ON THE UNSTEADY MOTION OF A VISCOUS HYDROMAGNETIC FLUID CONTAINED BETWEEN ROTATING COAXIAL CYLINDERS OF FINITE LENGTH

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Abstract

The problem of unsteady rotational motion of electrically conducting viscous incompressible fluid, contained within two asxially concentric cylinders of finite length in the presence of an axial symmetric magnetic field of constant strength, has been solved exactly using finite Hankel transform in combination with a technique presented in this paper. This paper presents a complete of the problem under consideration, which has been of interest for many years; moreover the Pneuman-Lykoudis solution in Magnetohydrodynamics and Childyat solution in hydrodynamics appears as a special case of this study. The analysis shows that the disturbance in the fluid disappears by increasing the magnetic field.

Keywords and phrases: Unsteady, motion, viscous, fluid, coaxial cylinders, finite length.

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1 Introduction

In recent years, the study of electrically conducting fluids has received much attention in areas such as rocket flight and high-speed reentry missiles. It is known that the motion of a conducting fluid in a magnetic field induces electric currents in the fluid, thereby modifying the field; at the same time, the flow in the magnetic field produces mechanical forces which in turn modify the motion.

Recently, C. D. Ghildyal [2] has presented a solution of the problem in hydrodynamics concerning the unsteady motion of a viscous, rotating fluid contained between two infinitely long coaxial cylinders. However, the motion of an electrically conducting fluid contained between two coaxial rotating cylinders in the presence of magnetic field becomes much more complicated. S. Chandrasekhar [1] has discussed the rotational and thermal instability of a viscous, rotating, electrically conducting fluid within two infinitely long coaxial cylinders in presence of a magnetic filed, while G. Pneuman and P. Lykoudis [6] have studied the steady motion of a viscous, rotating, electrically conducting fluid between two concentric cylinders of finite length.

This paper deals with the unsteady rotational motion of an electrically conducting fluid within two concentric cylinder of finite length. The unsteady motion is caused by an impulsive twist given to the inner cylinder so that it starts rotating suddenly in a previously steady fluid in the presence of an axial magnetic field.

The problem has attracted attention in practice; and since from point of experiment, infinite cylinders can not be used it would be advantageous to obtain an analytical solution for the case when the cylinders are finite. However, according to the knowledge of the author of this paper, such a solution has not yet been given in literature.

We assume that the material of which the cylinders are made is an electrical conductor. The surfaces of the cylinders in contact with the fluid will be electrically charged and there will be surface currents as well. The effect of these surface charges and the surface currents will be to prevent any electrical field in the radial direction and any magnetic field in the z-direction from penetrating into the cylinders $[1, 2]$. However, we suppose that the bounding planes are of insulating material, so that there will be static surface charge but not surface current.

In this paper, we are directly interested in the deformation of the fluid in the presence of a magnetic field; hence, we are searching for the velocity distribution in a magnetic field. That is the fundamental object of this study. After the fluid field is determined, we can study the electromagnetic field from the knowledge of the boundary and initial condition imposed upon the field. Not that a wide range of solutions compatible with these conditions are possible$[1-5]$.

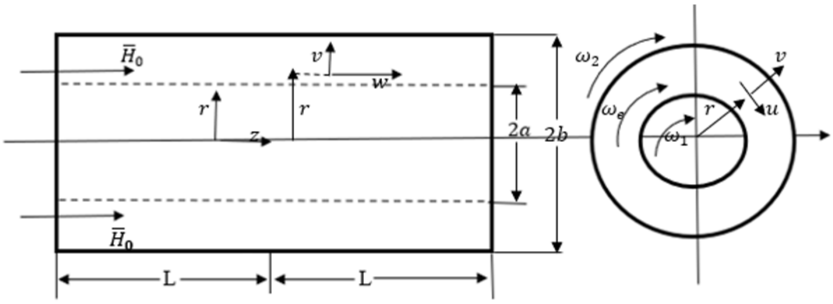

Figure 1: Geometry of the cylinders

2 Basic part

It is convenient for the study of this problem to introduce a cylindrical system (r, θ, z). a Let the length of the cylinders be $2L$, so that the origin of the system is located at the middle point on axis of cylinders. We assume that the velocity field is given by

$$\vec{V} = \vec{V}(\nu, u, w) \tag{1}$$

where ν, u, w are the components of the velocity in the radial, circumferential and axial directions, respectively. Since the motion of the fluid is primarily in the circumferential directions, the radial velocity v in the flow may be neglected on comparison with the circumferential velocity u. However, the assumption that the axial velocity w is zero is true only if the cylinders are infinitely long. If the cylinders are finite in length, the axial velocity is not zero. Therefore, the "end effects" should be considered. Pneman and Lykoudis [6] showed that the "end effects"will be confined to regions very close to the bounding planes and the axial velocity may be safely neglected in the region midway between these planes.

