ON THE INVESTIGATION OF ISOTROPIC THICK-WALLED SHELLS

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Abstract

We consider the problems of creating 2-dim models for thin-walled structures and satisfaction of boundary conditions when the generalized stress vector is given on the surfaces for elastic plates and shells. This problem was open also both for refined theories in the wide sense and hierarchical type models.

Keywords and phrases: Elastic thin-walled structures, problem of satisfaction of boundary condition, face of surfaces, Vekua type hierarchical models, refined theories in the wide sense.

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This paper is dedicated to the memory of a very dear person Alexander (Iasha) Khvoles. First, we'd like to recall some typical moments from his rich and exemplary life related to the activities around Ilya Vekuas Institute of Applied Mathematics. For the demonstration of his mathematical level, technique, and leadership talent, we believe the material of this paper is sufficient. Among the scientific disciplines of applied mathematics at our Institute are modeling and numerical implementation of problems in rigid body mechanics. Al. Khvoles was one of the active participants in this topic and achieved great success not only in solving, but also in implementing contractual topics beneficial for the Institute.

I. Let us consider the equilibrium equations of the elastic body in the form [1, 2]:

$$\partial_j(\sigma_{ij}+\sigma_{kj}u_{i,k})=f_j,\ x\in\Omega_h=D(x,y)\times]h^-(x,y),h^+(x,y)[, \tag{1}$$

Boundary conditions:

$$T_{i3}=\sigma_{i3}+\sigma_{j3}u_{i,j}=g_i^{\pm},\ x\in S^{\pm}=D\times h^{\pm},\ T_3=(T_{13},T_{23},T_{33})^T, \tag{2}$$

$$l[\partial_1,\partial_2,\partial_3](x,u)=g,\ x\in S=\partial D\times]h^-,h^+[. \tag{3}$$

Relation between the displacement vector $\mathbf{u} = (u_1, u_2, u_3)$, symmetrical strain ε and stress σ tensors satisfies the Cauchy formulae and Hooke's law:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{i,k}u_{j,k}), \quad \varepsilon = A\sigma, \quad \sigma = B\varepsilon. \tag{4}$$

The issues, we'll consider here, present the problem of satisfying the boundary conditions on surfaces $S^{\pm}$ of elastic plates. Although the main focus of research on the related problems is given in decision [2, Ch. I], some statements require a more careful attitude. This problem has to receive attention provisionally concerning all refined theories of von *Kármán*-Mindlin-Reissner (KMR)-type, except Reissner [3] and Ambartsumian [4] models. On the whole, these problems depend on the justification based on the variational method. As is known, depending on the "Dirichlet Principle" by Riemann, after substantial examples given by Weierstrass and Hadamard, this one was solved for the Dirichlet conditions by Hilbert[5] (for bilinear functional) and Razmadze [6, 7] (for 1-dim problems in general cases). With respect to (natural) Neumann conditions the resolving step was made by Rektorys [8]. Taking a step in this direction we constructed an example of elastic plates when the stress vector is given on the $S^{\pm}$. If we used the Legendre polynomials method of reduction, same to Vekua [9], as a basis, it would define an unstable process; at the same time Rektorys approach (when the differences from Legendre polynomials, with respect to indexes, were chosen as a basic system) gave the exact solution. This fact particularly demonstrates the existence of the "Vekua-type problem" concerning the satisfaction of boundary conditions on which Vekua studied carefully [9, ch.I,11,ch.II,2,] but incompletely. We studied this problem for the isotropic homogeneous elastic plate [2, part 6.3], the characteristic results are cited below.

Vekua spent more than 20 years considering the problem of creating 2-dim hierarchical models for an arbitrary integer N and separate ones when $N = 0, 1, 2$ without using any of physical and geometrical hypothesis,corresponding to the construction of different variances of the elastic plates and shells theory [9]. You can see, i.e. [10], sufficient reach corresponding references where Vekuas models are immediately used. Vekua worked the following way: thin-walled elastic structure (1)-(4) was considered for the linear, isotropic case; then the Galerkin method was applied for constructing corresponding models, using the Legendre polynomial system $\{p_n(x),\ p_n(\pm 1) = (\pm 1)^n\}$ for (1) and (3) by relations (4) used as the base systems; in addition introducing new expressions type (7.2c), [9, ch. I, point 7.2], named "the normalized moments of the field of stresses" which are coordinated with boundary conditions. On the basis of the expression (8.4 a,b), (8.9) and (8.4 a,b) ,(8.9) [9, point 8.1]which represents 2-dim boundary value problems, Vekua constructed the approximate solution of (1)-(4) in the form:

$$(u, \sigma) = \sum_{s=0}^{\infty} \left(\overset{s}{u}(x, y), \overset{s}{\sigma}(x, y) \right) p_s(z),$$

which "are not compatible with boundary data on face surfaces S^+, S^-. Therefore these approximations may prove to be rather rough values near the face surfaces" [9, page 79]. We called it the "Vekua problem". In [9] there were corrected the solution of corresponding to hierchical BVPs for any intejer N by additional functions satisfying also the approximate system of DEs. This function depends from the sum of differences of Legendre polinomials with respect to indexes in the form (11.7) ([9, ch. 1]:

$$U_0 = A_m(x, y)(p_{m+1}(\zeta) - p_{m-1}(\zeta)) + A_{m+1}(x, y)(p_{m+2}(\zeta) - p_m(\zeta)),$$

$$\zeta = \frac{z - \bar{h}}{2h}, \quad m > N + 2.$$

When N, m tends to infinity, the problem is open.

The another way for investigated this problem see [9, ch. II, §2]. Here for displacement vector and stress tensor are used Teylor series in point $z = 0$ and approximately satisfied BC on the surfaces and consider case when the approach has second order.

