Subrecursive Graphs of Representations of Irrational Numbers

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Introduction

We are interested in the multitude of ways to represent irrational numbers:

- base-b expansions
- Dedekind cuts
- Hurwitz characteristics
- continued fractions
- ...

Any one of these representations can be computably transformed into any other. Our main question: *is the transformation possible without using unbounded search*?

Subrecursive reducibility

Let R_1 and R_2 be representations of irrational numbers.

We will denote $R_1 \leq_S R_2$ (R_1 *is subrecursive in* R_2) if there exists an algorithm, which:

- uses no unbounded search;
- given an oracle, which is an R_2-representation of an irrational α, it produces an R_1-representation of α.

We will also denote

$$R_1 \equiv_S R_2 \text{ if } R_1 \leq_S R_2 \ \& \ R_2 \leq_S R_1$$

$$R_1 <_S R_2 \text{ if } R_1 \leq_S R_2 \ \& \ R_2 \not\leq_S R_1.$$

First Example

For an irrational number $\alpha \in (0,1)$:

- the *Dedekind cut* of α is the function $D : \mathbb{Q} \to \{0,1\}$, such that

$$D(q) = \begin{cases} 0, & \text{if } q < \alpha, \\ 1, & \text{if } q > \alpha. \end{cases}$$

- the *base-2 expansion* of α is the function $E : \mathbb{N} \to \{0,1\}$, such that

$$\alpha \;=\; \sum_{n=0}^{\infty} E(n) \cdot 2^{-n}.$$

$E \leq_S D$

Assume we have computed the digits $E(1), E(2), \ldots, E(n)$ and let

$$q_n \;=\; E(1)\cdot 2^{-1} + E(2)\cdot 2^{-2} + \ldots + E(n)\cdot 2^{-n}.$$

To compute $E(n+1)$ we ask the Dedekind cut:

$$E(n+1) = \begin{cases} 0, & \text{if } D(q_n + 2^{-n-1}) = 1, \\ 1, & \text{if } D(q_n + 2^{-n-1}) = 0. \end{cases}$$

No unbounded search is used in this algorithm!

$D \not\leq_S E$

We have access to the base-2 expansion E of α and we want to compute $D(q)$ for a given rational number q.
If q has a finite base-2 expansion of length n, then

$$q < \alpha \iff q \leq E(1) \cdot 2^{-1} + E(2) \cdot 2^{-2} + \ldots + E(n) \cdot 2^{-n}.$$

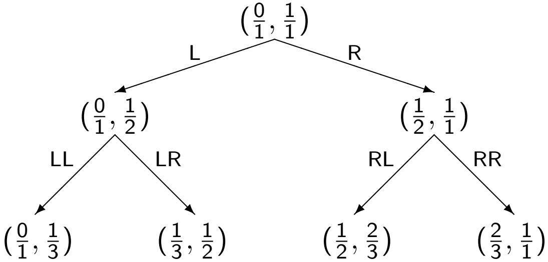
But what if q has an infinite base-2 expansion? For example, let $q = 1/3 = 0.(01)^{\omega}$. To decide whether $q < \alpha$ we must search for a position n, such that $E(n)$ is different from the n-th digit of q.

This algorithm requires unbounded search!

Hurwitz characteristic

Let us form the Farey pair tree of intervals:

- the root is $(\frac{0}{1}, \frac{1}{1})$;
- the left descendant of $(\frac{a}{b}, \frac{c}{d})$ is $(\frac{a}{b}, \frac{a+c}{b+d})$;
- the right descendant of $(\frac{a}{b}, \frac{c}{d})$ is $(\frac{a+c}{b+d}, \frac{c}{d})$.



The Hurwitz characteristic H of α is the unique infinite path in the tree, which consists of all intervals containing α.

$H \equiv_S D$

$H \leq_S D$: we compute $H(n)$ and the corresponding intervals recursively. We can decide whether we should go left or right by asking for the value $D(m)$, where m is the current mediant.

$D \leq_S H$: given a rational q, we compute the level s of its first occurrence in the tree. Let (a_s, b_s) be the interval on level s, which contains α. Then $D(q) = 0$ if $q \leq a_s$ and $D(q) = 1$ if $b_s \leq q$.

Both algorithms do not use unbounded search!

