ON THE 2-D NONLINEAR SYSTEMS OF EQUATIONS FOR NON-SHALLOW SHELLS (E. REISSNER, D. NAGHDI, W. KOITER, A. LURIE, I. VEKUA)

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Abstract

I. Vekua constructed several versions of the refined linear theory of thin and shallow shells, containing, the regular processes by means of the method of reduction of 3-D problems of elasticity to 2-D ones. In the present paper, by means of Vekua's method, the system of differential equations for the Geometrically nonlinear theory non-shallow shells is obtained.

Key words and phrases: Non-shallow shells, nonlinear theory.
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1. A complete system of equilibrium equation and the stress-strain relation of the 3-D geometrically nonlinear theory can be written in the vector form

$$\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}\boldsymbol{\sigma}^i}{\partial x^i}+\boldsymbol{\Phi}=0,\quad (i=1,2,3) \tag{1}$$

where g is the discriminant of the metric quadratic form of the 3-D domain, $\boldsymbol{\sigma}^i$ are the contravariant constituents of the stress vector, $\boldsymbol{\Phi}$ is an external force.

The stress-strain relation for the geometrically nonlinear theory of elasticity has the form

$$\boldsymbol{\sigma}^i=\sigma^{ij}(\boldsymbol{R}_j+\partial_j\boldsymbol{U})=E^{ijpq}e_{pq}(\boldsymbol{R}_j+\partial_j\boldsymbol{U}), \tag{2}$$

σ^{ij} are contravariant components of the stress tensor, e_{pq} are covariant components of the strain tensor, $\boldsymbol{U}$ is the displacement vector, E^{ijpq} and e_{ij} are defined by the formulas:

$$\begin{aligned}
&E^{ijpq}=\lambda g^{ij}g^{\mu q}+\mu(g^{ip}g^{jq}+g^{iq}g^{jp}),\\
&e_{ij}=\frac{1}{2}(\boldsymbol{R}_i\partial_j\boldsymbol{U}+\boldsymbol{R}_j\partial_i\boldsymbol{U}+\partial_i\boldsymbol{U}\partial_j\boldsymbol{U}),\\
&g^{ij}=\boldsymbol{R}^i\boldsymbol{R}^j,\quad (i,j,p,q=1,2,3).
\end{aligned} \tag{3}$$

2. To construct the theory of shells, we use more convenient coordinate system which is normally connected with the midsurface S. This means that the radius-vector $\boldsymbol{R}$ of any point of the domain Ω can be represented in the form

$$\boldsymbol{R}(x^1,x^2,x^3)=\boldsymbol{r}(x^1,x^2)+x^3\boldsymbol{n}(x^1,x^2)\ \ (x^3=x_3),$$

where $\boldsymbol{r}$ and $\boldsymbol{n}$ are respectively the radius-vector and the unit vector of the normal of the surface $S(x^3=0)$ and (x^1,x^2) are the Gaussian parameters of the midsurfaces S.

The covariant and contravariant basis vectors $\boldsymbol{R_i}$ and $\boldsymbol{R}^i$ of the surfaces $\hat{S}(x^3=\text{const})$ and the corresponding basis vectors $\boldsymbol{r}_i$ and $\boldsymbol{r}^{\,i}$ of the midsurface $S(x^3=0)$ are connected by the following relations:

$$\boldsymbol{R}_i=A_{i.}^{\cdot j}\boldsymbol{r}_j=A_{ij}\boldsymbol{r}^j,\quad \boldsymbol{R}^i=A_{\cdot j}^{i\cdot}\boldsymbol{r}^j=A^{ij}\boldsymbol{r}_j,\quad (i,j=1,2,3),$$

where

$$\begin{aligned}
&A_{\alpha.}^{\cdot\beta}=a_\alpha^\beta-x_3b_\alpha^\beta,\quad A_{\cdot\beta}^{\alpha\cdot}=\vartheta^{-1}[(1-2Hx_3)a_\beta^\alpha+x_3b_\beta^\alpha],\\
&A_3^i=A_i^3=\delta_3^i\vartheta=1-2Hx_3+Kx_3^2,\\
&\boldsymbol{R}_3=\boldsymbol{R}^3=\boldsymbol{r}_3=\boldsymbol{r}^3=\boldsymbol{n},\quad (\alpha,\beta=1,2).
\end{aligned}\tag{4}$$