Example 1. Let us consider the case when the boundary value problem of the theory of elasticity is a 1-dim problem and so, follows: $u_1 = u_2 = \varepsilon_{\alpha i} = \sigma_{\alpha i} = f_\alpha = 0$, $h = 1$, $\sigma_{33} = (\lambda + 2\mu)u_{3,3}$. Then we get the following boundary value problem:

$$-u''(x) = f(x), \quad u'(-1) = \alpha, \quad u'(1) = \beta. \tag{5}$$

The DEs (8.4 a, b) [9] in this simple case have the following form: As $z(x) = u(x) - \dfrac{\alpha + \beta}{2} x - \dfrac{\beta - \alpha}{4} x^2 + u_0$, problem (5) is equivalent to the following one:

$$-z''(x) = f(x) + \frac{\beta - \alpha}{2}, \quad z'(-1) = z'(1) = 0. \tag{6}$$

For simplicity we assume that $f(x) - \dfrac{\beta - \alpha}{2} = p_1(x)$ and consider the following coordinate system:

$$\begin{aligned} q_k(x) &= -(2k + 1) \int_{-1}^{x} (x - t) p_k(t) dt \\ &= \frac{1}{2k + 3}(p_{k+2} - p_k) - \frac{1}{2k - 1}(p_k - p_{k-2}), \quad k = 2, 3, ..., \end{aligned}$$

$$-q_0 = \frac{1}{3}(p_2 - p_0), \quad -q_1 = \frac{1}{5}(p_3 - p_1), \quad q'(\pm 1) = 0.$$

We will find the solution of (6) as the set: $z(x) = \sum\limits_{k=0}^{\infty} z_k q_k(x)$. Then by the projective method

$$\begin{aligned}
&(-z'', -q_0(x)) = z'(x)q_0(x)|_{-1}^{1} = 0,\\
&(-z'', -q_1) = \int_{-1}^{1} z'(p_3' - p_1')dx = (p_1, p_3 - p_1) \quad \Rightarrow\\
&-z_1 + \frac{3}{7}z_3 = -1,\\
&(-z'', -q_2) = z'(x)q_2|_{-1}^{1} + \textstyle\int_{-1}^{1} \sum\limits_{k=0}^{\infty} z_k q_k' p_1 dx \ \Rightarrow\\
&-\frac{1}{3}z_0 + 2\frac{3+7}{3\cdot 7}z_2 - \frac{1}{7}z_4 = 0,\\
&-\frac{1}{4n-1}z_{2n-2} + 2\frac{4n+1}{(4n-1)(4n+3)}z_{2n} - \frac{1}{4n+3}z_{2n+2} = 0, \ \ (n = 2, 4, ...),\\
&z_1 - z_3 = -\frac{1}{3}, \quad -\frac{1}{5}z_1 + \frac{14}{5+9}z_3 - \frac{1}{9}z_5 = \frac{1}{15},\\
&-\frac{1}{4n-1}z_{2n-1} + \frac{2(4n+1)}{(4n-1)(4n+3)}z_{2n} - \frac{1}{4n+3}z_{2n+1} = 0, \quad (n = 1, 3, 5, ...) \Rightarrow\\
&z_1 = -\frac{1}{3}, \quad z_0 = z_n = 0, \ (n = 2, 3, ...),
\end{aligned}$$

as matrices of both systems are irresoluble and by the generalising theorem of Olga Taussky-Todd are nonsingular ones. Thus the solution of problem (6) has the following form:

$$z(x) = \frac{1}{3}q_1(x), \text{ i.e. } -z''(x) = p_1, \ z'(\pm 1) = 0.$$

Example 2. If we assume in the initial boundary problem (5) that $f(x) = p_1(x), \ \alpha = \beta$ follow to [9] and pleriliminary his many other publications, using as basic system Legendre Polynomials and Projective method we have:

$$u(x) = \sum_{n=0}^{\infty} u_n p_n(x) = \left(\frac{2}{5} + \alpha\right) p_1(x) - \frac{1}{15}p_3(x).$$

Indeed

$$(-u", p_0) - (p_1, p_0) = -u'p_0|_{-1}^{1} + \int_{-1}^{1} u'p_0' dx = 0,$$

$$(-u", p_{2n}) - (p_1, p_{2n}) = 0 \ (n = 1, 2, ...) \Rightarrow$$

$$\sum_{k=0}^{n-1} k(2k+1)u_{2k} + n(2n+1)\sum_{k=n}^{\infty} u_{2k} = 0 \ \Rightarrow \ \sum_{k=n}^{\infty} u_{2k} = 0$$

$$\Rightarrow u_{2n} = 0 \ (n = 1, 2, ...), \forall u_0;$$

$$(-u", p_1) - (p_1, p_1) = 0 \ \Rightarrow \ \sum_{k=n}^{\infty} u_{2k+1} = \frac{1}{3} + \alpha,$$

$$(-u", p_{2n+1}) - (p_1, p_{2n+1}) = 0 \Rightarrow$$

$$\sum_{k=0}^{n-1}(k+1)(2k+1)u_{2k+1} + (n+1)(2n+1)\sum_{k=n}^{\infty} u_{2k+1} = \alpha,$$

$$\Rightarrow u_1 = \frac{2}{5} + \alpha, \ u_3 = -\frac{1}{15}, \ u_{2n+1} = 0 \ (n = 2, 3, ...).$$

Note that $v(x) = u(x) + c$, $\forall c = const$ determines the class of solutions of (5). When $|v - u| > c - \varepsilon$, $\forall c - \varepsilon > 0$, c, $\varepsilon > 0$, the V-process developed particular in §11 of [9] is unstable one and by well-known theorem of P. Lax, nonconvergent.

In the [2, ch. II, p. 6.3] we investigated the problem of construction and justification of Vekua type systems using methodology of [8] in case of native conditions. By using Galiorkin method to DE (1) we have: the components of stress vector σ_3 for systems of DEs corresponding to [9] and [2] are various; for models by [9] the condition (2) dont satisfy as underlined in §11 of [9]. Let us return to the initial problem (1)-(4) and consider the linear case. In the above-mentioned works there was considered a case where components of exterior vector tension σ_3 were given at $S^{\mp}$. The problem of satisfying these boundary conditions for any approximations were different among proposed systems: for some models they are natural, for others they appear to be the main ones, in the sense of variational methods (see, e.g., Rektorys [8]). We construct a class of operator equations, in fact, coinciding with systems (7.9 a,b),(7.18 h,i) or (8.16) [9], for brevity, we shall denote it as (V).

