The complexity of finding supergraphs

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Fix a countable graph G.

- Is G a(n induced) supergraph of an input graph H? (decision problems).
- If yes, can we find a copy of H in G? (search problems).

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In this talk, we always assume G to have a computable copy.

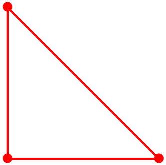
Graphs

The graphs $G = (V, E)$ we consider are countable, undirected, and without self-loops: that is, $V \subseteq \mathbb{N}$ and E satisfies anti-reflexivity and symmetry.

Definition

Given two graphs G and H we say that:

- G is a *supergraph* of H if $V(G) \supseteq V(H)$ and $E(G) \supseteq E(H)$;
- G is an *induced supergraph* of H if G is a supergraph of H and $E(G) = E(H) \cap (V(G) \times V(G))$.



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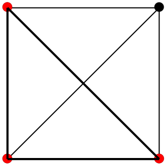

G_0 is an (induced) supergraph of G.

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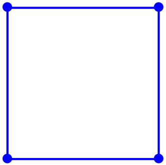

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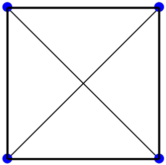

G_1 is a supergraph of G, but not an induced one.

Weihrauch reducibility and Effective Wadge reducibility

Computable analysis generalizes computability for functions on $\mathbb{N}$ to functions on $\mathbb{N}^{\mathbb{N}}$ (Baire space) and to represented spaces in general.

Definition

A *represented space* $\mathbf{X}$ is a pair (X, δ_X) where X is a set and $\delta_X :\subseteq \mathbb{N}^{\mathbb{N}} \to X$ is a (possibly partial) surjective function called *representation map*. We say that $p \in \mathbb{N}^{\mathbb{N}}$ is a *name* for x if $\delta_X(p) = x$.

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In this talk:

- **Gr** is the represented space of graphs, where a name for a graph is given by its characteristic function;
- **EGr** is the represented space of graphs, where a name for a graph is given by an enumeration of its vertices and edges.

f

Input: a name for an f-instance x.
Output: a name for (an element of) $f(x)$.

---

for a single input there may be multiple outputs!

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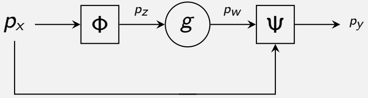
A problem f is *Weihrauch reducible* to g ($f \leq_W g$), if there are computable maps $\Phi, \Psi :\subseteq \mathbb{N}^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ s.t.

- for every name p_x for some input x of f, $\Phi(p_x) = p_z$, where p_z is a name for some input z for g and,
- for every name p_w for a solution w of $g(z)$, $\Psi(p_x \oplus p_w) = p_y$ where p_y is a name for a solution y of $f(x)$.



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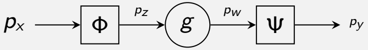

In case Ψ has no access to the original input of f (i.e. $\Psi(p_w) = p_y$), we say that the reduction is *strong* ($f \leq_{\mathrm{sW}} g$).

Wadge reducibility gives a notion of complexity between sets of topological spaces. Here we study its effective counterpart.

Definition

Let $A, B \subseteq \mathbb{N}^{\mathbb{N}}$. We say that B *effectively Wadge reduces* to A if there exists a computable function f such that $x \in B \iff f(x) \in A$.
For a (non-ambiguous) class Γ, we say that A is *Γ-complete* if $A \in \Gamma$ and, for every $B \in \Gamma$, B effectively Wadge reduces to A.

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Notation:

- $G \supseteq_{\mathsf{is}} H :\iff (\exists G' \cong G)(G'$ is an induced supergraph of $H)$;
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For a fixed countable graph G, we consider sets of (names of) graphs of the form

$$\{H \in (\mathbf{E})\mathbf{Gr} : G \supseteq_{\mathbf{(i)s}} H\} := \{p \in \operatorname{dom}(\delta_{(E)Gr}) : G \supseteq_{\mathbf{(i)s}} \delta_{(E)Gr}(p)\},$$

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N.B. all sets above are Σ^1_1.