Furthermore, we impose a magnetic field of constant strength. H_0, in axial direction and we assume that there are perturbations h_r, h_θ and h_z in the magnetic field and E_r, E_θ, E_z in the electrical field. Note that it can be assumed that $h_r << h_\theta$ and $h_r << h_z$, since the motion occurs in the $\theta-$direction. Therefore

$$\overrightarrow{H} = \overrightarrow{H}\left(0, h_0, H_0 + h_z\right). \tag{2}$$

Similarly,

$$\overrightarrow{E} = \overrightarrow{E}\left(0, E_\theta, E_z\right). \tag{3}$$

It has been assumed throughout this paper that physical and electromagnetic properties of the flow are known. Since the flow is incompressible with constant properties, the energy equation and characteristic equation may be omitted; therefore, the problem is reduced to the solution of the following system of equations, namely:

$$\text{Continuity} : \Delta \cdot \bar{V} = 0. \tag{4}$$

$$\text{Momentum} : \frac{\partial \vec{V}}{\partial t} + (\Delta \cdot \vec{V})\vec{V} + \frac{1}{\rho}\Delta p - \frac{\nu}{\rho}\Delta^2 \vec{V} - \frac{\mu_e}{\rho}(\vec{J} \times \vec{H}) = 0. \tag{5}$$

$$\text{Maxwell equations} : \Delta \cdot \vec{D} - \rho_e = 0. \tag{6}$$

$$\Delta \cdot \vec{H} = 0, \tag{7}$$

$$\Delta \times \vec{E} + \mu_e \frac{\partial \vec{H}}{dt} = 0, \tag{8}$$

$$\Delta \times \vec{H} - \vec{J} - \frac{\partial \vec{D}}{\partial t} = 0. \tag{9}$$

$$\Delta \cdot \vec{J} + \frac{\partial \rho_e}{dt} = 0 \tag{10}$$

Here are: $\vec{V}$ -velocity of the flow (not perturbed), ρ -density of the flow, p -pressure, μ -dynamical visco, $\vec{J}$ -current density, $\vec{H}$ -strength of magnetic field, μ_e -magnetic permeability, $\vec{E}$ -strength of electric field, p_e -charge density, t -time.
Note that

$$\vec{B} = \nu_e \vec{H}, \tag{11}$$

$$\vec{D} = \varepsilon_e \vec{E}, \tag{12}$$

$$\vec{J} = \sigma \left(\vec{E} + \nu_e \vec{q} \times \vec{H} \right), \tag{13}$$

where $\vec{B}$ is magnetic induction, $\vec{D}$ is displacement current, e_e is dielectric constant, s is electric conductivity. The system of eq. (4), (5), (6), (7), (8), (9), (10) is subjected to the conditions imposed upon the hydrodynamic field and those imposed upon the electromagnetic field. Assuming that both cylinders are rotating in the same direction, the no-slip condition at the walls of the cylinders requires that

$$u\,(a, \theta, z, t) = \omega_1 a, \tag{14}$$

$$u\,(b, \theta, z, t) = \omega_2 b, \tag{15}$$

$$u\,(r, \theta, \pm L, t) = \omega_e r, \tag{16}$$

where ω_1 and ω_2 are angular velocities of the cylinders, a is the radius of the inner cylinder, b is the radius of the outer cylinder, and ω_e is angular velocity of the bounding end planes.

Moreover, the initial condition, for $t = 0$, is prescribed, namely,

$$u\left(r, \theta, z, 0\right) = f\left(r, z\right), \tag{17}$$

where $f\left(r, z\right)$ is a regular function every where in the domain under consideration.

In addition to these conditions, we have the conditions imposed by electromagnetic field which require that on the surface of the discontinuity, such as $r = a$ and $r = b$, the normal and tangential components of the magnetic induction and electric field suffer a discontinuity, which is equal for the magnetic field to the components of the surface current density at the right angle to the field, and for the electric field to the components of the surface charge density [1, 2] perpendicular to the electric field. Note that the surface current density is measured in amper per meter, and surface charge density is Coulombs per square meter. However, in accordance with the assumption that the bounding end planes are of insulating material, we have that the normal and tangential components of the magnetic induction and electric field are continuous on those planes.

The main characteristic of the field equations (6), (7), (8) and (10) is that electric field depends upon the magnetic field through the time variation of $\vec{H}$; and the magnetic field depends on the electric field through the time variation of $\overrightarrow{D}$. Therefore, the initial condition in the electromagnetic field has to be introduced.

Evidently, the time derivative of $\vec{D}$ acts as a source for $\vec{H}$, and the derivative of $\vec{H}$ acts as a source for $\vec{E}$. Hence, coupling between the electric field and the magnetic field becomes bilateral. Note that the hydrodynamic field is coupled with the electromagnetic field through the electrobody force. At the first glance, the problem from the mathematical point of view represents a mixed boundary value problem of a very complicated nature. That is true. However, the problem can be solved using the method of the operational calculus outlined in this paper. From the continuity eq. (4), since circumferential symmetry is assumed, it follows that

$$u = u\left(r, z, t\right). \tag{18}$$

Then (5) and (17) lead to following equations in scalar form

$$-\rho\frac{u^2}{r} - F_r + \frac{\partial p}{\partial r} = 0, \tag{19}$$

$$\rho\frac{\partial u}{\partial t} - \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right) - F_\theta = 0, \tag{20}$$

$$F_z - \frac{\partial p}{\partial z} = 0, \tag{21}$$

where F_r, F_θ, F_z are coordinates of the electrobody force in r, θ and z direction, respectively. Then