Let us use this expansion into Fourier-Legendre for incomplete series components of stress tensor. By virtue of boundary conditions on $S^{\pm}$ we have:

$$\sigma_{\alpha\beta} = \sum_{k=0}^{\infty} \overset{s}{\sigma}_{\alpha\beta}\, p_s\left(\frac{z}{h}\right), \tag{7}$$

$$\sigma_3 = \frac{(h+z)g^+ + (h-z)g^-}{2h} + \sum_{s=1}^{\infty} \overset{s}{\sigma}_{3j}\left[p_{s+1}\left(\frac{z}{h}\right) - p_{s-1}\left(\frac{z}{h}\right)\right], \tag{8}$$

At first we construct the basic Vekua type hierarchical 2-dim model which approximates the linear boundary value problem for homogeneous isotropic plates (for details see [2, Ch. II, part 6.3]). Then equilibrium equations in terms of components of the stress tensor will be equivalent to the following infinite system

$$
\begin{aligned}
&c_m h \overset{m}{\sigma}_{\alpha\beta,\beta} + (2m+1) c_m \overset{m}{\sigma}_{\alpha\beta} = \overset{m}{f}_{\alpha} - h c_0 \delta_{m0} \frac{g_{\alpha}^{+} - g_{\alpha}^{-}}{2}, \\
&c_m h \left(\overset{m-1}{\sigma}_{\alpha 3,\alpha} - \overset{m+1}{\sigma}_{\alpha 3,\alpha} \right) + (2m+1) c_m \overset{m}{\sigma}_{33} = \overset{m}{f}_{3} - h c_0 \delta_{m0} \frac{g_{\alpha,\alpha}^{+} + g_{\alpha,\alpha}^{-}}{2} \\
&- h c_1 \delta_{m1} \frac{g_{\alpha,\alpha}^{+} - g_{\alpha,\alpha}^{-}}{2} - h c_0 \delta_{m0} \frac{g_{3}^{+} + g_{3}^{-}}{2}
\end{aligned} \tag{9}
$$

where

$$
\overset{m}{f} = \int_{-h}^{h} f(x_1, x_2, t) p_m \left(\frac{t}{h} \right) dt, \quad c_m = \frac{2}{2m+1}, \quad m = 0, 1, 2, \cdots.
$$

Hooke's law takes the following form:

$$
\begin{aligned}
&c_m h \overset{m}{\sigma}_{11} = (\lambda + 2\mu) c_m h \overset{m}{u}_{1,1} + \lambda c_m h \overset{m}{u}_{2,2} + \lambda (2m+1) c_m \sum_{k \geq m(2)} \overset{k+1}{u}_{3}, \\
&c_m h \overset{m}{\sigma}_{12} = \mu h c_m \left(\overset{m}{u}_{1,2} + \overset{m}{u}_{2,1} \right), \\
&c_m h \overset{m}{\sigma}_{22} = \lambda c_m h \overset{m}{u}_{1,1} + (\lambda + 2\mu) c_m h \overset{m}{u}_{2,2} + \lambda (2m+1) c_m \sum_{k \geq m(2)} \overset{k+1}{u}_{3}, \\
&c_m h \left(\overset{m-1}{\sigma}_{3\alpha} - \overset{m+1}{\sigma}_{3\alpha} \right) = \mu h c_m \overset{m}{u}_{3,\alpha} + \mu (2m+1) c_m \sum_{k \geq m(2)} \overset{k+1}{u}_{\alpha} \\
&- h c_0 \delta_{m0} \frac{g_{\alpha}^{+} + g_{\alpha}^{-}}{2} - h c_1 \delta_{m1} \frac{g_{\alpha}^{+} - g_{\alpha}^{-}}{2}, \\
&\quad c_m h \left(\overset{m-1}{\sigma}_{33} - \overset{m+1}{\sigma}_{33} \right) = \lambda h c_m \overset{m}{u}_{\alpha,\alpha} + (\lambda + 2\mu)(2m+1) c_m \sum_{k \geq m(2)} \overset{k+1}{u}_{3} \\
&\quad - h c_0 \delta_{m0} \frac{g_{3}^{+} + g_{3}^{-}}{2} - h c_1 \delta_{m1} \frac{g_{3}^{+} - g_{3}^{-}}{2}.
\end{aligned} \tag{10}
$$

Here and (often) below the following note is used:

$$
\sum_{k \geq i(s)} \overset{k}{u} = \overset{i}{u} + \overset{i+s}{u} + \overset{i+2s}{u} + \cdots, \quad \sum_{k \leq i(s)} \overset{k}{u} = \overset{i}{u} + \overset{i-s}{u} + \overset{i-2s}{u} + \cdots.
$$

Formulae (9) and (10) make it possible to obtain an explicit form of Vekua type system in displacement components. For this purpose we use Hooke's law for values σ_{3i} and condition (2). We shall have:

$$
g_{\alpha}^{+} = \mu \sum_{k=0}^{\infty} \left(\overset{k}{u}_{3,\alpha} + \frac{k(k+1)}{2h} \overset{k}{u}_{\alpha} \right),
$$

$$g_\alpha^- = \mu\sum_{k=0}^{\infty}(-1)^k\left(\overset{k}{u}_{3,\alpha} - \frac{k(k+1)}{2h}\overset{k}{u}_\alpha\right),$$

and

$$g_3^+ = \sum_{k=0}^{\infty}\left(\lambda\overset{k}{u}_{\alpha,\alpha} + (\lambda+2\mu)\frac{k(k+1)}{2h}\overset{k}{u}_3\right),$$

$$g_3^- = \sum_{k=0}^{\infty}(-1)^k\left(\lambda\overset{k}{u}_{\alpha,\alpha} - (\lambda+2\mu)\frac{k(k+1)}{2h}\overset{k}{u}_3\right).$$

We define values $g^+ \pm g^-$, entering (9). We shall have:

$$\begin{aligned}
g_\alpha^+ + g_\alpha^- &= 2\mu\sum_{k=0}^{\infty}\left(\overset{2k}{u}_{3,\alpha} + \frac{(k+1)(2k+1)}{h}\overset{2k+1}{u}_\alpha\right),\\
g_\alpha^+ - g_\alpha^- &= 2\mu\sum_{k=0}^{\infty}\left(\overset{2k+1}{u}_{3,\alpha} + \frac{k(2k+1)}{h}\overset{2k}{u}_\alpha\right),\\
g_3^+ + g_3^- &= 2\sum_{k=0}^{\infty}\left(\lambda\overset{2k}{u}_{\alpha,\alpha} + (\lambda+2\mu)\frac{(k+1)(2k+1)}{h}\overset{2k+1}{u}_3\right),\\
g_3^+ - g_3^- &= 2\sum_{k=0}^{\infty}\left(\lambda\overset{2k+1}{u}_{\alpha,\alpha} + (\lambda+2\mu)\frac{k(2k+1)}{h}\overset{2k}{u}_3\right),
\end{aligned} \tag{11}$$