H and K are a middle and Gaussian curvature of the midsurface S:

$$2H=b_\alpha^\alpha=b_1^1+b_2^2,\quad K=b_1^1b_2^2-b_2^1b_1^2.$$

The main quadratic forms of the midsurface $S(x_3=0)$ have the forms

$$I=ds^2=a_{\alpha\beta}dx^\alpha dx^\beta,\quad II=b_{\alpha\beta}dx^\alpha dx^\beta,$$

where

$$a_{\alpha\beta}=\boldsymbol{r}_\alpha\boldsymbol{r}_\beta,\ \ b_{\alpha\beta}=-\boldsymbol{n}_\alpha\boldsymbol{r}_\beta,\ \ (\alpha,\beta=1,2)$$

and for surfaces $\hat{S}(x^3=\text{const})$ we have

$$\hat{I}=d\hat{s}^2=g_{\alpha\beta}dx^\alpha dx^\beta,\ \ \hat{II}=\hat{K}_{\hat{S}}ds^2=\hat{b}_{\alpha\beta}dx^\alpha dx^\beta,$$

where

$$\begin{gathered}
g_{\alpha\beta}=a_{\alpha\beta}-2x_3b_{\alpha\beta}+x_3^2(2Hb_{\alpha\beta}-Ka_{\alpha\beta}),\\
\hat{b}_{\alpha\beta}=(1-2Hx_3)b_{\alpha\beta}+x_3Ka_{\alpha\beta}.
\end{gathered}$$

The equation of equilibrium of elastic shell-type bodies (1) can be written as

$$\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}\vartheta\boldsymbol{\sigma}^\alpha}{\partial x^\alpha}+\frac{\partial\vartheta\boldsymbol{\sigma}^3}{\partial x^3}+\vartheta\boldsymbol{\Phi}=0,\quad (a=a_{11}a_{22}-a_{12}^2).\tag{5}$$

where

$$\boldsymbol{\sigma}^i = \sigma^{ij}(\boldsymbol{R}_j + \partial_j \boldsymbol{U}) = E^{ijpq}_{pq})e_{pq}(\boldsymbol{R}_j + \partial_j \boldsymbol{U})] \;\; \Rightarrow$$

$$\boldsymbol{\sigma}^i = A^i_{i_1} A^p_{p_1} M^{i_1 j_1 p_1 q_1}\left[\boldsymbol{r}_{q_1}\partial_p \boldsymbol{U} + \frac{1}{2}A^q_{q_1}\partial_p \boldsymbol{U}\partial_q \boldsymbol{U}\right]\left(\boldsymbol{r}_{j_1} + A^j_{j_1}\partial_j \boldsymbol{U}\right), \quad (6)$$

$$M^{i_1 j_1 p_1 q_1} = \lambda a^{i_1 j_1} a^{p_1 q_1} + \mu(a^{i_1 p_1} a^{j_1 q_1} + a^{i_1 q_1} a^{j_1 p_1}), \; (a^{ij} = \boldsymbol{r}^i \boldsymbol{r}^j).$$

Note that sometimes under non-shallow shells be meant the following approximate equalities

$$\boldsymbol{R}^\alpha \cong (a^\alpha_\beta - x_3 b^\alpha_\beta)\boldsymbol{r}^\beta$$

$$(Reissner, Koiter, Haghdi, Lurie)$$

which are the first approximation of the general case (4).

3. The isometrical system of coordinates in the surface S is of the special interest, since in this system can be obtain basic equations of the theory of shells in a complex form, which in turn, allows one to construct for a rather wide class of problems complex representation of general solutions by means of analytic functions of one variable $z = x' + ix^2$. This circumstance makes it possible to apply the methods developed by N. Muskhelishvili and his disciples by means of the theory of functions of a complex variable and integral equations.