Decision problems

The following problems were introduced in [BHW21]. For a fixed graph G:

IS^G- (induced subgraph)

Input: $H \in \mathbf{Gr}$.
Output: 1 if $G \supseteq_{\mathbf{is}} H$, 0 otherwise.

If $G \in \mathbf{EGr}$, the corresponding problem is denoted by $e\mathsf{IS}_G$.

S^G - (subgraph)

Input: $H \in \mathbf{Gr}$.
Output: 1 if $G \supseteq_{\mathbf{s}} H$, 0 otherwise.

If $G \in \mathbf{EGr}$, the corresponding problem is denoted by $e\mathsf{S}_G$.

$G \in \mathbf{Gr} \rightarrow$ characteristic function, $G \in \mathbf{EGr} \rightarrow$ enumeration.

LPO

Input: $p \in 2^{\mathbb{N}}$.
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WF

Input: a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$.
Output: 1 if T is *well-founded* i.e., it has no infinite path, 0 otherwise.

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N.B. $\{H \in \mathbf{Gr} : G \supseteq_{\mathsf{is}} H\}$ is $\Sigma^1_1 \implies \mathsf{IS}^G \leq_{\mathrm{sW}} \mathsf{WF}$ (similarly for the other sets/problems).

Given a countable graph G, does $(e)\mathsf{IS}^G \equiv_{\mathrm{sW}} \mathsf{WF}$ or $(e)\mathsf{IS}^G <_{\mathrm{sW}} \mathsf{WF}$ (similarly for $(e)\mathsf{S}^G$)?

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In [CP22], we solved one of their open questions showing that,

- for the induced subgraph case, if the input graph is in **Gr**, these problems are either equivalent to LPO (if G is finite) or to WF (if G is infinite);
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A REMARK

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Observation: it is easy to find graphs for which the corresponding problem reaches WF (i.e., the infinite ray R).

We will show that for the supergraph problem "it's difficult being difficult".

Summary

	Wadge	Weihrauch
G finite	$\{H \in (\mathbf{E})\mathbf{Gr} : G \supseteq_{\mathsf{is}} H\}$ is Π^0_1-complete	$(e)\mathsf{IS}^G \equiv_{\mathrm{sW}} \mathsf{LPO}$
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K_ω	$\{H \in (\mathbf{E})\mathbf{Gr} : K_\omega \supseteq_{\mathsf{s}} H\}$ is computable	$(e)\mathsf{S}^{K_\omega} \equiv_{\mathrm{sW}} \mathsf{id}$
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K_ω denotes the complete graph on $\mathbb{N}$.

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$G = \bigotimes_{i\geq 1} R_i$	$\{H \in (\mathbf{E})\mathbf{Gr} : G \supseteq_{\mathsf{(i)s}} H\}$ is Π^0_3-complete	$(e)\mathsf{IS}^G \equiv_{\mathrm{sW}} (e)\mathsf{S}^G \equiv_{\mathrm{sW}} \mathsf{LPO}''$
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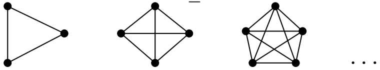
$\bigotimes_{i\geq 1} K_i$



The results in red answer positively a question left open in [BHW21], namely:

Is there a computable graph G such that $\mathsf{LPO} <_{\mathrm{sW}} \mathsf{IS}^G$? Yes.

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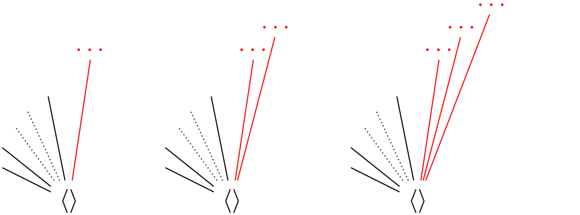
$\mathcal{S}$	$\{H \in (\mathbf{E})\mathbf{Gr} : \mathcal{S} \supseteq_{\mathsf{(i)s}} H\}$ is Π^0_5-complete	$(e)\mathsf{IS}^G \equiv_{\mathrm{sW}} (e)\mathsf{S}^G \equiv_{\mathrm{sW}} \mathsf{LPO}^{(4)}$.