From equations (10), summing up the three last formulae, for values $\overset{m}{\sigma}_{3\alpha}$ we obtain:

$$\sum_{s\le m(2)}\left(\overset{s-2}{\sigma}_{3\alpha} - \overset{s}{\sigma}_{3\alpha}\right) = -\overset{m}{\sigma}_{3\alpha} = \mu\sum_{s\le m(2)}\overset{s-1}{u}_{3,\alpha} + \frac{\mu}{h}\sum_{s\le m(2)}(2s-1)\sum_{k\ge s(2)}\overset{k}{u}_\alpha$$
$$-\frac{1}{2}(g_\alpha^+ + g_\alpha^-)\sum_{s\le m(2)}\delta_{s-1,0} - \frac{1}{2}(g_\alpha^+ - g_\alpha^-)\sum_{s\le m(2)}\delta_{s-1,1}.$$

Similarly

$$-\overset{m}{\sigma}_{33} = \mu\sum_{s\le m(2)}\overset{s-1}{u}_{\alpha,\alpha} + \frac{\lambda+2\mu}{h}\sum_{s\le m(2)}(2s-1)\sum_{k\ge s(2)}\overset{k}{u}_3$$
$$-\frac{1}{2}(g_3^+ + g_3^-)\sum_{s\le m(2)}\delta_{s-1,0} - \frac{1}{2}(g_3^+ - g_3^-)\sum_{s\le m(2)}\delta_{s-1,1}.$$

In these expressions $\overset{-1}{\sigma}_{3\alpha} = \overset{0}{\sigma}_{3\alpha} = 0$ is assumed.

Now, by using formulae (11) from the latter representations after some computations, we get

$$\overset{m}{\sigma}_{3\alpha} = \mu\sum_{s\ge(m+1)(2)}\left[\overset{s}{u}_{3,\alpha} + \frac{1}{2h}((s+1)(s+2) - m(m+1))\overset{s+1}{u}_\alpha\right],$$

$$\overset{m}{\sigma}_{33} = \sum_{s \geq (m+1)(2)} \left[\lambda \overset{s}{u}_{\alpha,\alpha} + \frac{1}{2h}(\lambda + 2\mu)((s+1)(s+2) - m(m+1)) \overset{s+1}{u}_3 \right].$$

Taking into account the last formulae, as well as (10), after obvious simplifications with respect to components of the displacement vector we obtain the following infinite system of Vekua's differential equations:

$$l_2 \overset{m}{u}_+ + (\lambda + \mu) h^{-1}(2m+1) \sum_{k \geq m(2)} \operatorname{grad} \overset{k+1}{u}_3 + \mu h^{-2} \frac{2m+1}{2}$$
$$\times \sum_{k \geq m(2)} [k(k+1) - m(m+1)] \overset{k}{u}_+ = \frac{1}{c_m h} \left[\overset{m}{f}_+ - \frac{g_+^+ - g_+^-}{2} \delta_{m0} \right],$$
$$\mu \Delta \overset{m}{u}_3 + (\lambda + \mu) h^{-1}(2m+1) \sum_{k \geq m(2)} \operatorname{div} \overset{k}{u}_+ + (\lambda + 2\mu) h^{-2} \frac{2m+1}{2}$$
$$\times \sum_{k \geq m(2)} [k(k+1) - m(m+1)] \overset{k}{u}_3 = \frac{1}{c_m h} \left[\overset{m}{f}_3 - \frac{g_3^+ - g_3^-}{2} \delta_{m0} \right],$$

Here

$$u_+ = (u_1, u_2)^T, \quad f_+ = (f_1, f_2)^T, \quad g_+ = (g_1, g_2)^T,$$
$$(l_2 u_+, u_+) = \mu(\Delta u_\alpha, u_\alpha) + (\lambda + \mu)(\operatorname{grad} \operatorname{div} u_+, u_+).$$

From system (10), evidently, for values $\overset{m}{\sigma}_{\alpha 3}$ we have:

$$\overset{m-1}{\sigma}_{\alpha 3} = \overset{m+1}{\sigma}_{\alpha 3} + \mu \overset{m}{u}_{3,\alpha} + \mu \frac{2m+1}{h} \sum_{k \geq m(2)} \overset{k+1}{u}_\alpha - \frac{1}{2} \delta_{m0}(g_\alpha^+ + g_\alpha^-) - \frac{1}{2} \delta_{m1}(g_\alpha^+ - g_\alpha^-)$$
$$= \overset{m+3}{\sigma}_{\alpha 3} + \mu \overset{m+2}{u}_{3,\alpha} + \mu \frac{2m+5}{h} \sum_{k \geq m(2)} \overset{k+3}{u}_\alpha + \mu \overset{m}{u}_{3,\alpha} + \mu \frac{2m+1}{h} \sum_{k \geq m(2)} \overset{k+1}{u}_\alpha$$
$$- \sum_{k \geq m(2)} \left(\frac{g_\alpha^+ + g_\alpha^-}{2} \delta_{k0} + \frac{g_\alpha^+ - g_\alpha^-}{2} \delta_{k1} \right),$$
$$\overset{m}{\sigma}_{\alpha 3} = \sum_{k \geq m(2)} \overset{k+1}{u}_{3,\alpha} + \mu h^{-1} \sum_{s \geq (m+1)(2)} (2s+1) \sum_{k \geq m(2)} \overset{k+2}{u}_\alpha$$
$$- \frac{1}{2} \sum_{k \geq (m+1)(2)} \left((g_\alpha^+ + g_\alpha^-) \delta_{k0} + (g_\alpha^+ - g_\alpha^-) \delta_{k1} \right), \quad m = 1, 2, ...$$
$$\overset{m}{\sigma}_{\alpha 3} = \mu \sum_{k \geq (m+1)(2)} \left(\overset{k}{u}_{3,\alpha} + \frac{1}{2}((k+1)(k+2) - m(m+1)) \overset{k+1}{u}_\alpha \right).$$

