CORRECTLY POSED BOUNDARY VALUE PROBLEMS FOR MAXWELL'S SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

Boundary value problems for Maxwell's system of partial differential equations are considered. Necessary and sufficient conditions, imposed on the boundary coefficients that ensure the correctness of the problem are found. It is shown what type of violation of the correctness of the problem occurs when these conditions are not fulfilled. It is also shown what changes in the initial conditions should be made to make the problem correct. In the case of a correctly posed problem, the solution is written out explicitly.

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In the space O_{xyz} consider Maxwell's system of equations [1]

$$E_t - rotH = 0, \quad H_t + rotE = 0, \tag{1}$$

where $E = (u_1, u_2, u_3)$ and $H = (u_4, u_5, u_6)$ are vectors of electric and magnetic fields, respectively, (the velocity of a light is taken as $c = 1$), $\mathrm{rot}E = (u_{3y} - u_{2z}, u_{1z} - u_{3x}, u_{2x} - u_{1y})$.

The system (1) can be rewritten as follows

$$\begin{cases} \dfrac{\partial u_1}{\partial t} - \dfrac{\partial u_6}{\partial y} + \dfrac{\partial u_5}{\partial z} = 0, & \dfrac{\partial u_4}{\partial t} + \dfrac{\partial u_3}{\partial y} - \dfrac{\partial u_2}{\partial z} = 0, \\ \dfrac{\partial u_2}{\partial t} - \dfrac{\partial u_4}{\partial z} + \dfrac{\partial u_6}{\partial x} = 0, & \dfrac{\partial u_5}{\partial t} + \dfrac{\partial u_1}{\partial z} - \dfrac{\partial u_3}{\partial x} = 0, \\ \dfrac{\partial u_3}{\partial t} - \dfrac{\partial u_5}{\partial x} + \dfrac{\partial u_4}{\partial y} = 0, & \dfrac{\partial u_6}{\partial t} + \dfrac{\partial u_2}{\partial x} - \dfrac{\partial u_1}{\partial y} = 0. \end{cases} \tag{2}$$

By notation $u = (u_1, u_2, u_3, u_4, u_5, u_6)$ the system (2) can be written in vectorial form

$$\frac{\partial u}{\partial t} + A\,\frac{\partial u}{\partial x} + B\,\frac{\partial u}{\partial y} + C\,\frac{\partial u}{\partial z} = 0, \tag{3}$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4}$$

Consider the case, when the vector u does not depend on the variables y and z, then the system (3) takes the form

$$\frac{\partial u}{\partial t} + A\,\frac{\partial u}{\partial x} = 0. \tag{5}$$

The Matrix A in the system (5) has real characteristic roots $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -1$, each of them of multiplicity two. Thus, the system (5) represents a non-split non-strictly hyperbolic system.

In the semi-strip $D:\ 0 < x < l,\ t > 0$ of the plane Oxt, for the system (5) consider the following boundary value problem: find in the domain D a regular solution $u \in C^1(\overline{D})$ to the system (5) which satisfies the following boundary conditions

$$(M_0 u)(x, 0) = \varphi(x), \;\; 0 \le x \le l, \tag{6}$$

$$(M_1 u)(0, t) = \mu_1(t), \;\; t \ge 0, \tag{7}$$

$$(M_2 u)(l, t) = \mu_2(t), \;\; t \ge 0, \tag{8}$$

where M_i are given $m_i \times 6$ dimensional matrices, $i = 0, 1, 2$, while $\varphi = (\varphi_1, \dots, \varphi_6)$ and $\mu_i = (\mu_{i1}, \dots, \mu_{i6})$, $i = 1, 2$, are given vector-functions from the class C^1, satisfying the corresponding agreement conditions at the points $O(0, 0)$ and $O'(l, 0)$. Here the number m_i, $0 \le m_i \le 6$, indicates the extent to which the boundary conditions occupy the corresponding part of the boundary of D. Particularly, $m_i = 0$ means that the corresponding part of the boundary of D is completely free from the boundary conditions. When the matrix M_0 in (6) is a unit one, then the problem (5)–(8) is called an initial-boundary problem.

It should be noted that the initial boundary problem for first-order hyperbolic systems, i.e., when M_0 is a unit matrix, has been investigated by many authors, but as a rule in the case when the system is split in the main part [1, 2, 3, 4, 5, 6].

Consider the initial boundary problem (5)–(8), i.e., when M_0 in (6) is a unit matrix. In this case, if we take into account the structure of the matrix A for the correctness of the problem it is necessary to require that the matrices M_1 and M_2 were of order 2×6, i.e. $m_1 = m_2 = 2$. Denote the elements of the matrix M_i by n^i_{kj}, $k = 1, 2$; $j = 1, \dots, 6$. Introduce two second-order square matrices

$$M_{11} = \begin{pmatrix} n^1_{12} + n^1_{16} & n^1_{13} - n^1_{15} \\ n^1_{22} + n^1_{26} & n^1_{23} - n^1_{25} \end{pmatrix},$$
$$M_{22} = \begin{pmatrix} n^2_{12} - n^2_{16} & n^2_{13} + n^2_{15} \\ n^2_{22} - n^2_{26} & n^2_{23} + n^2_{25} \end{pmatrix}.$$

Theorem. *For unique solvability of the initial boundary problem (5)–(8) with the unit matrix $M_0 = I$, $m_1 = m_2 = 2$, in the semi-strip D : $0 < x < l$, $t > 0$ for any vector-functions $\varphi = (\varphi_1, \dots, \varphi_6) \in C^1([0, l])$ and $\mu_i = (\mu_{i1}, \mu_{i2}) \in C^1([0, \infty])$, $i = 1, 2$, from the class $C^1(\overline{D})$, satisfying at the points $O(0, 0)$ and $O'(l, 0)$ the agreement conditions $M_1\varphi(0) = \mu_1(0)$, $M_2\varphi(l) = \mu_2(l)$, it is necessary and sufficient that*

$$\det M_{11} \neq 0, \quad \det M_{22} \neq 0. \tag{9}$$

Remark. Denote by m^i_j the columns of the matrix M_i, i.e., $M_i = (m^i_1, \dots, m^i_6)$ and introduce 2×6 order matrix $M^*_i = (m^i_2, m^i_3, m^i_5, m^i_6, m^i_1, m^i_4)$, $i = 1, 2$. Consider the sixth order quadratic matrix

$$K_0 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and introduce 2×6 order matrix $N_i = M^*_i K_0$, which will be represented as follows $N_i = (N_{i1}, N_{i2}, N_{i3})$, where N_{ij} are second order quadratic matrices. It is easy to verify that $N_{11} = M_{11}$ and $N_{22} = M_{22}$. When the conditions (9) are violated, then the problem (5)–(8) is not posed correctly. For example, when $M_{11} = 0$, $\det N_{12} \neq 0$, $\det N_{21} \neq 0$, $M_{22} = 0$, then the initial conditions (6) of this problem should be removed from the problem statement, particularly, $u_i(x, 0) = \varphi_i(x)$, $0 \leq x \leq l$, when $i = 2, 3, 5, 6$ is completely redundant, and for the correctness of the problem we should keep only the initial conditions $u_i(x, 0) = \varphi_i(x)$, $0 \leq x \leq l$, $i = 1, 4$. In the same conditions $M_{11} = 0$, $\det N_{12} \neq 0$, $\det N_{21} \neq 0$, $M_{22} = 0$, if we keep the initial conditions (6), then for the correctness of the problem we should consider the initial conditions (7) and (8) on that part of the boundary, where $t \geq l$.

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