SOLUTION OF SOME BOUNDARY VALUE PROBLEMS OF TENSION-COMPRESSION AND BENDING OF PLATES IN THE CASE OF THE $N = 1$ APPROXIMATION OF THE VEKUA THEORY

R. Janjgava[1], M. Narmania[2]

[1]I. Vekua Institute of Applied Mathematics of
I. Javakhishvili Tbilisi State University
11 University Str., Tbilisi 0186, Georgia
roman.janjgava@gmail.com
[2]University of Georgia
77 M. Kostava Str., Tbilisi 0175, Georgia
miranarma19@gmail.com

(Received 12.02.2022; accepted 16.06.2022)

Abstract

In the article, using the method of fundamental solutions, approximate solutions are constructed for some boundary-value problems of tension-compression and bending of homogeneous isotropic plates of constant thickness with holes. In this case, the elastic equilibrium of the plates is described by the refined system of equations of I. Vekua in the case of the $N = 1$ approximation.

Keywords and phrases: Theory of plates, Boundary-Value Problems, The method of fundamental solutions

AMS subject classification (2010): 74B20.

1 Introduction

The article considers a number of static, including mixed boundary value problems of tension-compression and bending of plates of constant thickness. The plates are considered homogeneous and isotropic, and their elastic equilibrium is described by the system of differential equations of the refined Vekua theory in the case of the approximation $N = 1$ [1-3].

The middle surface of the plates is a square with a circular hole. In all the problems under consideration, the contour of the hole is free from stresses ($\sigma_{rr} = 0, \sigma_{r\theta} = 0, \sigma_{r3} = 0$) and various boundary conditions are set on the sides of the square. The stress states of the plates under such boundary conditions are determined, and special attention is paid to studying the

$\sigma_{\theta\theta}$ on the hole contour. The formulated problems are solved approximately by the method of fundamental solutions [4, 5] and on the basis of works [6, 7].

2 Main Relations

Let $Oxyz$ be a rectangular Cartesian coordinate system in space. A homogeneous isotropic plate occupies an domain $\overline{\Omega} = \overline{\omega} \times [-h; h]$, where ω is a two-dimensional domain on a plane Oxy which is called the middle surface of the plate, h is the half-thickness of the plate, which is a positive constant.

As is known, for plates of constant thickness, the system of elastic equilibrium equations breaks down into two independent systems: tension-compression and bending. This article considers the case when displacements and stresses are linear functions of the coordinate along the thickness. This case corresponds to the approximation $N = 1$ of the Vekua theory.

In the case of approximation $N = 1$, the homogeneous system of equilibrium equations for plates in a two-dimensional domain ω has the form [1]:

System of Tension-Compression Equations

$$\begin{cases} \partial_\alpha \overset{(0)}{\sigma}_{\alpha\beta} = 0, & \beta = 1, 2 \\ \partial_\alpha \overset{(1)}{\sigma}_{\alpha 3} - \dfrac{3}{h} \overset{(0)}{\sigma}_{33} = 0; & \end{cases} \tag{1}$$

System of bending equations

$$\begin{cases} \partial_\alpha \overset{(1)}{\sigma}_{\alpha\beta} - \dfrac{3}{h} \overset{(0)}{\sigma}_{3\beta} = 0, & \beta = 1, 2 \\ \partial_\alpha \overset{(0)}{\sigma}_{\alpha 3} = 0, & \end{cases} \tag{2}$$

where the Greek indices take the values $1, 2$ and summation is assumed over the repeated index; $\partial_1 \equiv \partial_x \equiv \frac{\partial}{\partial x}$, $\partial_2 \equiv \partial_y \equiv \frac{\partial}{\partial y}$;

$$\overset{(0)}{\sigma}_{\alpha\beta} = \overset{(0)}{\sigma}_{\beta\alpha} = \frac{1}{2h} \int_{-h}^{h} \sigma_{\alpha\beta} dz, \quad \overset{(1)}{\sigma}_{3\beta} = \overset{(1)}{\sigma}_{\beta 3} = \frac{3}{2h^2} \int_{-h}^{h} z \sigma_{3\beta} dz,$$

$$\overset{(1)}{\sigma}_{\alpha\beta} = \overset{(1)}{\sigma}_{\beta\alpha} = \frac{3}{2h^2} \int_{-h}^{h} z \sigma_{\alpha\beta} dz, \quad \overset{(0)}{\sigma}_{3\beta} = \overset{(0)}{\sigma}_{\beta 3} = \frac{1}{2h} \int_{-h}^{h} \sigma_{3\beta} dz,$$

where $\sigma_{\alpha\beta}, \sigma_{3\beta}, \sigma_{33}$ are stress tensor components.

Hooke's law is written as follows

$$\begin{cases} \overset{(0)}{\sigma}_{\alpha\beta} = \lambda \left(\overset{(0)}{\theta} + \frac{1}{h}\overset{(1)}{u}_3 \right) \delta_{\alpha\beta} + \mu \left(\partial_\alpha \overset{(0)}{u}_\beta + \partial_\beta \overset{(0)}{u}_\alpha \right), \\ \overset{(1)}{\sigma}_{\alpha 3} = \mu \partial_\alpha \overset{(1)}{u}_3, \\ \overset{(0)}{\sigma}_{33} = \lambda \overset{(0)}{\theta} + \frac{1}{h}(\lambda + 2\mu)\overset{(1)}{u}_3; \end{cases} \tag{3}$$

$$\begin{cases} \overset{(1)}{\sigma}_{\alpha\beta} = \lambda \overset{(1)}{\theta} \delta_{\alpha\beta} + \mu \left(\partial_\alpha \overset{(1)}{u}_\beta + \partial_\beta \overset{(1)}{u}_\alpha \right), \\ \overset{(0)}{\sigma}_{\alpha 3} = \mu \left(\partial_\alpha \overset{(0)}{u}_3 + \frac{1}{h}\overset{(1)}{u}_\alpha \right), \\ \overset{(1)}{\sigma}_{33} = \lambda \overset{(1)}{\theta}, \end{cases} \tag{4}$$

where λ and μ are Lamé constants; $\delta_{\alpha\beta}$ is the Kronecker delta;

$$\overset{(0)}{u}_\alpha = \frac{1}{2h}\int_{-h}^{h} u_\alpha dz, \quad \overset{(1)}{u}_3 = \frac{3}{2h^2}\int_{-h}^{h} z u_3 dz, \quad \overset{(1)}{u}_\alpha = \frac{3}{2h^2}\int_{-h}^{h} z u_\alpha dz,$$

$$\overset{(0)}{u}_3 = \frac{1}{2h}\int_{-h}^{h} u_3 dz,$$

u_1, u_2, u_3 are components of the displacement vector, while

$$u_j \cong \overset{(0)}{u}_j + \frac{z}{h}\overset{(1)}{u}_j;$$

$$\overset{(k)}{\theta} = \partial_x \overset{(k)}{u}_1 + \partial_y \overset{(k)}{u}_2, \quad k = 0, 1.$$

Substituting relations (3) into the system of equations (1), we obtain the system of equilibrium equations of tension-compression of the plates in the components of the displacement vector

$$\begin{cases} \mu \Delta \overset{(0)}{u}_1 + (\lambda + \mu)\partial_x \overset{(0)}{\theta} + \frac{\lambda}{h}\partial_x \overset{(1)}{u}_3 = 0, \\ \mu \Delta \overset{(0)}{u}_2 + (\lambda + \mu)\partial_y \overset{(0)}{\theta} + \frac{\lambda}{h}\partial_y \overset{(1)}{u}_3 = 0, \\ \mu \Delta \overset{(1)}{u}_3 - \frac{3\lambda}{h}\overset{(0)}{\theta} - \frac{3(\lambda + 2\mu)}{h^2}\overset{(1)}{u}_3 = 0. \end{cases} \tag{5}$$

Substituting relations (4) into the system of equations (2), we obtain the system of equations of plate bending in the components of the displacement vector

$$\begin{cases} \mu\Delta\overset{(1)}{u}_1 + (\lambda+\mu)\partial_x\overset{(1)}{\theta} - \frac{3\mu}{h}\partial_x\overset{(0)}{u}_3 - \frac{3\mu}{h^2}\overset{(1)}{u}_1 = 0, \\ \mu\Delta\overset{(1)}{u}_2 + (\lambda+\mu)\partial_y\overset{(1)}{\theta} - \frac{3\mu}{h}\partial_y\overset{(0)}{u}_3 - \frac{3\mu}{h^2}\overset{(1)}{u}_2 = 0, \\ \mu\Delta\overset{(0)}{u}_3 + \frac{\mu}{h}\overset{(1)}{\theta} = 0. \end{cases} \tag{6}$$

Thus, in the case of tension-compression $\overset{(0)}{u}_\alpha, \overset{(1)}{u}_3, \overset{(0)}{\sigma}_{\alpha\beta}, \overset{(1)}{\sigma}_{\alpha 3}$ are the required quantities, and in case of bending the required quantities are $\overset{(1)}{u}_\alpha, \overset{(0)}{u}_3, \overset{(1)}{\sigma}_{\alpha\beta}, \overset{(0)}{\sigma}_{\alpha 3}$.