$$\overrightarrow{F} = \vec{J} \times \vec{B}, \tag{22}$$

where $\vec{J}$ and $\vec{B}$ are prescribed by eq. (11) and (13). However, it can be shown that h_z is zero everywhere in the field. The proof is simple. By circumferential symmetry eq. (7) becomes

$$\frac{\partial h_z}{\partial z} = 0, \tag{23}$$

or

$$h_z = h_z\left(r, t\right). \tag{24}$$

Moreover, on the boundary $z = \pm L$ we have that the magnetic field is constant, namely $\vec{H} = H_0 \vec{e}_z$. Hence, $h_z = 0$ at the planes $z = \pm L$, and therefore is zero everywhere. This completes the proof. Hereafter, $\vec{e}_r, \vec{e}_\theta, \vec{e}_z$ denote the unit vectors in r, θ and z-direction.

From eq. (1), (2) and (13) it follows

$$\vec{J} = \vec{J}\left(\ \sigma\mu_e H_0 u, \quad \sigma E_\theta, \sigma E_z \ \right). \tag{25}$$

Then, eq. (22) becomes

$$\vec{F} = \vec{F}\left[\mu_e \sigma\left(E_\theta H_0 - E_z h_\theta\right), \ -\sigma\mu_e^2 H_0^2, \ \sigma\mu_e^2 H_0 h_\theta u\right]. \tag{26}$$

Hence, eq. (19), (20), (21) and (26) lead to

$$-\rho\frac{u^2}{r} - \mu_e \sigma\left(E_\theta H_0 - E_z h_\theta\right) + \frac{\partial p}{\partial r} = 0 \tag{27}$$

$$\rho\frac{\partial u}{\partial t} - \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r} + \frac{\partial^2 u}{\partial z^2}\right) + \sigma\mu_e^2 H_0^2 u = 0, \tag{28}$$

$$\sigma\mu_e^2 H_o h_\theta u - \frac{\partial p}{\partial z} = 0. \tag{29}$$

From eq. (6) and (12), by virtue of circumferential symmetry, it follows

$$\frac{\partial E_z}{\partial z} = \frac{\rho_e}{\varepsilon_e}. \tag{30}$$

Moreover, from eq. (8) one obtains

$$\Delta \times \vec{F} + \mu_e \frac{\partial h_\theta}{\partial t}\vec{e}_\theta = 0. \tag{31}$$

Note that by taking the circumferential symmetry into account the first part of eq. (31) becomes

$$\Delta \times \vec{E} = \vec{\aleph}\left(-\frac{\partial E_\theta}{\partial z}, -\frac{\partial E_z}{\partial r}, \frac{1}{r}\frac{\partial\left(rE_\theta\right)}{\partial r}\right), \tag{32}$$

where $\vec{\aleph} = \Delta \times \vec{E}$.

Hence, eq. (31) lead to

$$\frac{\partial E_\theta}{\partial z} = 0, \tag{33}$$

$$\frac{\partial\left(rE_\theta\right)}{\partial r} = 0, \tag{34}$$

$$\frac{\partial E_z}{\partial r} - \mu_e\frac{\partial h_\theta}{\partial t} = 0. \tag{35}$$

From eq. (33) it follows that

$$E_\theta = \phi\left(r, t\right). \tag{36}$$

Moreover, eq. (34) yields

$$rE_\theta = K\left(t\right), \tag{37}$$

where $K\left(t\right)$ is time depending constant.

Therefore,

$$E_\theta = \frac{K\left(t\right)}{r}. \tag{38}$$

Now, we consider eq. (9). Note that

$$\Delta \times \vec{H} = \vec{H}\left(-\frac{\partial h_\theta}{\partial z}, 0, \frac{1}{r}\frac{\partial\left(rh_\theta\right)}{\partial r}\right). \tag{39}$$

Moreover, eq. (9), (12), (25) and (39) lead to

$$\sigma\mu_e H_0 u + \frac{\partial h_\theta}{\partial z} = 0, \tag{40}$$

$$\sigma E_\theta + \varepsilon_e\frac{\partial E_\theta}{\partial t} = 0, \tag{41}$$

$$\frac{\partial E_z}{\partial t} + \frac{\sigma}{\varepsilon_e}E_z - \frac{1}{r\varepsilon_e}\frac{\partial\left(rh_\theta\right)}{\partial r} = 0. \tag{42}$$

Evidently, from eq. (41) it follows

$$E_\theta = C_1 e^{-\frac{\sigma}{\varepsilon_e}t}, \tag{43}$$

where C_1 is a pure constant, not ye determined. Eq. (38) and (43) lead to

$$K(t) = C_1 r e^{-\frac{\sigma}{\varepsilon_e} t}. \tag{44}$$

However, at $t = 0$, $E_\theta(0) = 0$; hence, $C_1 = 0$. Therefore, $K(t) = 0$, and E_θ is zero everywhere for all t.