$\mathcal{S}$ is the disconnected union of $(T_n)_{n\in\mathbb{N}}$, where every T_n is a tree having finite paths of any length (in black) and $n+1$-many paths of infinite length (in red).



The proof of the fact that this set is complete was suggested by an anonymous referee of [CP23].

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Maybe with other "strange" graphs, we could go beyond Π^0_5: but is there some G such that $(e)\mathsf{IS}^G \equiv_{\mathrm{sW}} \mathsf{WF}$?

Search problems

(preliminary results!)

The problems below lead to 8 versions, depending on whether the input/output is given via characteristic function or enumeration (4 for the induced supergraph case and 4 for the supergraph one).

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The first result we obtain is that there exists a graph whose lower bound is $\mathsf{C}_{\mathbb{N}}$.

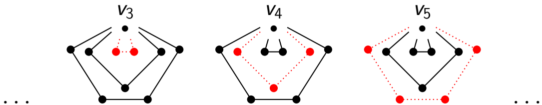
$\mathsf{C}_{\mathbb{N}}$

Input: an enumeration of the complement of a nonempty closed subset A of $\mathbb{N}$.
Output: some $p \in A$.

The graph H_{CN} has:

- as vertex set, for every $n \in \mathbb{N}$ a dedicated vertex v_n and
- for every $k \neq n$, a cycle of length k containing v_n. All the cycles are otherwise disjoint.

For example, H_{CN} on v_3, v_4 and v_5 looks like this (red vertices/edges are missing):



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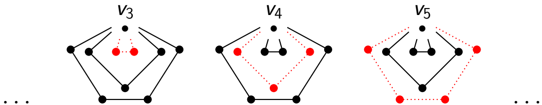

Theorem (C., Pauly)

$\mathsf{C}_{\mathbb{N}} \leq_{\mathrm{W}} \mathsf{SupCopy}_{H_{\mathrm{CN}}}$.

The graph R (the infinite ray)

We study the same problem for the more "natural" graph R.

$\mathsf{SupCopy}_\mathsf{R}$ can be rephrased as

- given in **input** either only finite line segments or finitely many line segments (possibly zero) plus a copy of R,
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The style of reasoning employed to in the study of $\mathsf{SupCopy}_{\mathsf{R}}$ is reminiscent of the study of the degrees of bi-embeddable categoricity of equivalence relations (as only the number and size of connected components matter).

Work in Progress

(Pt. 1)

Lower bounds for $\mathsf{SupCopy}_{\mathsf{R}}$

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Theorem (C., Pauly)

Let f be a pointed fractal. T.f.a.e.:

- $\mathsf{ACC}_{\mathbb{N}} \times f \le_{\mathrm{W}} \mathsf{SupCopy}_{\mathrm{R}}$;
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$\mathsf{ACC}_{\mathbb{N}}$ is the restriction of $\mathsf{C}_{\mathbb{N}}$ to sets of the form $\{\mathbb{N}\}$ or $\{\mathbb{N} \setminus \{n\} : n \in \mathbb{N}\}$.
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As a corollary, we obtain that $\lim \not\le_W \mathsf{SupCopy}_{\mathsf{R}}$ and Π^0_2-$\mathsf{C}_{\mathbb{N}} \not\le_W \mathsf{SupCopy}_{\mathsf{R}}$.

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Work in Progress

(Pt. 2)

So far, we have obtained sets of (names of) graphs being Γ-complete for some Γ being a Σ or Π class.

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Theorem (C., Marcone, Pauly)

$\{H \in (\mathbf{E})\mathbf{Gr} : \mathrm{R} \supseteq_{(\mathbf{i})\mathbf{s}} H\}$ and $\{H \in (\mathbf{E})\mathbf{Gr} : L \supseteq_{(\mathbf{i})\mathbf{s}} H\}$ are $\Sigma^0_3 \cup \Pi^0_3$-complete.

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Thanks for Your attention!