3 General solution of the system of equations (5)

The general solution of the system of equations (5) can be represented using two harmonic functions and one metaharmonic function of two variables.

Differentiating the first equation of system (5) with respect to x and the second with respect to y and summing the resulting equations, we will have

$$\Delta\left((\lambda+2\mu)\overset{(0)}{\theta} + \frac{\lambda}{h}\overset{(1)}{u}_3\right) = 0.$$

Differentiating the second equation of the system (5) by x and the first by y and considering their difference, we will have

$$\Delta\left(\mu\left(\partial_x\overset{(0)}{u}_2 - \partial_y\overset{(0)}{u}_1\right)\right) = 0.$$

We introduce the following notation

$$\Theta := (\lambda+2\mu)\overset{(0)}{\theta} + \frac{\lambda}{h}\overset{(1)}{u}_3, \quad K := \mu(\partial_x\overset{(0)}{u}_2 - \partial_y\overset{(0)}{u}_1). \tag{7}$$

Thus, the functions Θ and K are harmonic functions

$$\Delta\Theta = 0, \quad \Delta K = 0. \tag{8}$$

From notation (7) we obtain

$$\partial_x\overset{(0)}{u}_1 + \partial_y\overset{(0)}{u}_2 = -\frac{\lambda}{(\lambda+2\mu)h}\overset{(1)}{u}_3 + \frac{1}{\lambda+2\mu}\Theta, \tag{9}$$

$$\partial_x\overset{(0)}{u}_2 - \partial_y\overset{(0)}{u}_1 = \frac{1}{\mu}K. \tag{10}$$

From (9) and (10) we will have

$$\Delta\overset{(0)}{u}_1 = -\frac{\lambda}{(\lambda+2\mu)h}\partial_x\overset{(1)}{u}_3 + \frac{1}{\lambda+2\mu}\partial_x\Theta - \frac{1}{\mu}\partial_y K, \tag{11}$$

$$\Delta \overset{(0)}{u}_2 = -\frac{\lambda}{(\lambda+2\mu)h}\partial_y \overset{(1)}{u}_3 + \frac{1}{\lambda+2\mu}\partial_y \Theta + \frac{1}{\mu}\partial_x K. \tag{12}$$

Substituting relations (11), (12) and (9) into the first two equations of system (5), we obtain the following Cauchy-Riemann system

$$\begin{cases} \partial_x \Theta - \partial_y K = 0, \\ \partial_y \Theta + \partial_x K = 0. \end{cases} \tag{13}$$

Out of the system (13)

$$\Theta = a(\partial_x \varphi + \partial_y \psi), \quad K = a(\partial_x \psi - \partial_y \varphi), \tag{14}$$

where φ and ψ are arbitrary two-dimensional harmonic functions; a is an arbitrary non-zero real constant.

Taking into account (14), from formula (9) we obtain the following formula for $\overset{(0)}{\theta}$

$$\overset{(0)}{\theta} = -\frac{\lambda}{(\lambda+2\mu)h}\overset{(1)}{u}_3 + \frac{1}{\lambda+2\mu} a(\partial_x \varphi + \partial_y \psi). \tag{15}$$

Substituting expression (15) into the third equation of system (5), we obtain the following equation for $\overset{(1)}{u}_3$

$$\Delta \overset{(1)}{u}_3 - \frac{12(\lambda+\mu)}{(\lambda+2\mu)h^2}\overset{(1)}{u}_3 = \frac{3\lambda}{\mu(\lambda+2\mu)h} a(\partial_x \varphi + \partial_y \psi).$$

The general solution of the last equation is represented as

$$\overset{(1)}{u}_3 = c\chi - \frac{\lambda h}{4\mu(\lambda+\mu)} a(\partial_x \varphi + \partial_y \psi), \tag{16}$$

where χ is the general solution of the following Helmholtz equation

$$\Delta \chi - \frac{12(\lambda+\mu)}{(\lambda+2\mu)h^2}\chi = 0. \tag{17}$$

c is an arbitrary non-zero real constant.

As a result of substitution (16) into (15), we will have

$$\partial_x \overset{(0)}{u}_1 + \partial_y \overset{(0)}{u}_2 = -\frac{\lambda}{(\lambda+2\mu)h} c\chi + \frac{\lambda+2\mu}{4\mu(\lambda+\mu)} a(\partial_x \varphi + \partial_y \psi)$$

$$= -\frac{\lambda h}{12(\lambda+\mu)} c(\partial_{xx}\chi + \partial_{yy}\chi) + \frac{\lambda+2\mu}{4\mu(\lambda+\mu)} a(\partial_x \varphi + \partial_y \psi). \tag{18}$$

The second relation (14) is substituted into formula (10)

$$\partial_x \overset{(0)}{u}_2 - \partial_y \overset{(0)}{u}_1 = \frac{a}{\mu}(\partial_x \psi - \partial_y \varphi) \tag{19}$$

We combine (15) and (16) into the system

$$\left\{ \begin{array}{c} \partial_x \left(\overset{(0)}{u}_1 - \dfrac{(\lambda+2\mu)a}{4\mu(\lambda+\mu)}\varphi + \dfrac{\lambda hc}{12(\lambda+\mu)}\partial_x \chi \right) \\ +\partial_y \left(\overset{(0)}{u}_2 - \dfrac{(\lambda+2\mu)a}{4\mu(\lambda+\mu)}\psi + \dfrac{\lambda hc}{12(\lambda+\mu)}\partial_y \chi \right) = 0, \\ \partial_x \overset{(0)}{u}_2 - \partial_y \overset{(0)}{u}_1 = \dfrac{a}{\mu}(\partial_x \psi - \partial_y \varphi); \end{array} \right. \tag{20}$$

The second equation (20) is identically satisfied if $\overset{(0)}{u}_1$ and $\overset{(0)}{u}_2$ we take as follows

$$\overset{(0)}{u}_1 = \partial_x \Phi + \frac{a}{\mu}\varphi, \;\; \overset{(0)}{u}_2 = \partial_y \Phi + \frac{a}{\mu}\psi, \tag{21}$$

We substitute equalities (21) into the first equation of system (20)

$$\Delta \left(\Phi + \frac{\lambda hc}{12(\lambda+\mu)}\chi \right) = -\frac{(3\lambda+2\mu)a}{4\mu(\lambda+\mu)}(\partial_x \varphi + \partial_y \psi)$$

The general solution of the last equation can be written as

$$\Phi = -\frac{(3\lambda+2\mu)a}{8\mu(\lambda+\mu)}\left(x(\varphi + \partial_x \Psi) + y(\psi + \partial_y \Psi) \right) + \frac{a}{\mu}\Psi - \frac{\lambda hc}{12(\lambda+\mu)}\chi, \tag{22}$$

where Ψ is an arbitrary harmonic function.

We substitute (22) into formulas (21) $\left(a = \frac{4(\lambda+\mu)}{3\lambda+2\mu}, c = \frac{6(\lambda+\mu)}{\lambda\mu} \right)$

$$2\mu \overset{(0)}{u}_1 = \frac{5\lambda+6\mu}{3\lambda+2\mu}\varphi^* - x\partial_x \varphi^* - y\partial_x \psi^* - h\partial_x \chi,$$

$$2\mu \overset{(0)}{u}_2 = \frac{5\lambda+6\mu}{3\lambda+2\mu}\psi^* - y\partial_y \psi^* - x\partial_y \varphi^* - h\partial_y \chi,$$

where

$$\varphi^* = \varphi + \partial_x \Psi, \;\; \psi^* = \psi + \partial_y \Psi.$$

Since φ^* and ψ^* are arbitrary harmonic functions, instead of them we will again write φ and ψ , respectively

$$2\mu \overset{(0)}{u}_1 = \frac{5\lambda+6\mu}{3\lambda+2\mu}\varphi - x\partial_x \varphi - y\partial_x \psi - h\partial_x \chi,$$

$$2\mu\overset{(0)}{u}_2 = \frac{5\lambda+6\mu}{3\lambda+2\mu}\psi - y\partial_y\psi - x\partial_y\varphi - h\partial_y\chi, \tag{23}$$

$$2\mu\overset{(1)}{u}_3 = -\frac{2\lambda h}{3\lambda+2\mu}(\partial_x\varphi+\partial_y\psi) + \frac{12(\lambda+\mu)}{\lambda}\chi.$$

Thus, the general solution of the system of equations (5) is represented using two arbitrary harmonic functions φ and ψ, and one arbitrary solution χ of the Helmholtz equation (17).