Two more equations, eqs. (10) and (30), may be combined; namely, form eqs. (10) and (25), it follows that

$$\sigma\mu_e H_0 \frac{\partial u}{\partial r} + \sigma \frac{\partial E_z}{\partial z} + \frac{\partial \rho_e}{\partial t} = 0. \tag{45}$$

Eq. (30) and (45) lead to

$$\frac{\partial \rho_e}{\partial t} + \frac{\sigma}{\varepsilon_e}\rho_e + \sigma\mu_e H_0 \frac{\partial u}{\partial r} = 0. \tag{46}$$

Hence, we arrive at the following mixed boundary value problem; namely, to find u, h_θ, E_z, $\frac{\partial p}{\partial r}$, $\frac{\partial \rho}{\partial z}$ and ρ_e from the following system of equations.

$$\rho\frac{u^2}{r} - \mu_e\sigma E_z h_\theta - \frac{\partial p}{\partial r} = 0, \tag{47}$$

$$\rho\frac{\partial u}{\partial t} - \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r} + \frac{\partial^2 u}{\partial z^2}\right) + \sigma\mu_e^2 H_0^2 u = 0, \tag{48}$$

$$\sigma\mu_e^2 H_0 h_\theta u - \frac{\partial p}{\partial z} = 0, \tag{49}$$

$$\sigma\mu_e H_0 u + \frac{\partial h_\theta}{\partial z} = 0, \tag{50}$$

$$\frac{\partial E_z}{\partial t} + \frac{\sigma}{\varepsilon_e}E_z - \frac{1}{r\varepsilon_e}\frac{\partial (r h_\theta)}{\partial r} = 0, \tag{51}$$

$$\frac{\partial \rho_e}{\partial t} + \frac{\sigma}{\varepsilon_e}\rho_e + \sigma\mu H_0 \frac{\partial u}{\partial r} = 0. \tag{52}$$

Evidently, the key to the solution of this system is eq. (48); i. e. as soom as u is determined from eq. (48) then h_θ is evaluated from eq. (50). Then $\frac{\partial p}{\partial z}$, $\frac{\partial \rho}{\partial r}$, E_z, ρ_e follow directly from eq. (47) ,(47), (49), (51), (52).

3 Solution of the problem

The problem will be solved if eq. (48), subjected to the conditions eq. (14), (15), (16) and (17) is solved.

For such purposes, denote

$$\frac{\mu}{\rho} = \nu; \frac{\sigma\mu_e^2 H_0^2}{\mu} = M^2 \tag{53}$$

Then eq. (48) becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \left(\frac{1}{r^2} + M^2\right) u + \frac{\partial^2 u}{\partial z^2} - \frac{1}{\nu}\frac{\partial u}{\partial t} = 0. \tag{54}$$

Evidently, if $M = 0$ and in addition $L \to \infty$ then the solution is reduced to the simple solution, recently given in hydrodynamics for infinite long cylinders [1]. This will be later on evidently, from resulting velocity field.

Written in operator form, the boundary value problem, under consideration, can be written as

$$\left.\begin{array}{c} \left.\begin{array}{r} u(a,z,t) = \omega_1 a, \\ u(b,z,t) = \omega_2 b, \\ u(r,\pm L,t) = \omega_e r \end{array}\right\} \end{array} \begin{array}{c} D\{u(r,z,t)\} = 0, \\ u(r,z,0) = f(r,z), \\ \\ t > 0 \end{array}\right\}. \tag{55}$$

The solution to this problem follows directly by using method of superposition, namely:

$$u(r,z,t) = u_0(r,z,t) + u_1(r,z,t) \tag{56}$$

where u_0 and u_1 are the solutions of the following boundary value problems:
I.

$$\left.\begin{array}{c} \left.\begin{array}{r} u_0(a,z,t) = 0, \\ u_0(b,z,t) = 0, \\ u_0(r,\pm L,t) = 0 \end{array}\right\} \end{array} \begin{array}{c} D_0\{u_0(r,z,t)\} = 0, \\ u_0(r,z,0) = f(r,z), \\ \\ t > 0 \end{array}\right\}, \tag{57}$$

II.

$$\left.\begin{array}{c} \left.\begin{array}{r} u_1(a,z,t) = \omega_1 a, \\ u_1(b,z,t) = \omega_2 b, \\ u_1(r,\pm L,t) = \omega_e r \end{array}\right\} \end{array} \begin{array}{c} D_1\{u_1(r,z,t)\} = 0, \\ u_1(r,z,0) = 0, \\ \\ t > 0 \end{array}\right\}. \tag{58}$$

Note that before we start to find the solution of this problem the initial condition given by function $f(r,z)$ has to be specified. First, we assume that the external cylinder is rotating with $\omega_2$and the interior cylinder is fixed in stead motion [3]. The after a velocity $u(r,z)$ of steady fluid is obtained, an impulsive twist is given to the interior cylinder, so that it rotates with a constant angular velocity equal to velocity of steady fluid at the instant of application of the impulsive twist. This solution for $f(r,z)$ is obtained ([1], [2]) by means of Hankel Transform in the form of the Bessel functions the first and second kind $J_\tau()$, $Y_\tau()$, of the order $t = 1; 2$, namely

$$f(r,z) = \tfrac{\pi^2}{2}\sum_\beta \tfrac{\beta^2 J_1(\beta b)}{J_1^2(\beta a) - J_1^2(\beta b)} \times \left\{ \left[G_2(\beta) - G_1(\beta)\right] \tfrac{cosh\sqrt{\beta^2+a^2}}{cosh\sqrt{\beta^2+a^2}}\tfrac{z}{L} + G_1(\beta) \right\} B_1(\beta r) \tag{59}$$