The main quadratic forms in this system of coordinates are of the type

$$\begin{aligned} &I = ds^2 = \Lambda(x^1, x^2)[(dx^1)^2 + (dx^2)^2] = \Lambda(z, \bar{z})dzd\bar{z}, \quad (\Lambda > 0) \\ &II = k_1 ds^2 = b_{\alpha\beta}dx^\alpha dx^\beta = \frac{1}{2}\Lambda[\bar{Q}dz^2 + 2Hdzd\bar{z} + Qd\bar{z}^2], \end{aligned} \quad (7)$$

$$(2Q = b^1_1 - b^2_2 + 2ib^1_2)$$

Introducing the well-known differential operators

$$2\partial_z = \partial_1 - i\partial_2, \;\; 2\partial_{\bar{z}} = \partial_1 - i\partial_2,$$

for the nonlinear theory of non-shallow shells(5) and (6) we obtain the following complex writing for the system of equations of the equilibrium and "Hook's Law":

$$\frac{1}{\Lambda}\frac{\partial}{\partial z} = (\Lambda\bar{\boldsymbol{\sigma}} + \bar{\boldsymbol{r}}) + \frac{\partial}{\partial \bar{z}}(\Lambda\bar{\boldsymbol{\sigma}} + \bar{\boldsymbol{r}}_+) - \Lambda(H\sigma^+_3 + \partial\bar{\sigma}^+) + \partial_3\sigma^3_+ + F_+ = 0$$

$$\frac{1}{\Lambda}\left(\frac{\partial\Lambda\sigma^+_3}{\partial z} + \frac{\partial\Lambda\bar{\sigma}^+_3}{\partial\bar{z}}\right) + H(\sigma'_1 + \sigma^2_2) + Re(\bar{Q}\boldsymbol{\sigma}^+\boldsymbol{r}_+) + \partial_3\sigma^3_3 + F_3 = 0 \;\; (F_3 = Fn)$$

where

$$\boldsymbol{\sigma}^{+}\boldsymbol{r}_{+}=(\boldsymbol{\sigma}^{1}+i\boldsymbol{\sigma}^{2})(\boldsymbol{r}_{1}+i\boldsymbol{r}_{2})=\sigma_{1}^{1}-\sigma_{2}^{2}+i(\sigma_{2}^{1}+\sigma_{1}^{2})$$
$$=\vartheta\{\lambda\theta\mu(\boldsymbol{R}^{+}\partial_{z}\boldsymbol{U}+\bar{\boldsymbol{R}}^{+}\partial_{\bar{z}}-\boldsymbol{U}+\partial^{z}\boldsymbol{U}\partial^{\bar{z}}\boldsymbol{U})(\boldsymbol{R}^{+}+2\partial^{z}\boldsymbol{U})\boldsymbol{r}_{+}$$
$$+\mu[2(\boldsymbol{R}^{+}+\partial^{z}\boldsymbol{U})\partial_{\bar{z}}\boldsymbol{U}(\bar{\boldsymbol{R}}^{+}+2\partial^{z}\boldsymbol{U})\boldsymbol{r}_{+}+(\boldsymbol{R}_{+}\partial_{3}\boldsymbol{U}+2\boldsymbol{n}\partial^{z}\boldsymbol{U}+2\partial^{z}\boldsymbol{U}\partial_{3}\boldsymbol{U})\partial_{3}\boldsymbol{U}]\},$$

for $\bar{\sigma}^{+}\boldsymbol{r}_{+}+,\sigma_{3}^{+},\sigma_{3}^{3}$ we have analogous formulas, where