Substituting the last relation (23) into formula (15), we find

$$\overset{(0)}{\theta} = \frac{\lambda+2\mu}{\mu(3\lambda+2\mu)}(\partial_x\varphi+\partial_y\psi) - \frac{6(\lambda+\mu)}{\mu(\lambda+2\mu)h}\chi. \tag{24}$$

Substituting (23) and (24) into formulas (3), we obtain the representations of the stress components in terms of the functions φ , ψ and χ

$$\overset{(0)}{\sigma}_{11} = \frac{4(\lambda+\mu)}{3\lambda+2\mu}\partial_x\varphi - x\partial_{xx}\varphi + \frac{2\lambda}{3\lambda+2\mu}\partial_y\psi$$

$$-y\partial_{xx}\psi + \frac{12(\lambda+\mu)}{(\lambda+2\mu)h}\chi - h\partial_{xx}\chi,$$

$$\overset{(0)}{\sigma}_{22} = \frac{4(\lambda+\mu)}{3\lambda+2\mu}\partial_y\psi - y\partial_{yy}\psi + \frac{2\lambda}{3\lambda+2\mu}\partial_x\varphi$$

$$-x\partial_{yy}\varphi + \frac{12(\lambda+\mu)}{(\lambda+2\mu)h}\chi - h\partial_{yy}\chi,$$

$$\overset{(0)}{\sigma}_{12} = \frac{\lambda+2\mu}{3\lambda+2\mu}\partial_y\varphi - x\partial_{xy}\varphi + \frac{\lambda+2\mu}{3\lambda+2\mu}\partial_x\psi - y\partial_{xy}\psi - h\partial_{xy}\chi, \tag{25}$$

$$\overset{(1)}{\sigma}_{13} = \frac{6(\lambda+\mu)}{\lambda}\partial_x\chi - \frac{\lambda h}{3\lambda+2\mu}(\partial_{xx}\varphi+\partial_{xy}\psi),$$

$$\overset{(1)}{\sigma}_{23} = \frac{6(\lambda+\mu)}{\lambda}\partial_y\chi - \frac{\lambda h}{3\lambda+2\mu}(\partial_{xy}\varphi+\partial_{yy}\psi),$$

$$\overset{(0)}{\sigma}_{33} = \frac{24(\lambda+\mu)^2}{\lambda(\lambda+2\mu)h}\chi.$$

4 General solution of the system of equations (6)

Similarly, we will show that the general solution of the system of equations (6) can also be represented using two harmonic functions and one metaharmonic function of two variables.

Differentiating the first equation of system (6) with respect to x and the second with respect to y and summing the resulting equations, we have

$$(\lambda+2\mu)\Delta \overset{(1)}{\theta} - \frac{3\mu}{h}\left(\Delta \overset{(0)}{u}_3 + \frac{1}{h}\overset{(1)}{\theta}\right).$$

Taking into account the third equation of system (6), from the last equation we obtain

$$\Delta \overset{(1)}{\theta} = 0,$$

from here

$$\overset{(1)}{\theta} = a(\partial_x \varphi + \partial_y \psi), \tag{26}$$

where φ and ψ are arbitrary harmonic functions of two variables; a is an arbitrary non-zero real constant.

We differentiate the second equation of system (6) with respect to x and subtract from the resulting equation the first equation of the same system differentiated with respect to y . As a result, we obtain the following equation

$$\Delta\left(\partial_x \overset{(1)}{u}_2 - \partial_y \overset{(1)}{u}_1\right) - \frac{3}{h^2}\left(\partial_x \overset{(1)}{u}_2 - \partial_y \overset{(1)}{u}_1\right) = 0.$$

It follows from the last equation that

$$\partial_x \overset{(1)}{u}_2 - \partial_y \overset{(1)}{u}_1 = b\chi, \tag{27}$$

where χ is an arbitrary solution of the following Helmholtz equation

$$\Delta\chi - \frac{3}{h^2}\chi = 0; \tag{28}$$

b is an arbitrary non-zero real constant.

Comment. As in the case of a tension-compression system, in the case of a bending system, the harmonic functions are denoted by φ and ψ , and the metaharmonic function by χ But we will not confuse them, since the problems of tension-compression and bending will be solved independently from each other.

We combine equations (26) and (27) into the system

$$\begin{cases} \partial_x \overset{(1)}{u}_2 + \partial_y \overset{(1)}{u}_1 = a(\partial_x \varphi + \partial_y \psi), \\ \partial_x \overset{(1)}{u}_2 - \partial_y \overset{(1)}{u}_1 = b\chi. \end{cases} \tag{29}$$

Differentiating the first equation of system (29) with respect to x and the second with respect to y and subtracting the second equation from the first equation, we obtain

$$\Delta \overset{(1)}{u}_1 = a(\partial_{xx}\varphi + \partial_{xy}\psi) - b\partial_y \chi. \tag{30}$$

Differentiating the first equation of system (29) with respect to y and the second with respect to x and summing the resulting equations, we have

$$\Delta \overset{(1)}{u}_2 = a(\partial_{xy}\varphi + \partial_{yy}\psi) + b\partial_x\chi. \tag{31}$$

From the third equation of system (6), taking into account formula (26), we obtain the equation

$$\Delta \overset{(0)}{u}_3 = -\frac{1}{h}\overset{(1)}{\theta} = -\frac{a}{h}(\partial_x\varphi + \partial_y\psi). \tag{32}$$

The general solution of equation (32) can be written as follows

$$\overset{(0)}{u}_3 = -\frac{a}{2h}\left(x(\varphi + \partial_x\Phi) + y(\psi + \partial_y\Phi)\right) - \frac{h^2(\lambda + 2\mu)a}{3\mu}(\partial_{xx}\Phi + \partial_{yy}\Phi). \tag{33}$$

where Φ is an arbitrary harmonic function of two variables x and y.

Formulas (26), (30), (31), and (33) are substituted into the first and second equations of system (6). As a result, we get

$$\frac{3\mu}{h^2}\overset{(1)}{u}_1 = (\lambda + 2\mu)a(\partial_{xx}\varphi + \partial_{xy}\psi) + \frac{3\mu a}{h^2}\left((\varphi + \partial_x\Phi) + x\partial_x(\varphi + \partial_x\Phi)\right.$$
$$\left. -\mu b\partial_y\chi + y\partial_x(\psi + \partial_y\Phi)\right) + (\lambda + 2\mu)a\partial_x(\partial_{xx}\Phi + \partial_{yy}\Phi), \tag{34}$$
$$\frac{3\mu}{h^2}\overset{(1)}{u}_2 = (\lambda + 2\mu)a(\partial_{xy}\varphi + \partial_{yy}\psi) + \frac{3\mu a}{h^2}\left((\psi + \partial_y\Phi) + x\partial_y(\varphi + \partial_x\Phi)\right.$$
$$\left. +\mu b\partial_x\chi + y\partial_y(\psi + \partial_y\Phi)\right) + (\lambda + 2\mu)a\partial_y(\partial_{xx}\Phi + \partial_{yy}\Phi). \tag{35}$$

Let us introduce the notation

$$\varphi^* = \varphi + \partial_x\Phi, \quad \psi^* = \psi + \partial_y\Phi. \tag{36}$$

It is clear that φ^* and ψ^* are arbitrary harmonic functions.