where

$$G_1(\beta) = \frac{2\omega_2 b}{\pi(\beta^2+a^2)}\frac{J_1(\beta a)}{J_1(\beta b)}, \tag{60}$$

$$G_2(\beta) = \tfrac{\omega_e}{\beta}\left[b^2\left(J_2(\beta b)Y_1(\beta a) - Y_2(\beta b)J_1(\beta a)\right) - a^2\left(J_2(\beta a)Y_1(\beta a) - Y_2(\beta a)J_1(\beta a)\right)\right], \tag{61}$$

$$B_1(\beta r) = J_1(\beta r)Y_1(\beta a) - Y_1(\beta r)J_1(\beta a), \tag{62}$$

and β is positive root of equation:

$$J_1(\beta b)Y_1(\beta a) - Y_1(\beta b)J_1(\beta a) = 0. \tag{63}$$

Now, as soon as the velocity, given by eq. (59) has been reached, the impulsive twist is applied to the inner cylinder so that it starts to rotate with angular velocity ω_1 in the same direction as ω_2. Hence, we have determined the initial condition of the problem. Now, we are going to solve the problems given by eq. (57) and (58), which hereafter are referred as fundamental I and II. respectively:

Solution of the Fundamental Problem I

Let

$$u_0(r,z,t) = R(r)Z(z)T(t). \tag{64}$$

Then eq. (54) leads

$$\frac{\partial^2 Z}{\partial z^2} + A_n^2 Z = 0, \tag{65}$$

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + \left(k_m^2 - M^2 - \frac{1}{r^2}\right)R = 0, \tag{66}$$

and

$$\frac{\partial T}{\partial t} + \nu\left(k_m^2 + A_n^2\right)T = 0. \tag{67}$$

Eq. (65) with the boundary condition $z = \pm L$ lead to

$$Z = \sum_{n=0}^{\infty} A_n cos A_n \ z, \tag{68}$$

where A_n is a constant to be determined later and

$$A_n = \frac{2n+1}{2L}\pi, n = 0,\ 1,\ 2, ..., \tag{69}$$

Denote

$$k_m^2 - M^2 = a_m^2, m = 0,\ 1,\ 2, ..., \tag{70}$$

Then eq. (66) lead to

$$R\left(r\right) = D_1 J_1\left(a_m r\right) + D_2 Y_1\left(a_m r\right). \tag{71}$$

However,

$$R\left(a\right) = R\left(b\right) = 0. \tag{72}$$

Then

$$J_1\left(a_m a\right) Y_1\left(a_m b\right) + J_1\left(a_m b\right) Y_1\left(a_m a\right) = 0. \tag{73}$$

Hence, for given a and b, a_m are positive roots of eq. (73). Therefore, for any α_m we can find corresponding k_m given by eq. (70) for every m. Evidently, according to eq. (63) and (73) we have that $a_m = \beta$. Eq. (71) can be written in the form

$$R\left(r\right) = \sum_{m=0}^{\infty} D_m\left[Y_1\left(a_m a\right) J_1\left(a_m r\right) - J_1\left(a_m a\right) Y_1\left(a_m r\right)\right]. \tag{74}$$

Moreover, from eq. (67) it follows

$$T\left(t\right) = C e^{-r\left(k_m^2 + A_n^2\right)t}. \tag{75}$$

Hence, eq. (64) becomes

$$\begin{aligned} u_0\left(r, z, t\right) = \sum_{m}^{\infty}\sum_{n}^{\infty} A_{mn}\left[Y_1\left(a_m a\right) J_1\left(a_m r\right)\right. \\ \left. - J_1\left(a_m b\right) Y_1\left(a_m r\right)\right] cos A_n z e^{-\nu\left(k_m^2 + A_m^2\right)t}. \end{aligned} \tag{76}$$

The constant A_{mm} has to be determined in the usual way by using initial condition. Hence,

$$f\left(r, z\right) = \sum_{m}^{\infty}\sum_{n}^{\infty} A_{mn}\left[Y_1\left(a_m a\right) J_1\left(a_m r\right) - J_1\left(a_m b\right) Y_1\left(a_m r\right)\right] cos A_n \ z. \tag{77}$$

Note that the solutions of Bessel's equation satisfying the boundary conditions of the
Sturm-Liouville type form an orthogonal system. [5], with respect to the weight function $r$over interval $[a, b]$. Evidently,

$$\int_a^b rB_1^2\left(a_m r\right)dr = \frac{b^2}{2}B_2^2\left(a_m b\right) - \frac{a^2}{2}B_2^2\left(a_m a\right), \tag{78}$$

where

$$\left.\begin{array}{r} B_2\left(a_m b\right) = Y_1\left(a_m a\right)J_2\left(a_m b\right) - J_1\left(a_m a\right)Y_2\left(a_m b\right) \\ B_1\left(a_m a\right) = Y_1\left(a_m a\right)J_2\left(a_m a\right) - J_1\left(a_m a\right)Y_2\left(a_m a\right) \end{array}\right\}. \tag{79}$$