$$\Theta=\boldsymbol{R}^{+}\partial_{z}\boldsymbol{U}+\bar{\boldsymbol{R}}^{+}\partial_{\bar{z}}\boldsymbol{U}+2\partial_{z}\boldsymbol{U}\partial^{\bar{z}}\boldsymbol{U}+\partial_{3}U_{3}+\frac{1}{2}(\partial_{3}\boldsymbol{U})^{2},$$
$$\boldsymbol{R}^{+}=\boldsymbol{R}^{1}+i\boldsymbol{R}^{2},\ \ \partial^{z}\boldsymbol{U}=\frac{1}{2}[(\boldsymbol{R}^{+}\bar{\boldsymbol{R}}^{+})\partial_{z}\boldsymbol{U}_{+}+(\bar{\boldsymbol{R}}^{+}\boldsymbol{R}^{+})_{\partial_{\bar{z}}}\boldsymbol{U}],$$
$$\boldsymbol{R}^{+}=\vartheta^{-1}[(1-Hx_{3})\boldsymbol{r}^{+}+x_{3}Q\bar{\boldsymbol{r}}^{+}],\ \ \boldsymbol{r}^{+}=\boldsymbol{r}^{1}+i\boldsymbol{r}^{2},\ \ \boldsymbol{r}_{+}=\boldsymbol{r}_{1}+i\boldsymbol{r}_{2},$$

Further

$$\boldsymbol{R}^{+}\boldsymbol{R}^{+}=\frac{4x_{3}}{\Lambda}\frac{1-Hx_{3}}{\vartheta^{2}}Q,\ \ \boldsymbol{R}^{+}\bar{\boldsymbol{R}}^{+}=\frac{2}{\Lambda}\frac{\vartheta+2x_{3}^{2}Q}{\vartheta^{2}},$$
$$\boldsymbol{R}^{+}=\boldsymbol{r}_{+}+2\frac{Q}{\vartheta}x_{3},\ \ \bar{\boldsymbol{R}}^{+}\boldsymbol{r}_{+}=2\frac{1-Hx_{3}}{\vartheta},\ \ \boldsymbol{r}^{+}\bar{\boldsymbol{r}}^{+}=\frac{2}{\Lambda},\ \ \boldsymbol{r}^{+}\bar{\boldsymbol{r}}_{+}=2,$$
$$\boldsymbol{r}^{+}\partial_{z}\boldsymbol{U}=\frac{1}{\Lambda}\partial_{z}\boldsymbol{U}_{+}-Hu_{3},\ \ \boldsymbol{r}^{+}\partial_{z}\boldsymbol{U}=\partial_{z}u-Qu_{3},$$
$$\boldsymbol{n}\partial_{z}\boldsymbol{U}=\partial_{z}u_{3}+\frac{\bar{Q}u_{+}+H\bar{u}_{+}}{2},$$

The displacement vector $\boldsymbol{U}$ representable in the form

$$\boldsymbol{U}=u^{\alpha}\boldsymbol{r}_{\gamma}+U^{3}\boldsymbol{n}=\frac{1}{2}(\boldsymbol{U}^{+}\bar{\boldsymbol{r}}_{+}+\bar{\boldsymbol{U}}^{+}\boldsymbol{r}_{+})+U_{3}\boldsymbol{n}=I_{m}\left[(U_{(e)}+iU_{(s)})\frac{dz}{ds}\boldsymbol{r}\right]+U_{3}\boldsymbol{n}$$

where

$$u_{+}=u_{1}+iu_{2}=\boldsymbol{U}\boldsymbol{r}_{+},\ \ U^{+}=Ur^{+},\ \ u_{(e)}=\boldsymbol{U}\boldsymbol{e},\ \ \boldsymbol{U}_{(s)}=\boldsymbol{U}_{s},\ \ l\times s=n$$

4. I. Vekua's method reduction. There are many different methods of reducing 3D problems of the theory of elasticity to 2D ones of the theory shells. Since the system of Legendre polynomials $P-m(\frac{x_3}{h})$ is complete in the interval $[-h,h]$ for equation (5) we obtain the equivalent infinite system of 2D equations

$$\int_{-h}^{h}\left[\frac{1}{\sqrt{a}}\frac{\partial\sqrt{a}\theta\boldsymbol{\sigma}^{\alpha}}{\partial x^{\alpha}}+\frac{\partial\theta\boldsymbol{\sigma}^{3}}{\partial x^{3}}+\theta\boldsymbol{\Phi}\right]P_{m}\left(\frac{x_{3}}{h}\right)dx_{3}=0,$$