Taking into account notation (36), relations (34) and (35) can be written as

$$\frac{3\mu}{h^2}\overset{(1)}{u}_1 = (\lambda + 2\mu)a(\partial_{xx}\varphi^* + \partial_{xy}\psi^*) + \frac{3\mu a}{2h^2}\left(\varphi^* + x\partial_x\varphi^* + y\partial_x\psi^*\right)$$
$$-\mu b\partial_y\chi,$$
$$\frac{3\mu}{h^2}\overset{(1)}{u}_2 = (\lambda + 2\mu)a(\partial_{yy}\psi^* + \partial_{xy}\varphi^*) + \frac{3\mu a}{2h^2}\left(\psi^* + y\partial_y\psi^* + x\partial_y\varphi^*\right)$$
$$+\mu b\partial_x\chi.$$

Since ϕ^* and ψ^* are arbitrary harmonic functions, we will write them below without asterisks. Then from the last two relations we easily obtain

$$2\mu\overset{(1)}{u}_1 = \frac{2(\lambda+\mu)h^2}{3}a(\partial_{xx}\varphi + \partial_{xy}\psi) + \mu a\left(\varphi + x\partial_x\varphi + y\partial_x\psi\right) - \frac{2\mu h^2}{3}b\partial_y\chi, \tag{37}$$

$$2\mu\overset{(1)}{u}_2 = \frac{2(\lambda+\mu)h^2}{3}a(\partial_{yy}\psi + \partial_{xy}\varphi) + \mu a\left(\psi + y\partial_y\psi + x\partial_y\varphi\right) + \frac{2\mu h^2}{3}\mu b\partial_x\chi. \tag{38}$$

We define arbitrary constants a and b as follows

$$a = \frac{1}{\mu}, \quad b = \frac{3}{2\mu h^2}.$$

Then formulas (37), (38), and (33) take the form

$$\begin{gathered} 2\mu\overset{(1)}{u}_1 = \varphi + x\partial_x\varphi + y\partial_x\psi + \frac{2(\lambda+2\mu)h^2}{3\mu}(\partial_{xx}\varphi + \partial_{xy}\psi) - \partial_y\chi, \\ 2\mu\overset{(1)}{u}_2 = \psi + y\partial_y\psi + x\partial_y\varphi + \frac{2(\lambda+2\mu)h^2}{3\mu}(\partial_{yy}\psi + \partial_{xy}\varphi) + \partial_x\chi, \\ 2\mu\overset{(0)}{u}_3 = -\frac{1}{h}(x\varphi + y\psi). \end{gathered} \tag{39}$$

Formula (26) will be written as follows

$$\overset{(1)}{\theta} = \frac{1}{\mu}(\partial_x\varphi + \partial_y\psi). \tag{40}$$

Thus, we have represented the general solution of the system of equations (6) using two arbitrary harmonic functions φ and ψ and one arbitrary solution of the Helmholtz equation χ using formulas (39).

Substituting formulas (39), (40) into relations (4), we can express the quantities $\overset{(1)}{\sigma}_{\alpha\beta}, \overset{(0)}{\sigma}_{\alpha 3}$ and $\overset{(1)}{\sigma}_{33}$ using the functions φ, ψ and χ

$$\overset{(1)}{\sigma}_{11} = \frac{\lambda+2\mu}{\mu}\partial_x\varphi + \frac{\lambda}{\mu}\partial_y\psi + x\partial_{xx}\varphi + y\partial_{xx}\psi + \frac{2(\lambda+2\mu)h^2}{3\mu}\partial_{xx}(\partial_x\varphi + \partial_y\psi) - \partial_{xy}\chi,$$

$$\overset{(1)}{\sigma}_{22} = \frac{\lambda+2\mu}{\mu}\partial_y\psi + \frac{\lambda}{\mu}\partial_x\varphi + y\partial_{yy}\psi + x\partial_{yy}\varphi$$

$$+\frac{2(\lambda+2\mu)h^2}{3\mu}\partial_{yy}(\partial_x\varphi+\partial_y\psi)+\partial_{xy}\chi,$$

$$\overset{(1)}{\sigma}_{12}=\partial_y\varphi+\partial_x\psi+x\partial_{xy}\varphi+y\partial_{xy}\psi+\frac{2(\lambda+2\mu)h^2}{3\mu}\partial_{xx}(\partial_y\varphi-\partial_x\psi)$$

$$+\frac{1}{2}(\partial_{xx}\chi-\partial_{yy}\chi), \tag{41}$$

$$\overset{(0)}{\sigma}_{13}=\frac{(\lambda+2\mu)h}{3\mu}(\partial_{xx}\varphi+\partial_{xy}\psi)-\frac{1}{2h}\partial_y\chi,$$

$$\overset{(0)}{\sigma}_{23}=\frac{(\lambda+2\mu)h}{3\mu}(\partial_{yy}\psi+\partial_{xy}\varphi)+\frac{1}{2h}\partial_x\chi,$$

$$\overset{(1)}{\sigma}_{33}=\frac{\lambda}{\mu}(\partial_x\varphi+\partial_y\psi).$$

5 Using a General Solution to Construct Approximate Solutions to Boundary Value Problems of Tension-Compression and Bending of Plates

The representation of general solutions found in the previous sections can be used to construct approximate solutions to the boundary value problems of tension-compression and plate bending. For this purpose, we use the results of [5].

In the general solutions (23), (25) and (39), (41) for each index $j = 1, 2, ..., n$ we take the harmonic functions φ_j and ψ_j as follows

$$(\varphi_j,\psi_j)=(a_j,b_j)\ln\sqrt{x^2+y^2}, \tag{42}$$

and the solution of the Helmholtz equation χ_j can be represented as follows

$$\chi_j=c_jK_0(\eta\sqrt{x^2+y^2}), \tag{43}$$

where $K_0(\eta\sqrt{x^2+y^2})$ is the zero-order Macdonald function; $\eta=\frac{2\sqrt{3}}{h}\sqrt{\frac{\lambda+\mu}{\lambda+2\mu}}$ in the case of tension-compression and $\eta=\frac{\sqrt{3}}{h}$ in the case of bending.

According to formulas (42), the partial derivatives of the functions φ_j and ψ_j will have the form

$$(\partial_x\varphi_j,\partial_x\psi_j)=(a_j,b_j)\frac{x}{x^2+y^2},\ \ (\partial_y\varphi_j,\partial_y\psi_j)=(a_j,b_j)\frac{y}{x^2+y^2}. \tag{44}$$

The partial derivatives of the functions χ_j are written as follows

$$\partial_x\chi_j=-c_j\frac{\eta xK_1(\eta\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}},\ \ \partial_y\chi_j=-c_j\frac{\eta yK_1(\eta\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}} \tag{45}$$

In addition, we need the second order derivatives of the functions φ_j, ψ_j and χ_j

$$(\partial_{xx}\varphi_j, \partial_{xx}\psi_j) = -(\partial_{yy}\varphi_j, \partial_{yy}\psi_j) = (a_j, b_j)\frac{y^2 - x^2}{(x^2 + y^2)^2},$$

$$(\partial_{xy}\varphi_j, \partial_{xy}\psi_j) = -(a_j, b_j)\frac{2xy}{(x^2 + y^2)^2},$$

$$\partial_{xx}\chi_j = c_j\frac{\eta^2}{2}\left(K_0(\eta\sqrt{x^2 + y^2}) - \frac{y^2 - x^2}{x^2 + y^2}K_2(\eta\sqrt{x^2 + y^2})\right), \tag{46}$$

$$\partial_{yy}\chi_j = c_j\frac{\eta^2}{2}\left(K_0(\eta\sqrt{x^2 + y^2}) + \frac{y^2 - x^2}{x^2 + y^2}K_2(\eta\sqrt{x^2 + y^2})\right),$$

$$\partial_{xy}\chi_j = c_j\eta^2\frac{xy}{x^2 + y^2}K_2(\eta\sqrt{x^2 + y^2}).$$

Substituting the corresponding formulas (42)-(46) into relations (23), we obtain the following expressions for displacements in the case of tension-compression

$$2\mu\overset{(0)}{u}_{1j} = \left(\frac{5\lambda + 6\mu}{3\lambda + 2\mu}\ln r - \frac{x^2}{r^2}\right)a_j - \frac{xy}{r^2}b_j + \frac{\eta h x K_1(\eta r)}{r}c_j,$$

$$2\mu\overset{(0)}{u}_{2j} = \left(\frac{5\lambda + 6\mu}{3\lambda + 2\mu}\ln r - \frac{y^2}{r^2}\right)b_j - \frac{xy}{r^2}a_j + \frac{\eta h y K_1(\eta r)}{r}c_j, \tag{47}$$

$$2\mu\overset{(1)}{u}_{3j} = -\frac{2\lambda h}{3\lambda + 2\mu}\frac{x}{r^2}a_j - \frac{2\lambda h}{3\lambda + 2\mu}\frac{y}{r^2}b_j + \frac{12(\lambda + \mu)}{\lambda}K_0(\eta r)c_j, \tag{48}$$

where

$$r = \sqrt{x^2 + y^2}.$$

Substituting the corresponding formulas (42)-(46) into relations (25), we obtain the following expressions for stresses in the case of tension-compression

