathcal{S}$.

It turns out that the degree structure has a top and bottom degree.

Let $\mathbf{0}$ denote the degree of the representation by Weirauch intersections (nested intervals).

Let $\mathbf{1}$ denote the degree of the representation by continued fractions.

Theorem

We have

$$\mathbf{0} \leq \mathbf{a} \leq \mathbf{1}$$

for any degree $\mathbf{a}$.

Definition

A function $I : \mathbb{N} \longrightarrow \mathbb{Q} \times \mathbb{Q}$ is a *Weihrauch intersection* for the real number α if the left component of the pair $I(i)$ is strictly less that the right component of the pair $I(i)$ (for all $i \in \mathbb{N}$) and

$$\{ \alpha \} = \bigcap_{i=0}^{\infty} I_i^O$$

where I_i^O denotes the open interval given by the the pair $I(i)$.

If we have a Weihrauch intersection for an irrational number α, the we can compute the Dedekind cut of α (we will need unbounded search). If we have the Dedekind cut of α, we can obviously compute a Weihrauch intersection for α (we do not need unbounded search).

The class of all Weihrauch intersections for irrationals in the intervall $(0, 1)$ is a representation.

Definition

Let α be an irrational in the interval $(0, 1)$. The *continued fraction of* α is the unique function $f : \mathbb{N}^+ \longrightarrow \mathbb{N}^+$ such that $\alpha = [0; f(1), f(2), \ldots]$ where

$$[\ 0; a_1, a_2, a_3 \ldots\ ] \ = \ 0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}}$$

The class of all continued fractions of irrationals in the intervall $(0, 1)$ is a representation.

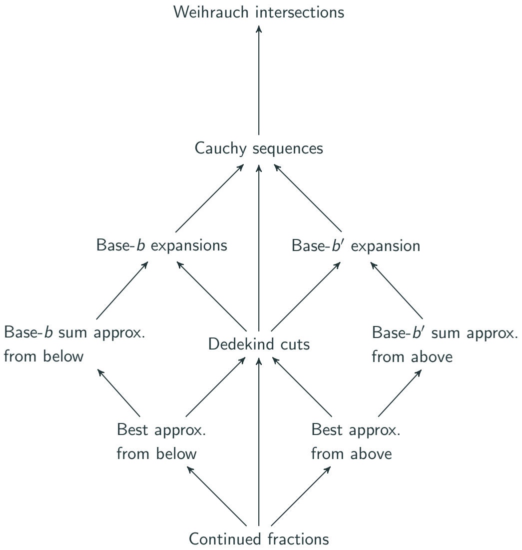

Figure 1: Overview.

`arXiv:2304.07227 [pdf, ps, other] math.LO cs.CC`

On representations of real numbers and the computational complexity of converting between such representations.

Authors: Amir M. Ben-Amram, Lars Kristiansen, Jakob Grue Simonsen

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Another paper recently submitted to a journal (but not to `arXive`):

A Degree Structure on Representations of Irrational Numbers

Authors: Amir M. Ben-Amram, Lars Kristiansen


That's all Folks!