Hence, by means of generalized Fourier Analysis, it follows

$$A_{mn} = \frac{2}{L}\frac{\int_a^b\int_{-L}^{+L} rf\left(r,z\right)B_1\left(a_m r\right)cosA_n\ zdrdz}{b^2B_2^2\left(a_m b\right) - a^2B_2^2\left(a_m a\right)}. \tag{80}$$

Therefore, eq. (76) and (80) lead to

$$u_0\left(r,z,t\right) = \sum_m^{\infty}\sum_n^{\infty} A_{mn}B_1\left(a_m r\right)cosA_n\ ze^{-\nu\left(k_m^2+A_n^2\right)t}, \tag{81}$$

where A_{mn} is given by eq. (80), and $B_1\left(a_m r\right)$ by eq. (62). Thus, the solution of the problem defined by eq. (57) is determined.

The Solution of the Fundamental Problem II

We have to solve the boundary value problem eq. (58). For such purposes, we use finite Havkel. Transform with respect to r. Define

$$\overline{u}\left(a_m,z,t\right) = \int_a^b ru\left(r,z,t\right)B_1\left(a_m r\right)dr, \tag{82}$$

where a_m are positive roots of eq, (73). Then eq. (58) becomes

$$\begin{aligned} &\tfrac{\partial^2\overline{u_1}(a_m,z,t)}{\partial z^2} - k_m^2\overline{u_1}\left(a_m,z,t\right) - \tfrac{1}{\nu}\tfrac{\partial\overline{u_1}(a_m,z,t)}{\partial t} \\ &= -\tfrac{2}{\pi}\left[\omega_2 b\tfrac{J_1(a_m a)}{J_1[a_m b]} - \omega_2 a\right], \end{aligned} \tag{83}$$

with

$$\overline{u_1}\left(a_m,z,t\right) = 0, \tag{84}$$

and

$$\overline{u_1}\left(a_m,\pm L,t\right) = G_2\left(a_m\right)t > 0. \tag{85}$$

The solution of eq. (83) subjected to the conditions eq. (84) and (85) can be obtained using Laplace Transform. In this paper, we will not use Laplace Transform in order not to confuse problem with transforms; moreover, the inverse of Laplace Transform will be very difficult to manipulate. However, a technique will be demonstrated here which willbring the solution of the problem under consideration to be in closed form [1-5]. This technique, in fact, is based upon the known principle of superposition. Namely, a nonhomogeneous equation with homogeneous initial condition and nonhomogeneous boundary conditions can be reduced by substitution to a homogeneous equation with nonhomogeneous initial and boundary conditions.

By such a procedure, we relax the partial differential equation but complicate the initial and boundary conditions. In order to relax the boundary conditions, we use a substitution prescribed in this paper which permits the function to vanish on the boundary; however, the initial condition is now in functional form

Therefore, through two types o substitutions the nonhomogeneous partial differential equation with homogeneous initial condition and nonhomogeneous boundary conditions is reduced to a homogeneous partial differential equation with homogeneous boundary conditions and nonhomogeneous initial condition. Under such a procedure, the problem under consideration is solved exactly.

Let

$$\overline{u_1}\left(a_m, z, t\right) = \overline{u}_1^*\left(a_m, z, t\right) - \frac{A^*\left(a_m\right)}{k_m^2}, \tag{86}$$

where

$$A^*\left(a_m\right) = -\frac{2}{\pi}\left[\omega_2 b\frac{J_1\left(a_m a\right)}{J_1\left(a_m b\right)} - \omega_2 a\right]. \tag{87}$$

Then, the boundary value problem II, can be written as

$$\left.\begin{array}{r} \overline{D}_1^*\left\{\overline{u}_1^*\left(a_m, z, t\right)\right\} = 0, \\ \overline{u}_1^*\left(a_m, z, t\right) = \frac{A^*\left(a_m\right)}{k_m^2}, \\ \overline{u}_1^*\left(a_m, \pm L, t\right) = G_2\left(a_m\right) + \frac{A^*\left(a_m\right)}{k_m^2}. \end{array}\right\} \tag{88}$$

However, the boundary value problems given by eq. (88) possesses nonhomogeneous boundary and initial condition. Let

$$\overline{u}_1^*\left(a_m, z, t\right) = \phi\left(a_m, z, t\right) + Scoshk_m\, z, \tag{89}$$

where S has to be chosen so that $F\left(a_m, z, t\right)$ vanishes at the boundary, $z = \pm L$.