$$\overset{(0)}{\sigma}_{11j} = \left(\frac{4(\lambda + \mu)}{3\lambda + 2\mu}\frac{x}{r^2} - \frac{x(y^2 - x^2)}{r^4}\right)a_j + \left(\frac{2\lambda}{3\lambda + 2\mu}\frac{y}{r^2} - \frac{y(y^2 - x^2)}{r^4}\right)b_j$$
$$+\frac{6(\lambda + \mu)}{(\lambda + 2\mu)h}\left(K_0(\eta r) + \frac{y^2 - x^2}{r^2}K_2(\eta r)\right)c_j,$$

$$\overset{(0)}{\sigma}_{22j} = \left(\frac{4(\lambda + \mu)}{3\lambda + 2\mu}\frac{y}{r^2} + \frac{y(y^2 - x^2)}{r^4}\right)b_j + \left(\frac{2\lambda}{3\lambda + 2\mu}\frac{x}{r^2} + \frac{x(y^2 - x^2)}{r^4}\right)a_j$$
$$+\frac{6(\lambda + \mu)}{(\lambda + 2\mu)h}\left(K_0(\eta r) - \frac{y^2 - x^2}{r^2}K_2(\eta r)\right)c_j,$$

$$\overset{(0)}{\sigma}_{12j} = \left(\frac{\lambda+2\mu}{3\lambda+2\mu} \frac{y}{r^2} + \frac{2x^2y}{r^4} \right) a_j + \left(\frac{\lambda+2\mu}{3\lambda+2\mu} \frac{x}{r^2} + \frac{2xy^2}{r^4} \right) b_j - \eta^2 h \frac{xy}{r^2} K_2(\eta r) c_j, \tag{49}$$

$$\overset{(1)}{\sigma}_{13j} = -\frac{\lambda h}{3\lambda+2\mu} \frac{y^2-x^2}{r^4} a_j + \frac{\lambda h}{3\lambda+2\mu} \frac{2xy}{r^4} b_j - \frac{6(\lambda+\mu)}{\lambda} \frac{\eta x K_1(\eta r)}{r} c_j,$$

$$\overset{(1)}{\sigma}_{23j} = \frac{\lambda h}{3\lambda+2\mu} \frac{2xy}{r^4} a_j + \frac{\lambda h}{3\lambda+2\mu} \frac{y^2-x^2}{r^4} b_j - \frac{6(\lambda+\mu)}{\lambda} \frac{\eta y K_1(\eta r)}{r} c_j,$$

$$\overset{(0)}{\sigma}_{33j} = \frac{24(\lambda+\mu)^2}{\lambda(\lambda+2\mu)h} K_0(\eta r) c_j.$$

Substituting the corresponding formulas (42)-(47) into relations (39), we obtain the following expressions for displacements in the case of plate bending

$$2\mu \overset{(1)}{u}_{1j}(x,y) = \left(\ln r + \frac{x^2}{r^2} + \frac{2(\lambda+2\mu)h^2}{3\mu} \frac{y^2-x^2}{r^4} \right) a_j + \left(\frac{xy}{r^2} - \frac{4(\lambda+2\mu)h^2}{3\mu} \frac{xy}{r^4} \right) b_j + \frac{\sqrt{3} y K_1(\sqrt{3} h^{-1} r)}{hr} c_j,$$

$$2\mu \overset{(1)}{u}_{2j}(x,y) = \left(\frac{xy}{r^2} - \frac{4(\lambda+2\mu)h^2}{3\mu} \frac{xy}{r^4} \right) a_j + \left(\ln r + \frac{y^2}{r^2} - \frac{2(\lambda+2\mu)h^2}{3\mu} \frac{y^2-x^2}{r^4} \right) b_j - \frac{\sqrt{3} x K_1(\sqrt{3} h^{-1} r)}{hr} c_j, \tag{50}$$

$$2\mu \overset{(0)}{u}_{3j}(x,y) = -\frac{1}{h}(x \ln r) a_j - \frac{1}{h}(y \ln r) b_j.$$

By substituting the corresponding formulas (42)-(47) into relations (41), we obtain the following expressions for stresses in the case of bending

$$\overset{(1)}{\sigma}_{11j}(x,y) = x \left(\frac{\lambda+2\mu}{\mu} \frac{1}{r^2} + \frac{y^2-x^2}{r^4} - \frac{4(\lambda+2\mu)h^2}{3\mu} \frac{-x^4+3y^4+2x^2y^2}{r^8} \right) a_j + y \left(\frac{\lambda}{\mu} \frac{1}{r^2} + \frac{y^2-x^2}{r^4} + \frac{4(\lambda+2\mu)h^2}{3\mu} \frac{3x^4-y^4+2x^2y^2}{r^8} \right) b_j - \frac{3xy}{h^2 r^2} K_2(\sqrt{3} h^{-1} r) c_j,$$

$$\overset{(1)}{\sigma}_{22j}(x,y) = x \left(\frac{\lambda}{\mu} \frac{1}{r^2} - \frac{y^2-x^2}{r^4} + \frac{4(\lambda+2\mu)h^2}{3\mu} \frac{-x^4+3y^4+2x^2y^2}{r^8} \right) a_j + y \left(\frac{\lambda+2\mu}{\mu} \frac{1}{r^2} - \frac{y^2-x^2}{r^4} - \frac{4(\lambda+2\mu)h^2}{3\mu} \frac{3x^4-y^4+2x^2y^2}{r^8} \right) b_j$$

$$+\frac{3xy}{h^2r^2}K_2(\sqrt{3}h^{-1}r)c_j$$

$$\overset{(1)}{\sigma}_{12j}(x,y)=y\left(\frac{1}{r^2}-\frac{2x^2}{r^4}+\frac{4(\lambda+2\mu)h^2}{3\mu}\frac{3x^4-y^4+2x^2y^2}{r^8}\right)a_j$$

$$+x\left(\frac{1}{r^2}-\frac{2y^2}{r^4}+\frac{4(\lambda+2\mu)h^2}{3\mu}\frac{-x^4+3y^4+2x^2y^2}{r^8}\right)b_j \tag{51}$$

$$-\frac{3}{2h^2}\frac{y^2-x^2}{r^2}K_2(\sqrt{3}h^{-1}r)c_j,$$

$$\overset{(0)}{\sigma}_{13j}(x,y)=\frac{(\lambda+2\mu)h}{3\mu}\frac{y^2-x^2}{r^4}a_j-\frac{2(\lambda+2\mu)h}{3\mu}\frac{xy}{r^4}b_j+\frac{\sqrt{3}}{2h^2}\frac{yK_1(\sqrt{3}h^{-1}r)}{r}c_j,$$

$$\overset{(0)}{\sigma}_{23j}(x,y)=-\frac{2(\lambda+2\mu)h}{3\mu}\frac{xy}{r^4}a_j-\frac{(\lambda+2\mu)h}{3\mu}\frac{y^2-x^2}{r^4}b_j-\frac{\sqrt{3}}{2h^2}\frac{xK_1(\sqrt{3}h^{-1}r)}{r}c_j,$$

$$\overset{(1)}{\sigma}_{33j}\ (x,y)=\frac{\lambda}{\mu}\frac{x}{r^2}a_j+\frac{\lambda}{\mu}\frac{y}{r^2}b_j.$$

After that, each j-th function (48)-(51) is shifted by the value (ξ_j,ζ_j) (fig.1). To do this, in formulas (48)-(51) ithe variables x and y are replaced by $x-\xi_j$ and $y-\zeta_j$, respectively. The functions $2\mu\overset{(k)}{u}_{1j}(x-\xi_j,y-\zeta_j),...,\overset{(k)}{\sigma}_{33j}\ (x-\xi_j,y-\zeta_j)$, $j=1,2,...,n$ $k=0,1$, will have a singularity at the point (ξ_j,ζ_j).