Analogously,

$$\overset{m}{\sigma}_{33} = \sum_{k \geq (m+1)(2)} \left(\lambda \overset{k}{u}_{3,\alpha} + \frac{1}{2h}(\lambda + 2\mu)((k+1)(k+2) - m(m+1)) \overset{k}{u}_3 \right).$$

Taking into account these formulae we obtain

$$\frac{1}{2}(g_\alpha^+ + g_\alpha^-)\sum_{m=1}^{\infty}\delta_{m-1,0} + \frac{1}{2}(g_\alpha^+ - g_\alpha^-)\sum_{m=1}^{\infty}\delta_{m-1,1} - \overset{m}{\sigma}_{3\alpha}$$
$$= \mu \sum_{k<(m+1)(2)} \left(\overset{k}{u}_{3,\alpha} + \frac{1}{2h}(k+1)(k+2)\overset{k+1}{u}_\alpha\right).$$

Hence for values $\overset{m}{\sigma}_{\alpha 3}$ we have:

$$\overset{m}{\sigma}_{\alpha 3} = \frac{1}{2}(g_\alpha^+ + g_\alpha^-)\sum_{m\geq 1(1)}^{\infty}\delta_{m-1,0} + \frac{1}{2}(g_\alpha^+ - g_\alpha^-)\sum_{m\geq 1(1)}^{\infty}\delta_{m-1,1}$$
$$-\mu \sum_{k\leq(m-1)(2)} \left(\overset{k}{u}_{3,\alpha} + \frac{1}{2h}(k+1)(k+2)\overset{k+1}{u}_\alpha\right).$$

Similarly for $\overset{m}{\sigma}_{33}$ we shall have:

$$\overset{m}{\sigma}_{33} = \frac{1}{2}(g_3^+ + g_3^-)\sum_{m\geq 1(1)}^{\infty}\delta_{m-1,0} + \frac{1}{2}(g_3^+ - g_3^-)\sum_{m\geq 1(1)}^{\infty}\delta_{m-1,1}$$
$$-\sum_{k\leq(m-1)(2)} \left[\lambda \overset{k}{u}_{\alpha,\alpha} + \frac{\lambda+2\mu}{2h}(k+1)(k+2)\overset{k+1}{u}_3\right].$$

Taking into account these expression in (9) we obtain the infinite system of differential equations according to Vekua's system (V) in the following form:

$$\begin{aligned}
&l_2 \overset{n}{u}_+ + h^{-1}(2n+1)\mathrm{grad}\left(\lambda \sum_{i\geq n(2)} \overset{i+1}{u}_3 - \mu \sum_{i\leq n(2)} \overset{i+1}{u}_3\right)\\
&-\mu h^{-2}\frac{2n+1}{2}\sum_{i\leq n(2)} i(i+1)\overset{i}{u}_+ = \frac{1}{hc_n}\Bigg[\overset{n}{f}_+\\
&-\left(\frac{g_+^+ - g_+^-}{2}\delta_{n0} + (g_+^+ + g_+^-)\sum_{i\geq 1(1)}\delta_{i-1,0} + (g_+^+ - g_+^-)\sum_{i\geq 1(1)}\delta_{i-1,1}\right)\Bigg],\\
&\mu\Delta \overset{n}{u}_3 + h^{-1}(2n+1)\mathrm{div}\left(\mu \sum_{i\geq n(2)} \overset{i+1}{u}_+ - \lambda \sum_{i\leq n(2)} \overset{i-1}{u}_+\right)\\
&-(\lambda+2\mu)h^{-2}\frac{2n+1}{2}\sum_{i\leq n(2)} i(i+1)\overset{i}{u}_3 = \frac{1}{hc_n}\Bigg[\overset{n}{f}_3\\
&-\left(\frac{g_3^+ - g_3^-}{2}\delta_{n0} + (g_3^+ + g_3^-)\sum_{i\geq 1(1)}\delta_{i-1,0} + (g_3^+ - g_3^-)\sum_{i\geq 1(1)}\delta_{i-1,1}\right)\Bigg],
\end{aligned} \tag{12}$$

$$n = 0, 1, 2, ..., N.$$

The comparison of equations (12), named as (V_1), with the system (V) proves their identity for $N = 0, 1, 2$. When $N \geq 3$, the main parts (containing only second order partial derivatives) of systems (7.18 h, i) [9] and (12) are different. Then [9, page 52] we read: the (7.18 h, i) is a strong elliptic system of PDEs for $N \geq 3$, "but we do not rewrite this one in a more expanded form and shall not deal with the investigation of problems of existence and uniqueness in the general form".

Evidently, in order to obtain effective values a priori in the form of energy inequalities for Vekua's operator with fixed N together with highest derivatives, we should pay attention to the explicit form of summands with derivatives of zero and first order from unknown moments $\overset{n}{u}_i(x_1, x_2)$ $(n = 0, 1, 2, ...)$ appearing in system (12). Thus, we constructed (12) corresponding to the equations (1). Reduced boundary conditions, originated by the data on the lateral surfaces S and the construction of which is not difficult, should be added to these systems. For this purpose we should multiply equalities (3) by Legendre polynomials $p_i\left(\frac{z}{h}\right)$ and integrate them between $-h$ and h. If Hooke's law and other representations from (4) are used, then we come up to the finite reduced boundary conditions, defined on ∂D.

II. This part dedicated to investgation of Vekua theory for isotropic thick-walled shells of non-homogeneous structure are sited from [2, pp. 136-141] (see [12, 14-16] too).

Let on S the coordinate lines be lines of curvature:

$$\mathbf{R}_\alpha = (1 - k_\alpha x^3)\mathbf{r}_\alpha, \;\; \mathbf{R}_3 = \mathbf{n}, \;\; \mathbf{R}^\alpha = \frac{1}{1 - k_\alpha x^3}\mathbf{r}^\alpha,$$

where k_α denote principal curvatures of the midsurface, $\mathbf{R}_i$ and $\mathbf{R}^i$ are covariant and contravariant base vectors of the space, $\mathbf{r}_\alpha$ and $\mathbf{r}^\alpha$ are covariant and contravariant base vectors of the midsurface, $\mathbf{n}$ the unit vector of the normal of the surface S.