Continued fraction

The continued fraction of α is the unique sequence $c : \mathbb{N} \to \mathbb{N} \setminus \{0\}$, such that

$$\alpha = 0 + \cfrac{1}{c(0) + \cfrac{1}{c(1) + \cfrac{1}{\ddots}}}$$

We will also denote $c = [\,]$.
The following equality relates the continued fraction to the Hurwitz characteristic:

$$H = \underbrace{\mathsf{LL}\ldots\mathsf{L}}_{c(0)-1}\underbrace{\mathsf{RR}\ldots\mathsf{R}}_{c(1)}\underbrace{\mathsf{LL}\ldots\mathsf{L}}_{c(2)}\underbrace{\mathsf{RR}\ldots\mathsf{R}}_{c(3)}\ldots$$

$H \leq_S [\,]$

Therefore, we can compute H from the continued fraction $c = [\,]$: given n, compute the unique $x \leq n+1$, such that

$$c(0)+c(1)+\ldots+c(x-1) \;<\; n+2 \;\leq\; c(0)+c(1)+\ldots+c(x).$$

$$\text{Then } H(n) \;=\; \begin{cases} \text{L}, & \text{if } x \text{ is even,} \\ \text{R}, & \text{if } x \text{ is odd.} \end{cases}$$

But we can do better. The two inequalities may be checked using the graph of the bounded sum of the continued fraction and not the continued fraction itself.

Main goal

This leads us to the main goal of the current research.

For any representation R (considered as a function) we define a new representation $\mathcal{G}(R)$ by:

$$\mathcal{G}(R)(x,y) = \begin{cases} 0, & \text{if } R(x) = y, \\ 1, & \text{if } R(x) \neq y. \end{cases}$$

Question: Is $\mathcal{G}(R)$ subrecursively equivalent to a known representation, or it gives rise to a new subrecursive degree?

Two technical tools

(Tool 1) : There exists a function $t : \mathbb{N} \to \mathbb{N} \setminus \{0\}$, such that $\mathcal{G}(t) <_S t$.

Informally, t is a complex function, but its graph is simple.

For a function s, let s^Σ be the bounded sum of s, $s^\Sigma(x) = \sum_{y=0}^{x} s(y)$.

(Tool 2) : There exists a function $s : \mathbb{N} \to \mathbb{N} \setminus \{0\}$, such that $\mathcal{G}(s^\Sigma) <_S \mathcal{G}(s)$.

Informally, the graph of s is complex, but the graph of its bounded sum is simple.

Applications

Let us take α to be the irrational number with continued fraction t, where t is the function given by Tool 1. We obtain

$$\mathcal{G}([\,]) <_S [\,].$$

Let us take β to be the irrational number with continued fraction s, where s is the function given by Tool 2. Then

$$\mathcal{G}([\,]^\Sigma) <_S \mathcal{G}([\,]).$$

We also have shown: $H \leq_S \mathcal{G}([\,]^\Sigma)$ (in fact, $H \equiv_S \mathcal{G}([\,]^\Sigma)$). Combining these results we obtain:

Theorem

$$D \equiv_S H <_S \mathcal{G}([\,]) <_S [\,].$$

Left and right best approximations

Let $\alpha \in (0,1)$ be irrational and $(l_1, r_1), (l_2, r_2), \ldots, (l_n, r_n), \ldots$ be its sequence of intervals in the Farey pair tree.

The unique strictly increasing function $L : \mathbb{N} \to \mathbb{Q}$, such that $Ran(L) = \{l_i \mid i \in \mathbb{N}\}$, will be called the complete left best approximation of α.
The unique strictly decreasing function $R : \mathbb{N} \to \mathbb{Q}$, such that $Ran(R) = \{r_i \mid i \in \mathbb{N}\}$, will be called the complete right best approximation of α.
It is known that

$$D <_S L <_S [\,], \quad D <_S R <_S [\,], \quad \{L, R\} \equiv_S [\,],$$

in particular, L and R are subrecursively incomparable.

Theorem

$$\mathcal{G}(L) \equiv_S D \equiv_S \mathcal{G}(R).$$

$\mathcal{G}([\,]), L, R$

Theorem

$$\mathcal{G}([\,]) \not\leq_S L, \quad \mathcal{G}([\,]) \not\leq_S R.$$

Proof: take $H = \mathrm{R}^{s(0)}\mathrm{L}\mathrm{R}^{s(1)}\mathrm{L}\mathrm{R}^{s(2)}\mathrm{L}\ldots$, where s is the function from Tool 2.

Theorem

$$L \not\leq_S \{R, \mathcal{G}([\,])\}, \quad R \not\leq_S \{L, \mathcal{G}([\,])\}.$$

Proof: take $H = \mathrm{R}^{t(0)}\mathrm{L}\mathrm{R}^{t(1)}\mathrm{L}\mathrm{R}^{t(2)}\mathrm{L}\ldots$, where t is the function from Tool 1.

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Thanks for your attention!