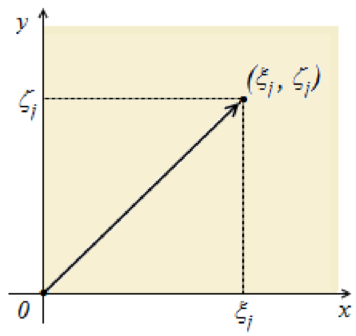

Figure 1: Shift of singular point

Next, we apply the method developed in [5]. Approximate solutions of boundary value problems are sought in the form

$$\mu\overset{(k)}{u}_p(x;y)=\sum_{j=1}^{n}\overset{(k)}{u}_{pj}\ (x-\xi_j,y-\zeta_j), \tag{52}$$

$$\mu\overset{(k)}{\sigma}_{pq}(x;y)=\sum_{j=1}^{n}\overset{(k)}{\sigma}_{pqj}(x-\xi_j,y-\zeta_j),\quad p,q=1,2,3;\;\; k=0,1. \tag{53}$$

Satisfying the corresponding boundary conditions at the ponts (x_i, y_j), $i = 1, 2, ..., n$ of the boundary of the considered two-dimensional domain ω (fig.2), for the desired coefficients $a_1, a_2, ..., a_n,\; b_1, b_2, ...b_n, c_1, c_2, ..., c_n$, we obtain a system of 3n linear algebraic equations with 3n unknowns

$$\begin{cases}\sum_{j=1}^{n}\{A_{1ij}a_j+B_{1ij}b_j+C_{1ij}c_j\}=f_1(x_i,y_i),\\ \sum_{j=1}^{n}\{A_{2ij}a_j+B_{2ij}b_j+C_{2ij}c_j\}=f_2(x_i,y_i),\\ \sum_{j=1}^{n}\{A_{3ij}a_j+B_{3ij}b_j+C_{3ij}c_j\}=f_3(x_i,y_i),\quad i=1,2,...,n,\end{cases} \tag{54}$$

where $f_p(x_i, y_i)$, $p = 1, 2, 3$, values of functions specified on the boundary of the domain ω at points $(x_i, y_i) \in \partial\omega$.

Having solved the system of equations (54) and substituting the found values of the coefficients $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n, c_1, c_2, ..., c_n$ into formulas (53) and (54), we obtain analytical expressions for approximate solutions of boundary value problems. Naturally, we can compare the boundary values of the obtained approximate solution with the values of the corresponding boundary function.

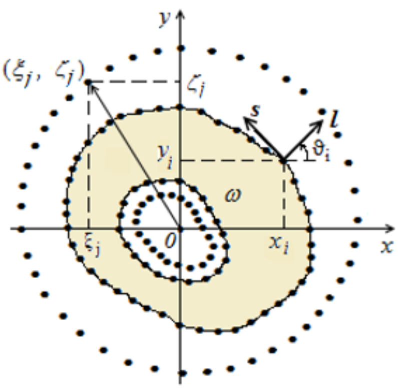

Figure 2: Domain surrounded by singular points

If the accuracy of approximation of the boundary functions satisfies us, then based on the correctness of the solved boundary problems, we can conclude that the constructed approximate solution that exactly satisfies the system of equilibrium equations in the domain ω, is a good enough approximation for the exact solution of the corresponding boundary value problem.

6 Solving boundary value problems for a system of tension-compression equations

We will consider the domain $\omega = \omega_1 \setminus \overline{\omega_2}$, where

$$\omega_1 = \{(x,y)| -4 < x < 4, -4 < y < 4\},$$

$$\omega_2 = \left\{(x,y)|x^2 + y^2 < 2.25\right\}.$$

In all the considered problems $h = 0.2$; We will assume that the plate is made of steel and the Lamé constants are $\lambda = 107.13$ GPa, $\mu = 79.3$ GPa. In all problems $n = 88$ and singular points (ξ_j, ζ_j) $j = 1, 2, ..., 88$ are located as shown in Fig. 3. Thus, in the case of each problem, the system of equations (54) will consist of 264 equations with the same number of unknowns.

In addition, in all the problems considered in this section, it is assumed that the contour of the hole is free from stresses, i.e., the following boundary conditions are satisfied

$$\begin{cases} \overset{(0)}{\sigma}_{rr} = 0, \\ \overset{(0)}{\sigma}_{r\theta} = 0, \quad on \;\; r = 1.5, \\ \overset{(1)}{\sigma}_{r3} = 0, \end{cases} \tag{55}$$

where (see fig. 4)

$$\overset{(0)}{\sigma}_{rr} = \overset{(0)}{\sigma}_{11} \cos^2\theta + 2\overset{(0)}{\sigma}_{12} \sin\theta\cos\theta + \overset{(0)}{\sigma}_{22} \sin^2\theta,$$

$$\overset{(0)}{\sigma}_{r\theta} = \left(\overset{(0)}{\sigma}_{22} - \overset{(0)}{\sigma}_{11}\right) \sin\theta\cos\theta + \overset{(0)}{\sigma}_{12} \left(\cos^2\theta - \sin^2\theta\right),$$

$$\overset{(1)}{\sigma}_{r3} = \overset{(1)}{\sigma}_{13} \cos\theta + \overset{(1)}{\sigma}_{23} \sin\theta.$$

Of particular interest to us will be the distribution of stress $\overset{(0)}{\sigma}_{\theta\theta}$ along the contour of the hole

$$\overset{(0)}{\sigma}_{\theta\theta} = \overset{(0)}{\sigma}_{11} \sin^2\theta - 2\overset{(0)}{\sigma}_{12} \sin\theta\cos\theta + \overset{(0)}{\sigma}_{22} \cos^2\theta.$$

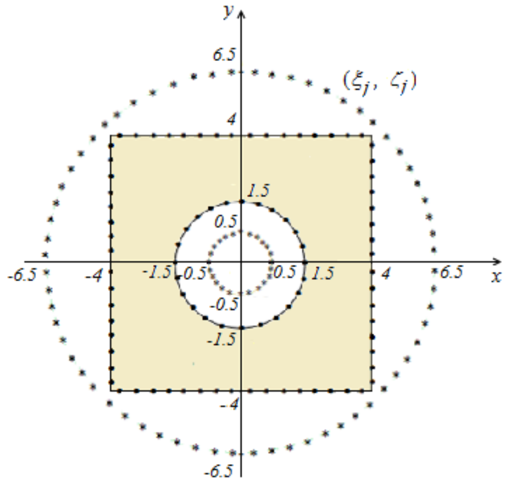

Figure 3: The considered domain ω

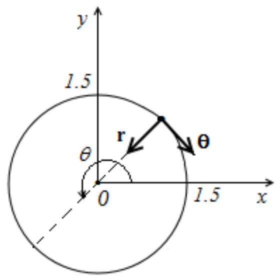

Figure 4: Vectors $\boldsymbol{r}$ and $\boldsymbol{\theta}$ and angle θ

Problem 1. One-sided stretching of a plate or the Kirsch problem (Fig. 5). Conditions (55) are set on the hole contour, and the following boundary conditions are set on the sides of the square

$$x = \pm 4, \quad -4 < y < 4: \quad \overset{(0)}{\sigma}_{11} = 1.0, \quad \overset{(0)}{\sigma}_{12} = 0, \quad \overset{(1)}{\sigma}_{13} = 0;$$

$$y = \pm 4, \quad -4 < x < 4: \quad \overset{(0)}{\sigma}_{22} = 0, \quad \overset{(0)}{\sigma}_{21} = 0, \quad \overset{(1)}{\sigma}_{23} = 0;$$

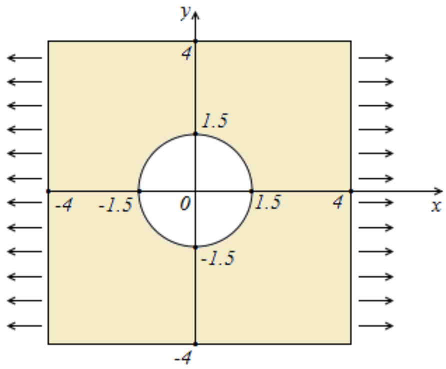

Figure 5: Problem 1. One-sided stretching of the plate

Solving the formulated boundary value problem by the approximate method described above, we obtain analytical expressions for all stress components. After that, it is not difficult to construct graphs of their boundary values. Figures 6-8 show graphs of the boundary values of functions $\overset{(0)}{\sigma}_{11}$, $\overset{(0)}{\sigma}_{22}$ and $\overset{(0)}{\sigma}_{12}$. As can be seen from these figures, any significant deviations of these stresses from the given boundary conditions are observed only near the corner points. Figure 9 shows graphs of the boundary values of stresses $\overset{(0)}{\sigma}_{rr}$, $\overset{(0)}{\sigma}_{r\theta}$ and $\overset{(1)}{\sigma}_{r3}$ on the contour of the hole.