Then the equilibrium equations of shells have such form [9]:

$$\frac{1}{\sqrt{a}}\partial_\alpha(\sqrt{a}P^\alpha) - \partial_3 P^3 + \vartheta F = 0, \; \vartheta = (1 - k_1 x^3)(1 - k_2 x^3). \tag{13}$$

Using method of [9] in our case we have:

$$\overset{(k)}{P}{}^{\alpha\beta} = \sum_{m=0}^{\infty}\left\{\lambda^{km}_{3-\alpha,\gamma} a^{\alpha\beta}\left(\nabla_\gamma \overset{m}{u}{}^\gamma - b^\gamma_\gamma \overset{m}{u}_3\right) + \frac{1}{h}\lambda^{km}_{3-\alpha} a^{\alpha\beta}\overset{m}{u}_3\right.$$

$$\left. + \mu^{km}_{3-\alpha,\gamma}\left[a^{\alpha\lambda}\left(\nabla_\gamma \overset{m}{u}{}^\beta - b^\beta_\gamma \overset{m}{u}_3\right) + a^{\beta\lambda}\left(\nabla_\gamma \overset{m}{u}{}^\alpha - b^\alpha_\gamma \overset{m}{u}_3\right)\right]\right\}$$

$$\overset{(k)}{P}{}^{\alpha 3} = \sum_{m=0}^{\infty} \left[\mu^{km}_{3-\alpha,\gamma} a^{\alpha\gamma} \left(\nabla_\gamma \overset{m}{u}_3 + b_{\gamma\beta} \overset{m}{u}{}^\beta \right) + \frac{1}{h} \mu^{km}_{3-\alpha} \overset{m}{u}{}'^\alpha \right]$$

$$\overset{(k)}{P}{}^{3\alpha} = \sum_{m=0}^{\infty} \left[\mu^{km}_{3-\alpha,\gamma} a^{\alpha\gamma} \left(\nabla_\gamma \overset{m}{u}_3 + b_{\gamma\beta} \overset{m}{u}{}^\beta \right) + \frac{1}{h} \mu^{km}_\bullet \overset{m}{u}{}'^\alpha \right]$$

$$\overset{(k)}{P}{}^{33} = \sum_{m=0}^{\infty} \left[\lambda^{km}_{3-\gamma} \left(\nabla_\gamma \overset{m}{u}{}^\gamma - b^\gamma_\gamma \overset{m}{u}_3 \right) + \frac{1}{h} (\lambda^{km}_\bullet + 2\mu^{km}_\bullet) \overset{m}{u}{}'_3 \right],$$

where

$$\lambda^{km}_{\alpha\beta} = \left(k + \frac{1}{2}\right) \frac{1}{h} \int_{-h}^{h} \lambda A^\beta_\alpha P_k(x^3/h) P_m(x^3/h) dx^3,$$

$$\mu^{km}_{\alpha\beta} = \left(k + \frac{1}{2}\right) \frac{1}{h} \int_{-h}^{h} \mu A^\beta_\alpha P_k(x^3/h) P_m(x^3/h) dx^3,$$

$$\lambda^{km}_{\alpha} = \left(k + \frac{1}{2}\right) \frac{1}{h} \int_{-h}^{h} \lambda A_\alpha P_k(x^3/h) P_m(x^3/h) dx^3,$$

$$\mu^{km}_{\alpha} = \left(k + \frac{1}{2}\right) \frac{1}{h} \int_{-h}^{h} \mu A_\alpha P_k(x^3/h) P_m(x^3/h) dx^3,$$

$$\lambda^{km}_\bullet = \left(k + \frac{1}{2}\right) \frac{1}{h} \int_{-h}^{h} \lambda \vartheta P_k(x^3/h) P_m(x^3/h) dx^3,$$

$$\mu^{km}_\bullet = \left(k + \frac{1}{2}\right) \frac{1}{h} \int_{-h}^{h} \mu \vartheta P_k(x^3/h) P_m(x^3/h) dx^3,$$

$$A_\alpha = 1 - k_\alpha x^3, \quad A^\beta_\alpha = \frac{A_\alpha}{A_\beta}.$$

Let $g^\pm$ are defined by [13, part 1.2]. Then we have

$$P^{3(+)} = \vartheta^{(+)} g^+, \quad P^{3(-)} = \vartheta^{(-)} g^-, \quad \vartheta^{(\pm)} = (1 \pm k_1 h)(1 \pm k_2 h).$$

The solution of (13), according to [12] has the form

$$P^\alpha = \sum_{k=0}^{\infty} \overset{(k)}{P}{}^\alpha P_k(x^3/h),$$

$$P^3 = \frac{1}{2} \left[\left(1 + \frac{x^3}{h}\right) \vartheta^{(+)} g^+ + \left(1 - \frac{x^3}{h}\right) \vartheta^{(-)} g^- \right]$$
$$+ \sum_{k=0}^{\infty} \overset{(k)}{P}{}^3_* [P_k(x^3/h) - P_{k+2}(x^3/h)].$$

If we use the method of Vekua [9] we have

$$\frac{1}{h}\partial_\alpha\left(\sqrt{a}\overset{(k)}{P^\alpha}\right)-\frac{2k+1}{h}\sum_{s=0}^{[\frac{k-1}{2}]}\overset{(k-2S-1)}{P^3}+\overset{(k)}{F}=0,\ (k=0,1,...) \tag{14}$$

where

$$\overset{(k)}{F}=\frac{2k+1}{2}(\vartheta^+g_3^+-(-1)^k\vartheta^-g_3^-).$$

Cutting the infinite system (14) we have:

$$\begin{gathered}\nabla_\alpha\overset{(k)}{P^{\alpha\beta}}-b_\alpha^\beta\overset{(k)}{P^{\alpha3}}-\frac{2k+1}{h}\sum_{s=0}^{[\frac{k-1}{2}]}\overset{(k-2S-1)}{P^{3\beta}}+\overset{(k)}{F^\beta}=0,\\ \nabla_\alpha\overset{(k)}{P^{\alpha3}}+b_{\alpha\beta}\overset{(k)}{P^{\alpha\beta}}-\frac{2k+1}{h}\sum_{s=0}^{[\frac{k-1}{2}]}\overset{(k-2S-1)}{P_{33}}+\overset{(k)}{F^3}=0,\\ (k=0,1,...,N)\end{gathered} \tag{15}$$