$$(\boldsymbol{\sigma}^i, \boldsymbol{U}, \boldsymbol{\Phi}) = \sum_{m=0}^{\infty} \overset{(m)}{\sigma^i} \left(\overset{(m)}{U} \overset{(m)}{\Phi} \right) P_m \left(\frac{x_3}{h} \right),$$

or in the complex for m, for approximation of order N we have

$$\begin{aligned}
&\frac{h}{\Lambda}\frac{\partial}{\partial z}\left(\overset{(m)}{\boldsymbol{\sigma}^{+}\boldsymbol{r}_{+}}\right) + h\frac{\partial}{\partial \bar z}\left(\overset{(m)}{\bar{\boldsymbol{\sigma}}^{+}\boldsymbol{r}_{+}}\right) - \varepsilon\left(H\overset{(m)}{\boldsymbol{\sigma}_{\mathbf{3}}^{+}} + Q\overset{(m)}{\bar{\boldsymbol{\sigma}}_{\mathbf{3}}^{+}}\right) R \\
&-(2m+1)\left(\overset{(m-1)}{\boldsymbol{\sigma}_{+}^{\mathbf{3}}} + \overset{(m+3)}{\boldsymbol{\sigma}_{+}^{\mathbf{3}}} + \dots\right) + h\overset{(m)}{F_1} = 0 \\
&\frac{h}{\Lambda}\left(\frac{\partial \overset{(m)}{\sigma_{+}^{3}}}{\partial z} + \frac{\partial \overset{(m)}{\bar\sigma_{+}^{3}}}{\partial \bar z}\right) + \varepsilon\{H\overset{(m)}{\sigma_{\alpha}^{\alpha}} + Re[\bar Q \overset{(m)}{(\boldsymbol{\sigma}^{+}\boldsymbol{r}_{+})}]\}R \\
&-(2m+1)\left(\overset{(m-1)}{\boldsymbol{\sigma}_{+}^{\mathbf{3}}} + \overset{(m+3)}{\boldsymbol{\sigma}_{+}^{\mathbf{3}}} + \dots\right) + h\overset{(m)}{F_3} = 0 \\
&(m = 0.1.2.\cdots, N)
\end{aligned} \tag{8}$$

where (now we write only linear part in explicit form)

$$\begin{aligned}
&h\sigma^{+}r_{+} = 4\mu\Lambda\left(h\partial_z\overset{(m)}{U^{+}} - \varepsilon Q\overset{(s)}{U_3}R\right) \\
&+2\lambda\sum_{s=0}^{N}\left(\overset{(m,s)}{I_1} - H\overset{(m,s)}{I_2}\right)Q\Big[t(\lambda+\mu)\left(h\overset{(s)}{\theta} - H\varepsilon\overset{(s)}{U_3}R\right)\Big] \\
&+2\mu\left(\tfrac{h}{\Lambda}\partial_{\bar z}\overset{(s)}{U_\alpha} - \varepsilon H\overset{(s)}{U_3}R\right) + \overset{(m,s)}{I_2}Q\Big[\left(h\partial_z\overset{(s)}{\bar U^{+}} - \varepsilon\bar\partial\overset{(s)}{U_3}R\right)(\lambda+\mu) \\
&+(\lambda+3\mu)\bar Q\left(h\partial_z\overset{(s)}{\bar U^{+}} - \varepsilon Q\overset{(s)}{U_3}R\right)\Big] + 2\lambda\overset{(m,s)}{I_3}Q\overset{(s)}{U_3} + \overset{(s)}{L_1}\overset{(s)}{U3}\}.
\end{aligned} \tag{9}$$