Evidently, from eq. (89) it follows

$$S=\left[G_2\left(a_m\right)+\frac{A^*\left(a_m\right)}{k_m^2}\right]\frac{1}{coshk_m\ L}. \tag{90}$$

Therefore,

$$\overline{u}_1^*\left(a_m,z,t\right)=\phi\left(a_m,z,t\right)+\left[G_2\left(a_m\right)+\frac{A^*\left(a_m\right)}{k_m^2}\right]\frac{coshk_m\ z}{coshk_m\ L}. \tag{91}$$

Substituting qe. (91) into eq. (88) it follows

$$\left.\begin{array}{c} D_\phi\left\{\phi\left(a_m,z,t\right)\right\}=0, \\ \phi\left(a_m,z,0\right)=\dfrac{A^*\left(a_m\right)}{k_m^2}-\left[G_2\left(a_m\right)+\dfrac{A^*\left(a_m\right)}{k_m^2}\right]\dfrac{coshk_m\ z}{coshk_m\ L,} \\ \phi\left(a_m,\pm L,t\right)=0, \qquad t>0 \end{array}\right\} \tag{92}$$

Hence,

$$\left.\begin{array}{c} \dfrac{\partial^2\phi\left(a_m,z,t\right)}{\partial z^2}-k_m^2\phi\left(a_m,z,t\right)-\dfrac{1}{\nu}\dfrac{\partial\Phi}{\partial t}\left[a_m,z,t\right]=0, \\ \phi\left[a_m,z,0\right]=\dfrac{A^*\left(a_m\right)}{k_m^2}-\left[G_2\left(a_m\right)+\dfrac{A^*\left(a_m\right)}{k_m^2}\right]\dfrac{coshk_m\ z}{coshk_m\ L}, \\ \phi\left(a_m,\pm L,t\right)=0, \qquad t>0 \end{array}\right\} \tag{93}$$

Using

$$\phi\left(a_m,z,t\right)=\Psi\left(a_m,z\right)T\left(t\right) \tag{94}$$

then, the partial differential Eq. (93) becomes

$$\left.\begin{array}{c} \frac{\partial^2\Psi\left(a_m,z\right)}{\Psi z^2}-\left(k_m^2-j^2\right)\Psi\left(a_m,z\right)=0, \\ \frac{\partial T}{\partial t}+j^2\nu T=0. \end{array}\right\} \tag{95}$$

From eq. (67) we have that

$$\frac{\partial T}{\partial t}+\left(k_m^2+A_n^2\right)\nu T=0. \tag{96}$$

Therefore,

$$j^2=\left(k_m^2+A_n^2\right). \tag{97}$$

Hence,

$$\frac{\partial^2\Psi\left(a_m,z\right)}{\partial z^2}+A_n^2\Psi\left(a_m,z\right)=0, \tag{98}$$

and

$$T\,(t) = Me^{-\nu\left(k_m^2 + A_n^2\right)t}, \tag{99}$$

where M is a constant.
Evidently,

$$\Phi_m\,(a_m, z, t) = \sum_n^{\infty} M_{mn} cosA_n\ ze^{-\nu\left(k_m^2 + A_n^2\right)t}. \tag{100}$$

Using generalized Fourier Analysis it follows that

$$M_{mn} = \frac{2sinA_n\ L}{A_n L}\left\{\frac{A^*\,(a_m)}{k_m^2} - \left[G_2\,(a_m) + \frac{A^*\,(a_m)}{k_m^2}\right]\frac{A_n^2}{k_m^2 + A_n^2}\right\}. \tag{101}$$

Then eq. (82), (83). (86), (89) lead to

$$\begin{aligned} u_1\,(r, z, t) = {} & \frac{p^2}{2}\sum_m^{\infty}\sum_n^{\infty}\frac{a_m^2 J_1\,(a_m b)}{J_1^2\,(a_m a) - J_1^2\,(a_m b)} \\ & \times M_{mn} B_1\,(a_m r)\,cosA_n\ ze^{-\nu\left(k_m^2 + A_n^2\right)t} \\ & + \frac{p^2}{2}\sum_m^{\infty}\frac{a_m^2 J_1\,(a_m b)}{J_1^2\,(a_m a) - J_1^2\,(a_m b)} B_1\,(a_m r) \\ & \left[\left(G_2\,(a_m) + \frac{A^*\,(a_m)}{k_m^2}\right)\frac{coshk_m\ z}{coshk_m\ L} - \frac{A^*\,(a_m)}{k_m^2}\right]. \end{aligned} \tag{102}$$

Hence, eq. (56), (81) and (102) lead to

$$\begin{aligned} u\,(r, z, t) = {} & \sum_m^{\infty}\sum_n^{\infty}\left[A_{mn} + \frac{\pi^2}{2}\ \frac{a_m^2 J_1(a_m b)}{J_1^2(a_m a) - J_1^2(a_m b)} M_{mn}\ \right] \\ & \times B_1\,(a_m r)\,cosA_n\ ze^{-\nu\left(k_m^2 + A_n^2\right)t} \\ & + \frac{\pi^2}{2}\sum_m^{\infty}\frac{a_m^2 J_1(a_m b)}{J_1^2(a_m a) - J_1^2(a_m b)} B_1\,(a_m r) \\ & \times\left[\left(G_2\,(a_m) + \frac{A^*(a_m)}{k_m^2}\right)\frac{coshk_m\ z}{coshk_m\ L} - \frac{A^*(a_m)}{k_m^2}\right] = T^* + S^*. \end{aligned} \tag{103}$$

4 Discussion of the solution

Eq. (103) represents the principal object of this paper, since this equation is the mathematical key of the problem. More important, this equation represents the general solution for the velocity field, from which other solutions previously given in literature ([1]) follow directly as special cases of this equation. At the outset, we recognized that Eq. (103) consists of two parts: The first part, T^*, represents the transient velocity and the second part, S^*, represents the steady velocity which was given previously in literature in literature [2].