where

$$\overset{(k)}{P^{\alpha\beta}}=\sum_{m=0}^N\left\{\lambda_{3-\alpha,\gamma}^{km}a^{\alpha\beta}\left(\nabla_\gamma\overset{m}{u^\gamma}-b_\gamma^\gamma\overset{m}{u_3}\right)+\frac{2m+1}{h}\lambda_{3-\alpha}^{km}a^{\alpha\beta}\sum_{S=0}^{\frac{[N-m-1]}{2}}\overset{m+2S+1}{u_3}\right.$$

$$\left.+\mu_{3-\alpha,\gamma}^{km}\left[a^{\alpha\lambda}\left(\nabla_\gamma\overset{m}{u^\beta}-b_\gamma^\beta\overset{m}{u_3}\right)+a^{\beta\lambda}\left(\nabla_\gamma\overset{m}{u^\alpha}-b_\gamma^\alpha\overset{m}{u_3}\right)\right]\right\} \tag{16}$$

$$\overset{(k)}{P^{\alpha3}}=\sum_{m=0}^N\left[\mu_{3-\alpha,\gamma}^{km}a^{\alpha\gamma}\left(\nabla_\gamma\overset{m}{u_3}+b_{\alpha\beta}\overset{m}{u^\beta}\right)+\frac{2m+1}{h}\mu_{3-\alpha}^{km}\sum_{S=0}^{\frac{[N-m-1]}{2}}\overset{m+2S+1}{u^\alpha}\right],$$

$$\overset{(k)}{P^{3\alpha}}=\sum_{m=0}^N\left[\mu_{3-\alpha,\gamma}^{km}a^{\alpha\gamma}\left(\nabla_\gamma\overset{m}{u_3}+b_{\gamma\beta}\overset{m}{u^\beta}\right)+\frac{2m+1}{h}\mu_\bullet^{km}\sum_{S=0}^{\frac{[N-m-1]}{2}}\overset{m+2S+1}{u^\alpha}\right]$$

$$\overset{(k)}{P^{33}}=\sum_{m=0}^N\left[\lambda_{3-\gamma}^{km}\left(\nabla_\gamma\overset{m}{u^\gamma}-b_\gamma^\gamma\overset{m}{u_3}\right)+\frac{2m+1}{h}(\lambda_\bullet^{km}+2\mu_\bullet^{km})\sum_{S=0}^{\frac{[N-m-1]}{2}}\overset{m+2S+1}{u_3}\right].$$

These equations (15-16) represent the closed system of differential equations with respect to $\overset{k}{u^j}$ functions.

Consider now when Ω_h is the spherical shell with constant thickness $2h$, R denotes the radius. Let $x^1 = x$, $x^2 = y$, $x^3 = x_3$. In our case we evidently have:

$$a_{11} = a_{22} = \Lambda; \;\; a_{12} = a_{21} = 0.$$

If ν, φ are geographical coordinates of a sphere, then the isometrical coordinate system it may be have

$$x = \tan\frac{\nu}{2}\cos\varphi, \;\; y = \tan\frac{\nu}{2}\sin\varphi,$$

and

$$\Lambda = R^2\Lambda_0^2, \;\; \Lambda_0 = \frac{2}{1+x^2+y^2} = 2\cos^2\frac{\nu}{2}.$$

The system of equations (15-16) following to Vekua [9] has the following complex form:

$$\begin{gathered}
\frac{1}{\Lambda}\frac{\partial}{\partial z}\left(\overset{(k)}{P}_{11} - \overset{(k)}{P}_{22} + 2i\overset{(k)}{P}_{21}\right) + \frac{\partial}{\partial \bar{z}}\left(\overset{(k)}{P}{}_1^1 + \overset{(k)}{P}{}_2^2\right) \\
+\frac{1}{R}\frac{\partial}{\partial z}\left(\overset{(k)}{P}_{13} + i\overset{(k)}{P}_{23}\right) \\
-\frac{2k+1}{h}\sum_{s=0}^{\left[\frac{k-1}{2}\right]}\left(\overset{(k-2S-1)}{P_{31}} + i\overset{(k-2S-1)}{P_{32}}\right) + \overset{(k)}{F}_1 + i\overset{(k)}{F}_2 = 0, \qquad (17)\\
2\mathrm{Re}\left[\frac{1}{\Lambda}\frac{\partial}{\partial z}\left(\overset{(k)}{P}_{13} + i\overset{(k)}{P}_{23}\right) - \frac{1}{R}\left(\overset{(k)}{P}{}_1^1 + \overset{(k)}{P}{}_2^2\right)\right] \\
-\frac{2k+1}{h}\sum_{s=0}^{\left[\frac{k-1}{2}\right]}\overset{(k-2S-1)}{P_{33}} + \overset{(k)}{F}_3 = 0, \;\; (k = 0,1,...,N)
\end{gathered}$$

$$\begin{gathered}
\overset{(k)}{P}_{11} - \overset{(k)}{P}_{22} + 2i\overset{(k)}{P}_{21} = 4\sum_{s=0}^{N}\mu_{km}\Lambda\frac{\partial}{\partial\bar{z}}\frac{1}{\Lambda}\left(\overset{m}{u}_1 + i\overset{m}{u}_2\right), \\
\left(\overset{(k)}{P}{}_1^1 + \overset{(k)}{P}{}_2^2\right) = 2\sum_{s=0}^{N}\left\{(\lambda_{km} + \mu_{km})\left(\overset{m}{\theta} + \frac{2}{R}\overset{m}{u}_3\right)\right. \\
\left. + \frac{2m+1}{h}\lambda_{km}^{(1)}\sum_{s=0}^{\left[\frac{N-m-1}{2}\right]}\overset{m+2S+1}{u_3}\right\}, \\
\overset{(k)}{P}_{13} + i\overset{(k)}{P}_{23} = \sum_{m=0}^{N}\left\{\mu_{km}\left[2\frac{\partial \overset{m}{u_3}}{\partial\bar{z}} - \frac{1}{R}\left(\overset{m}{u}_1 + i\overset{m}{u}_2\right)\right]\right. \qquad (18)
\end{gathered}$$

