# Primitive recursive degree spectra of structures

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- All operations can be equivalently transformed into ralations, e.g., $+1$ can be replaced by the successor relation.

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for every $X$. Then for some fixed parameters $\vec{a} \in \mathcal{A}$ there are computable mappings into quantifier-free formulae $n \mapsto \Phi_n$ and $n \mapsto \Psi_n$ such that

$$n \in P \iff \mathcal{A} \models (\exists \vec{x}) \Phi_n(\vec{x}, \vec{a});$$

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- No.
- Because every $f$-punctual structure can be isomorphically transformed into an $f$-punctual structure whose operations are primitively recursively bounded , i.e., there is a set $X \leq_{PR} f$ such that the structure has an $X$-punctual presentation.

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- (Folklore). If $P \mid_T Q$ then the collection $\{X : P \leq_T X\} \cup \{X : Q \leq_T X\}$ is not a Turing degree spectrum.

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- (AKKLMM, 2016). The collection $\{X : X \not\leq_T \emptyset''\}$ is not a Turing degree spectra.

# The idea of the Wehner's proof

- We can code into a structure the family of finite sets

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- Some modification of the family gives the degree spectrum of the hyperimmune degrees (CK, 2010).

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- (Folklore). If $P \mid_{PR} Q$ then the collection $\{f : P \leq_{PR} f\} \cup \{f : Q \leq_{PR} f\}$ is not a $PR$-degree spectrum.
- (Set Basis Property). If $f \in Sp_{PR}(\mathcal{A})$ then $X \in Sp_{PR}(\mathcal{A})$ for some $X \leq_{PR} f$.

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- Note that the proof can be made uniform.

# Punctual families

- A countable family $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is punctual if there is a $\{0,1\}$-valued primitive recursive function $p$ such that

$$X \in \mathcal{F} \iff (\exists i)(\forall x)[x \in X \leftrightarrow p\langle i, x\rangle = 1]$$

  for all $X$.

- For a function $f$ a countable family $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is $f$-punctual if there is a $\{0,1\}$-valued function $p \leq_{PR} f$ such that

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# Coding families

- For a countable family $\mathcal{F} \subseteq 2^{\mathbb{N}}$ define the algebraic structure $\mathcal{A}(\mathcal{F})$ in the language $\{r, s, l\}$ on the domain $\mathcal{F} \times \mathbb{N} \times \mathbb{N}$ such that the unary operations $r$ and $s$ are defined by
$$r(X, x, y) = (X, 0, y);$$
$$s(X, x, y) = (X, x + 1, y);$$
and the unary predicate $l$ is defined by
$$l(X, x, y) \leftrightarrow x \in X.$$
- A countable family $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is $f$-punctual if and only if $f \in Sp_{PR}(\mathcal{A}(\mathcal{F}))$.

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  is punctual.
- Just because we can start to enumerate the sets $\{n\} \oplus F$ with $F(i) \neq p_n(i)$ after the value $p_n(i)$ become computed.

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- Indeed, if $\mathcal{V}$ is punctual then for every $n$ we can computably find an $s(n)$ such that $p_{s(n)} \neq p_{\varphi_n(n)}$. But then $s(n) \neq \varphi_n(n)$. A contradiction.

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- If $X \not\leq_{PR} \emptyset$ we can use $X$ as a sample to extend every unsucessfull guess $F \neq p_{\varphi_n(n)}$.
- Now it is enough to apply Set Basis Property for $\{f : f \not\leq_{PR} \emptyset\}$.

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- Let $h$ be the Ackerman's function, $P_n^h$ be the $n$-th set $\leq_{PR} h$, $P^h = \oplus_n P_n^h$. Then the collection $\{f : f \not\leq_{PR} P^h\}$ has no Set Basis Property

# $PR$-Spectra of Slaman-Wehner type

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- Let $h$ be the Ackerman's function, $P^h_n$ be the $n$-th set $\leq_{PR} h$, $P^h = \oplus_n P^h_n$. Then the collection $\{f : f \not\leq_{PR} P^h\}$ has no Set Basis Property, and so is not a $PR$-degree spectra.

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- The $T$-degree spectra are closed under jumps and jump-inversions, so we have collections of the high degrees $\{X : \emptyset'' \leq_T X'\}$ and of the non-low degrees $\{X : X' \not\leq_T \emptyset'\}$. Are there $PR$-analogues of these?

Thank you!