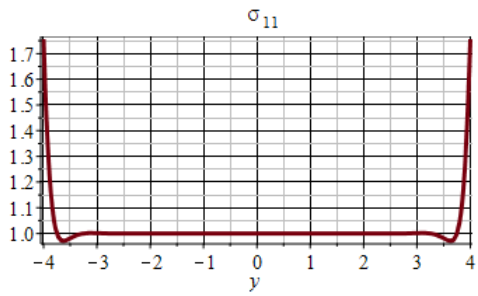

Figure 6: Problem 1. Stress component $\overset{(0)}{\sigma}_{11}$ at the boundary $x = 4.0, \quad -4.0 < y < 4.0$

If the accuracy of approximation of the boundary functions is considered satisfactory, then, based on the correctness of the solved boundary problem, we can conclude that the constructed approximate solution, which exactly satisfies the system of equations (5) inside the domain, is a fairly good approximation of the exact solution of the boundary problem under consideration.

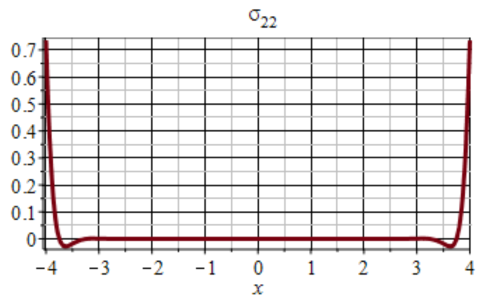

Figure 7: Problem 1. Stress component $\overset{(0)}{\sigma}_{22}$ at the boundary $x = 4.0, \quad -4.0 < y < 4.0$

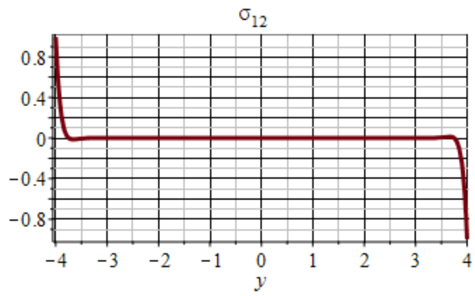

Figure 8: Problem 1. Stress component $\overset{(0)}{\sigma}_{12}$ at the boundary $x = 4.0, \quad -4.0 < y < 4.0$

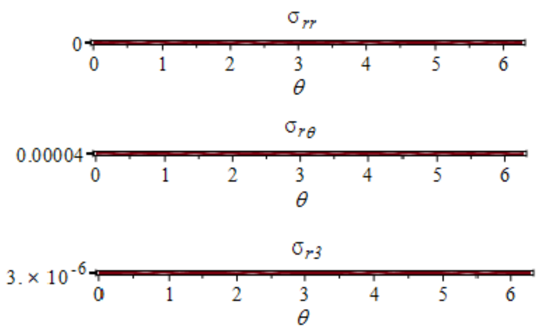

Figure 9: Problem 1. Stress components $\overset{(0)}{\sigma}_{rr}$, $\overset{(0)}{\sigma}_{r\theta}$, $\overset{(1)}{\sigma}_{r3}$ on the hole contour

Figure 10 shows the $\overset{(0)}{\sigma}_{\theta\theta}$ stress distribution on the hole contour. As expected, the maximum values $\sigma_{\theta\theta}$ are obtained at the $\theta = \pm\frac{\pi}{2}$. The stress concentration factor $\left(k = \frac{max|\overset{(0)}{\sigma}_{\theta\theta}|}{p},\ p = 1\right)$ as seen from figure 10, equals

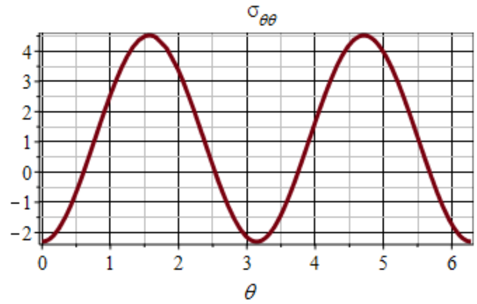

Figure 10: Problem 1. Distribution of stress $\overset{(0)}{\sigma}_{\theta\theta}$ on the contour of the hole

$k = 4.5$.

Problem 2. Conditions (55) are set on the hole contour, and the following boundary conditions are set on the sides of the square

$$x = \pm 4, \quad -4 < y < 4: \quad \overset{(0)}{\sigma}_{11} = 1.0, \ \overset{(0)}{\sigma}_{12} = 0, \ \overset{(1)}{\sigma}_{13} = 0;$$

$$y = \pm 4, \quad -4 < x < 4: \quad \overset{(0)}{u}_{2} = 0, \ \overset{(0)}{u}_{1} = 0, \ \overset{(1)}{u}_{3} = 0;$$

This problem differs from the previous problem in that in this case the upper and lower faces of the plate are rigidly clamped.

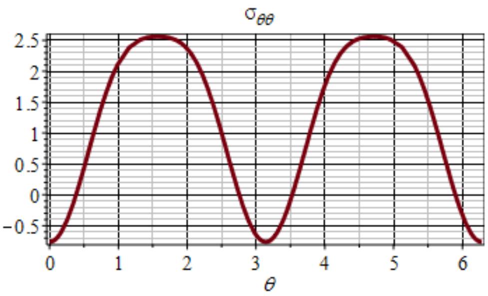

Figure 11: Problem 2. Distribution of stress $\overset{(0)}{\sigma}_{\theta\theta}$ on the contour of the hole

As expected, the stress concentration on the hole contour decreases significantly. The maximum value $\overset{(0)}{\sigma}_{\theta\theta} = 2.6$ is reached at $\theta = \pm\frac{\pi}{2}$ (see Fig. 11).

Problem 3. Consider the problem when two adjacent faces are pinched, the third face is stress-free, and a constant normal stress is set on the fourth face

$$x = +4, \quad -4 < y < 4: \quad \overset{(0)}{\sigma}_{11} = 1.0, \ \overset{(0)}{\sigma}_{12} = 0, \ \overset{(1)}{\sigma}_{13} = 0;$$

$$y = +4, \quad -4 < y < 4: \quad \overset{(0)}{\sigma}_{22} = 0, \quad \overset{(0)}{\sigma}_{12} = 0, \quad \overset{(1)}{\sigma}_{23} = 0;$$

$$x = -4, \quad -4 < y < 4: \quad \overset{(0)}{u}_{1} = 0, \quad \overset{(0)}{u}_{2} = 0, \quad \overset{(1)}{u}_{3} = 0;$$

$$y = -4, \quad -4 < y < 4: \quad \overset{(0)}{u}_{2} = 0, \quad \overset{(0)}{u}_{1} = 0, \quad \overset{(1)}{u}_{3} = 0;$$

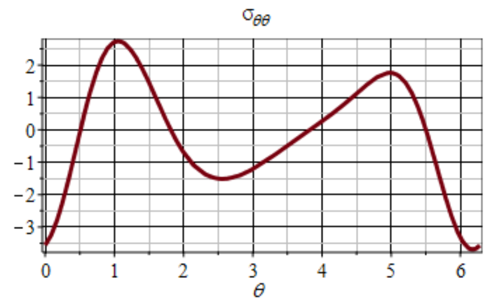

Figure 12: Problem 3. Distribution of stress $\overset{(0)}{\sigma}_{\theta\theta}$ on the contour of the hole

As can be seen from Figure 12, the maximum value $\overset{(0)}{\sigma}_{\theta\theta} = 2.8$ is reached at is reached at $\theta \approx 62°$.

Problem 4. In this problem, three faces of the plate are clamped, and a constant normal stress is set on the fourth face

$$x = +4, \quad -4 < y < 4: \quad \overset{(0)}{\sigma}_{11} = 1.0, \quad \overset{(0)}{\sigma}_{12} = 0, \quad \overset{(1)}{\sigma}_{13} = 0;$$

$$y = \pm 4, \quad -4 < x < 4: \quad \overset{(0)}{u}_{2} = 0, \quad \overset{(0)}{u}_{1} = 0, \quad \overset{(1)}{u}_{3} = 0;$$

$$x = -4, \quad -4 < y < 4: \quad \overset{(0)}{u}_{1} = 0, \quad \overset{(0)}{u}_{2} = 0, \quad \overset{(1)}{u}_{3} = 0;$$

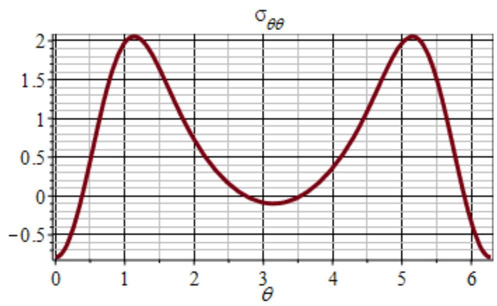

Figure 13: Problem 4. Distribution of stress $\overset{(0)}{\sigma}_{\theta\theta}$ on the contour of the hole

As can be seen from Figure 13, the maximum value $\overset{(0)}{\sigma}_{\theta\theta} = 2.1$ is reached at $\theta \approx \pm 69°$.