Moreover, the transient part itself, T^*, consists of two parts, namely $T^* = T_1^* + T_2^*$, where T_1^* is the transient part resulting from suddenly applied initial velocity on the interior cylinder, and T_2^* is the part added to T_1^* when the steady state is finally attained.

Hence, Eq. (103) can be written as

$$u(r,z,t) = T_1^* + T_2^* + S^*. \tag{104}$$

Evidently T^*decreases when t increases. Moreover, by increasing the magnetic field we increase the value of M as can be seen from eq. (53). However, from eq. (70) it follows that $k_m^2 = a_m^2 + M^2$.

Therefore by increasing the magnetic field the values of k_m will increase also. Hence, by large increases of the magnetic field the transient part disappears quickly and the duration of disturbance is of shorter time; consequently the magnetic field has stabilized motion. Note that T_1^* results from the initial velocity given to the system and therefore is dependent on the function $f(r,z)$. However, T_2^* is a pure transient part, since it generates a source of disturbance, and as can be seen from eq. (103) is dependent on the ration $\frac{b}{a}$, namely:

1. If $\frac{b}{a} \to 1$ then from eq. (103) it follows $J_1^2(a_m a) \approx J_1^2(a_m b)$, hence the steady state is attained very soon after disturbance;

2. If $\frac{b}{a} \gg 1$ then the disturbance will continue for a longer period, and

3. If the value of $\frac{b}{a}$ is between the above considered values, then a concerning the duration of the disturbance can be obtained using Bessel's inequality, which says that for an orthonormal system $\{B_n(a_m r)\}$, whether closed or not, we have that

$$\sum_m^{\infty} C_m^2 \le \int_a^b [\phi(a_m r)]^2 dr, \tag{105}$$

where C's is all the coefficients in the generalized Fourier expansion of $\Phi(\alpha_m r)$ in terms of $B_n(a_m r)$, and $[a,b]$ is the interval of orthogonality.

Evidently,

$$|T^*| = |T_1^* + T_2^*| = |T_1^*| + |T_2^*|. \tag{106}$$

Denote by a the smallest root of eq. (63), then eq. (70) gives the smallest value of k_m^2 say k^{*2}; moreover, the smallest root A_n off eq. (69) is given by $A_0 = \frac{\pi}{2L}$.

Then

$$\eta = k^{*2} + A_0^2 \tag{107}$$

is the smallest value of $k_m^2 + A_n^2$. Therefore,

$$|T_2^*| \leq H \left| \sum_m^\infty \frac{J_1(a_m b)\, B_1(a_m r)}{J_1^2(a_m a) - J_1^2(a_m b)} \right| cos\frac{\pi z}{2L}\ e^{-\nu\eta t}, \tag{108}$$

where

$$H = \frac{\pi^2}{2} a_m^2 M_{mn}, \tag{109}$$

is the smallest value of $\frac{\pi^2}{2} a_m^2 M_{mn}$.

Hence, by means of Bessel, inequality

$$|T_2^*| = H\frac{a^2}{2r}\left(\frac{b^2 - r^2}{b^2 - a^2}\right) cos\frac{\pi z}{2L}\ e^{-\nu\eta t}. \tag{110}$$

Therefore, for any fixed $z = z_1$ and $t = t_1$. T_2^* decreases when r increases, in other words the duration of disturbance continue longer in the region closer to outer cylinder.

Hence, in order to reduce the duration of the disturbance in to hydromagnetic fluid under consideration we have either to increase the magnetic field or to allow the ration of the radii of cylinders, $\frac{b}{a}$ approach 1, or to do both together.

Further mathematical treatment of the problem now becomes trivial, since other unknown quantities can be easily evaluated. In this paper, because of the limitation of the space and the triviality of the solution of the remaining equations, the technique of the solution will not be presented. Evidently, integrating eq. (52) it follows

$$\rho_e(r, z, t) = e^{-\frac{\sigma}{\varepsilon_e}t}\left[C(r, z) - \sigma\mu_e H_0 \int^t \frac{\partial u(r, z, \xi)}{\partial r} e^{-\frac{\sigma}{\varepsilon_e}\xi} d\xi\right] \tag{111}$$

where the constant $C(r, z)$ is determined by condition that $\rho_e(r, z, 0) = 0$. Similarly, from eq. (47), (49), (50) and (51) one can easily obtain the values of p, and E_z. The friction couple per unit length of cylinders and that at that at the end of the bounding planes can be evaluated without any difficulty as was shown in ref [3] and [4,5].

5 Conclusion

The problem of unsteady rotational motion of electrically conducting, viscous, incompressible fluid contained within two axially concentric cylinders of finite length in the presence of an axial symmetric magnetic field, has been studied and solved by methods of methods of the operational calculus. The solution for the velocity field is obtained in an exact from, since the corresponding partial differential equations are solved exactly. It has been shown that the distu-rbance is of very short time duration if $\frac{b}{a} \to 1$. However, if $\frac{b}{a} \gg 1$ then disturbance is longer in the region closer to the outer cylinder. Moreover, as the magnetic field is increased the disturbance disappears more quickly. Finally, this solution is more general, since other solutions which have previously appeared in literature on the subject are merely special cases of this investigation.

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