$$+\frac{2m+1}{h}\lambda_{km}^{(1)}\sum_{s=0}^{[\frac{N-m-1}{2}]}\left(\overset{m+2S+1}{u_1}+i\overset{m+2S+1}{u_2}\right)\Bigg\},$$

$$\overset{(k)}{P}_{31}+i\overset{(k)}{P}_{32}=\sum_{m=0}^{N}\left\{\mu_{km}^{(1)}\left[2\frac{\partial \overset{m}{u_3}}{\partial z}-\frac{1}{R}\left(\overset{m}{u_1}+i\overset{m}{u_2}\right)\right]\right.$$

$$\left.+\frac{2m+1}{h}\lambda_{km}^{(2)}\sum_{s=0}^{[\frac{N-m-1}{2}]}\left(\overset{m+2S+1}{u_1}+i\overset{m+2S+1}{u_2}\right)\right\},$$

$$\overset{(k)}{P}_{33}=2\sum_{s=0}^{N}\left\{\lambda_{km}^{(1)}\left(\overset{m}{\theta}+\frac{2}{R}\overset{m}{u_3}\right)+\frac{2m+1}{h}(\lambda_{km}^{(2)}+2\mu_{km}^{(2)})\sum_{s=0}^{[\frac{N-m-1}{2}]}\overset{m+2S+1}{u_3}\right\},$$

where

$$\overset{m}{\theta}=\nabla_\gamma \overset{m}{u}^{\gamma}=2\mathrm{Re}\left[\frac{1}{\Lambda}\frac{\partial}{\partial z}\left(\overset{m}{u_1}+i\overset{m}{u_2}\right)\right],$$

$$z=x+iy,\ \ \bar{z}=x-iy,$$

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),\ \ \frac{\partial}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right),$$

$$\lambda_{km}=\lambda_{11}^{km},\ \ \lambda_{km}^{(\alpha)}=\left(k+\frac{1}{2}\right)\frac{1}{2}\int_{-h}^{h}\lambda\left(1+\frac{x^3}{R}\right)^{\alpha}P_k(x_3/h)P_m(x_3/h)dx_3,$$

$$\mu_{km}=\mu_{11}^{km},\ \ \mu_{km}^{(\alpha)}=\left(k+\frac{1}{2}\right)\frac{1}{2}\int_{-h}^{h}\mu\left(1+\frac{x^3}{R}\right)^{\alpha}P_k(x_3/h)P_m(x_3/h)dx_3.$$

Assume that $\lambda_{km}^{(\alpha)}$, $\mu_{km}^{(\alpha)}$ are constants and

$$\overset{(k)}{F}_1=\overset{(k)}{F}_2=\overset{(k)}{F}_3=0.$$

Below we use the equality:

$$\frac{1}{\Lambda}\frac{\partial}{\partial z}\Lambda\frac{\partial}{\partial \bar{z}}\frac{1}{\Lambda}\frac{\partial}{\partial \bar{z}}(\cdot)=\frac{1}{4R^2}(\nabla^2+2)(\cdot),$$

where ∇^2 is Laplace's operator on the sphere with unit radius:

$$\nabla^2=\frac{1}{\Lambda_0^2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=\frac{4}{\Lambda_0^2}\frac{\partial^2}{\partial z\partial \bar{z}}.$$

The solution of equations (17-18) we find in the following form:

$$\overset{m}{u_1}+i\overset{m}{u_2}=R^2\frac{\partial}{\partial \bar{z}}(W_{k+1}+i\Omega_{k+1}),\ (k=0,1,...,N)$$

where W_{k+1}, Ω_{k+1} are arbitrary real functions.

Let us introduce the definitions:

$$\overset{k}{u}_3 = hW_{N+2+k}, \ (k = 0, 1, ..., N).$$

Using these representations we finally have such a system:

$$\sum_{j=1}^{N+1} D_{kj} \nabla^2 W_j - \sum_{j=1}^{N+1} L_{kj} W_j = 0, \ (k = 1, 2, ..., 2N + 2)$$

$$\sum_{j=1}^{N+1} d_{kj} \nabla^2 \Omega_j - \sum_{j=1}^{N+1} l_{kj} \Omega_j = 0, \ (k = 1, 2, ..., 2N + 1),$$

or in the matrix form:

$$D\nabla^2 W - LW = 0, \tag{19}$$

$$d\nabla^2 \Omega - l\Omega = 0, \tag{20}$$

where W, Ω are columns vectors. we can reduce matrix equations (19-20) to the form:

$$\nabla^2 W - AW = 0, \ \ A = D^{-1}L,$$

$$\nabla^2 \Omega - B\Omega = 0, \ \ B = d^{-1}l.$$

We remark that for finding the general solution of this system it is possible to apply Vekua's method [17]. If we assume that matrices A and B have simple eigenvalue numbers and vectors according to[14]: $\alpha_1, ..., \alpha_{2N+2}$, $\beta_1, ..., \beta_{2N+1}$, $X^{(1)}, ..., X^{(2N+2)}$, $Y^{(1)}, ..., Y^{(N+1)}$ respectively, then general solutions of these equations have the form:

$$W = \sum_{m=1}^{2N+2} X^{(m)} \psi_m, \ \ \Omega = \sum_{m=1}^{N+1} Y^{(m)} \chi_m,$$

where ψ_m, χ_m are arbitrary solutions of the following scalar equations:

$$\nabla^2 \psi_m - \alpha_m \psi_m = 0, \ \ (m = 1, ..., 2N + 2),$$

$$\nabla^2 \chi_m - \beta_m \chi_m = 0, \ \ (m = 1, ..., N + 1).$$

Finally for $\overset{k}{u}_j$ we have:

$$\overset{k}{u}_1 + i\overset{k}{u}_2 = R^2 \frac{\partial}{\partial \bar{z}} \left(\sum_{m=1}^{2N+2} X_{k+1}^{(m)} \psi_m + i \sum_{m=1}^{N+1} Y_{k+1}^{(m)} \chi_m \right),$$

$$\overset{k}{u}_3 = h \sum_{m=1}^{N+2+k} X_{k+1}^{(m)} \psi_m, \ \ (k = 0, 1, ..., N).$$

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