Problem 5. In this problem, one face of the plate is clamped, a normal stress is set on the opposite face, and the other two faces are stress-free.

$$x = +4, \quad -4 < y < 4: \quad \overset{(0)}{\sigma}_{11} = 1.0, \quad \overset{(0)}{\sigma}_{12} = 0, \quad \overset{(1)}{\sigma}_{13} = 0;$$

$$y = \pm 4, \quad -4 < x < 4: \quad \overset{(0)}{\sigma}_{22} = 0, \quad \overset{(0)}{\sigma}_{21} = 0, \quad \overset{(1)}{\sigma}_{23} = 0;$$

$$x = -4, \quad -4 < y < 4: \quad \overset{(0)}{u}_{1} = 0, \quad \overset{(0)}{u}_{2} = 0, \quad \overset{(1)}{u}_{3} = 0;$$

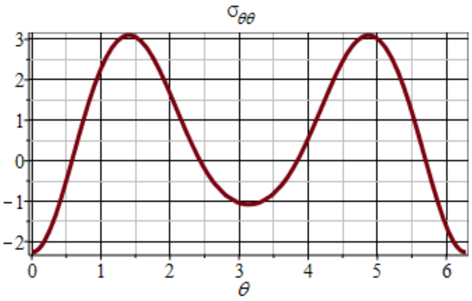

Figure 14: Problem 5. Distribution of stress $\overset{(0)}{\sigma}_{\theta\theta}$ on the contour of the hole

As can be seen from Figure 14, the maximum value is $\overset{(0)}{\sigma}_{\theta\theta} = 3.1$ reached at $\theta \approx \pm 75°$.

Problem 6. Plate bending in its own plane (Fig. 15). Conditions (55) are again set on the hole contour, and the following boundary conditions are set on the sides of the square

$$x = \pm 4, \quad -4 < y < 4: \quad \overset{(0)}{\sigma}_{11} = -0.5y, \quad \overset{(0)}{\sigma}_{12} = 0, \quad \overset{(1)}{\sigma}_{13} = 0;$$

$$y = \pm 4, \quad -4 < x < 4: \quad \overset{(0)}{\sigma}_{22} = 0, \quad \overset{(0)}{\sigma}_{21} = 0, \quad \overset{(1)}{\sigma}_{23} = 0.$$

Figure 15 shows a graph of the boundary value of the function $\overset{(0)}{\sigma}_{11}$ on the side of the square: $y = \pm 4, \quad -4 < x < 4$. We will not give the boundary values of the remaining stress components at the boundary of the domain under consideration; we only note that the found approximate solution satisfies the given boundary conditions with sufficient accuracy.

As can be seen from Figure 16, the maximum values modulo $\sigma_{\theta\theta}$ are obtained when $\theta = \pm\frac{\pi}{2}$. In this case $\overset{(0)}{\sigma}_{\theta\theta}|_{\theta=\frac{\pi}{2}} = -0.8$ and $\overset{(0)}{\sigma}|_{\theta=-\frac{\pi}{2}} = 1.6$

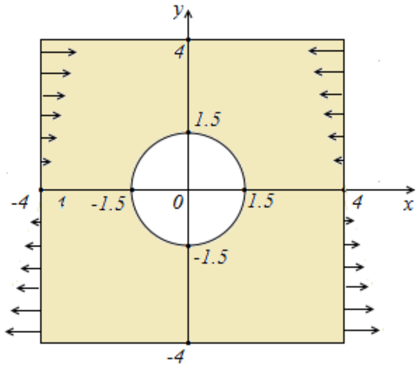

Figure 15: Problem 6. Plate bending in its own plane

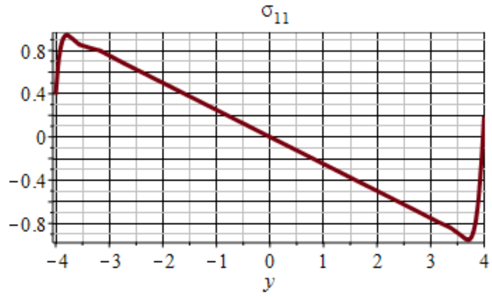

Figure 16: Stress component $\overset{(0)}{\sigma}_{11}$ at the boundary $x = 4.0, \quad -4.0 < y < 4.0$

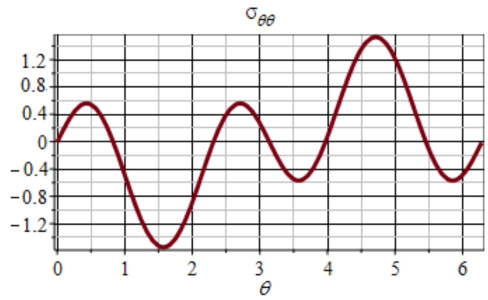

Figure 17: Problem 6. Distribution of stress $\overset{(0)}{\sigma}_{\theta\theta}$ on the contour of the hole

7 Solving boundary value problems for a system of bending equations

In this section, we consider one problem of plate bending. The middle surface of the plate is the domain ω, considered in the previous section. In both cases, the following homogeneous boundary conditions will be set on the hole contour

$$\begin{cases} \overset{(1)}{\sigma}_{rr} = 0, \\ \overset{(1)}{\sigma}_{r\theta} = 0, \quad on \;\; r = 1.5, \\ \overset{(0)}{\sigma}_{r3} = 0 \end{cases} \tag{56}$$

Problem 7. Consider now the problem of symmetrical bending of a rectangular plate. In this case, boundary conditions (56) are specified on the hole contour, and the following boundary conditions are specified on the sides of the square

$$x = \pm 4, \quad -4 < y < 4: \quad \overset{(1)}{\sigma}_{11} = 1.0, \;\; \overset{(1)}{\sigma}_{12} = 0, \;\; \overset{(0)}{\sigma}_{13} = 0;$$

$$y = \pm 4, \quad -4 < x < 4: \quad \overset{(1)}{\sigma}_{22} = 0, \;\; \overset{(1)}{\sigma}_{21} = 0, \;\; \overset{(0)}{\sigma}_{23} = 0;$$

The problem posed is solved by the method described in this article.

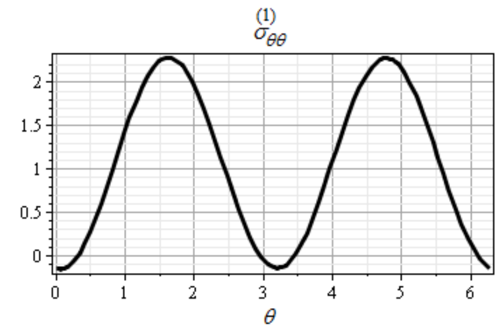

Figure 18: Problem 7. Distribution $\overset{(1)}{\sigma}_{\theta\theta}$ on the contour of the hole

Figure 18 shows distribution $\overset{(1)}{\sigma}_{\theta\theta}$ on the hole contour. As expected, the maximum values $\overset{(1)}{\sigma}_{\theta\theta}$ are obtained at the $\theta = \pm\frac{\pi}{2}$. The stress concentration factor as seen from figure 17, equals $k = 2.28$.

References

1. I. N. Vekua, On one method of calculating of prismatic shells (Russian), Trudy Tbilis. Mat. Inst., 21 (1955), 191–259

2. Vekua I.N. *Shell Theory: General Methods of onstruction*, Pitman Advanced Publishing Program, Boston-London-Melbourne (1985).

3. T. Meunargia, Some problems of stresses concentration for anisotropic plates (Different Theories), in Russian, 1977, Tbilisi State University Publishing House, 20 p

4. V. D. Kupradze , M. A. Aleksidze, The method of functional equations for the approximate solution of certain boundary value problems. USSR Computational Mathematics and Mathematical Physics, 4, 4, 82–126 (1964)

5. M. A. Aleksidze, *Solution of Boundary Value Problems by Expansion in Non-Orthogonal Functions*, Nauka, Moscow, Russia, 1978 (Russian)

6. R. Janjgava, The Approximate Solution of Some Plane Boundary Value Problems of the Moment Theory of Elasticity, Advances in Mathematical Physics, (2016), Article ID 3845362, 12 pageshttp://dx.doi.org/10.1155/2016/3845362.

7. R. Janjgava, Approximate Solution of Some Plane Boundary Value Problems for Perforated Cosserat Elastic Bodies, Adv. Appl. Math. Mech. (2019), 1064-1083; doi:10.4208/aamm.OA-2018-0019