For $\left(\overset{(m)}{\bar{\boldsymbol{\sigma}}^{+}\boldsymbol{r}_{+}}\right), \overset{(m)}{\boldsymbol{\sigma}_3^{+}} = \left(\overset{(m)}{\boldsymbol{\sigma}_3}, \boldsymbol{n}\right), \overset{(m)}{\sigma_{+}^{3}} = \left(\overset{(m)}{\boldsymbol{\sigma}_{+}}\boldsymbol{n}\right), \overset{(m)}{\sigma_{+}^{3}} = \overset{(m)}{\boldsymbol{\sigma}_3}\boldsymbol{n}$ we have analogous formulas, where $\overset{(m)}{\boldsymbol{L_i}}(\overset{s}{U})$ $(i = 1, \cdots, s)$ are the nonlinear parts of relations (4.2)

Then we have

$$\overset{(m,s)}{Q} = \frac{1}{\lambda}(\partial_z\overset{(m)}{U_{+}} + \partial_{\bar z}\overset{(m)}{\bar U_{+}}) + \overset{(m)}{U_3'}, \quad \overset{(m)}{U_i'} = (2m+1)(\overset{(m+1)}{U_i} + \overset{(m+3)}{U_i} + \cdots),$$

$$
\begin{aligned}
\overset{(m,s)}{I_1} &= \frac{2m+1}{2h}\int_{-h}^{h}\frac{x_3 p_m p_s dx_3}{1-2Hx_3+kx_3^2},\\
\overset{(m,s)}{I_2} &= \frac{2m+1}{2h}\int_{-h}^{h}\frac{x_3^2 p_m p_s dx_3}{1-2Hx_3+kx_3^2},\\
\overset{(m,s)}{I_3} &= \frac{2m+1}{2h}\int_{-h}^{h} x_3 p_m p_s dx_3,\\
\overset{(m,s)}{I_4} &= \frac{2m+1}{2h}\int_{-h}^{h}(1-2Hx_3+kx_3^2) p_m p_s dx_3,
\end{aligned}
\tag{10}
$$

The above integrals can be calculated explicitly and their expressions with regard to ξ have the form, for example

$$
\overset{(m,s)}{I_3} = \frac{2m+1}{2\sqrt{E}}\sum_{r=0}^{[\frac{m}{2}]}\sum_{p=0}^{\infty}\overset{(m,s)}{M_r p}\,\varepsilon^{s-m+2(p+q)}\{[(H+\sqrt{E})R]^{s-m+2(p+q)}
$$

$$
-[(H-\sqrt{E})R]^{s-m+2(p+q)}]\},
$$

where

$$
\overset{(m,s)}{M_{rp}} = 2^{s-m}\frac{(-1)^r(2m+2r)!(s+p)!(s+2p)}{r!(m-r)(m-2r)!p!(2s+2p+1)!},\quad (E=H^2-k\geq 0)
$$

and ε is a small parameter is expressed in the form

$$
\varepsilon = \frac{h}{R} \leq q < 1
$$

Here h is the semi thickness of the shell and R is a certain characteristic radius of curvature of curvature of the midsurface S.

Now we assume the validity of the expansions for approximation of order N:

$$
\left(\overset{(m)}{\boldsymbol{\sigma}^i},\overset{(m)}{\boldsymbol{U}},\overset{(n)}{\boldsymbol{F}}\right) = \sum_{n=1}^{\infty}\left(\overset{(m,n)}{\boldsymbol{\sigma}^i},\overset{(m,n)}{\boldsymbol{U}},\overset{(m,n)}{\boldsymbol{F}}\right)\varepsilon^n,\quad (m=0,1,\cdots,N)
$$

Substituting the above expansion into the (4.1) and (4.2), then equalizing the coefficients of expansion for $\varepsilon\varepsilon^n$ we obtain the following 2D infinite system of equilibrium equations with respect to components of displacement vector in the isometric coordinates:

$$
\begin{gathered}
4\mu\partial_z(\lambda^{-1}\partial_z \overset{(m,n)}{U_+} +2(\lambda+\mu)\partial_{\bar z} \overset{(m,n)}{\theta} +\frac{2\lambda}{h}\partial_{\bar z} \overset{(m,n)}{U_3'} \\
-\frac{2m+1}{h}\mu\Big[2\lambda_{\bar z}\left(\overset{(m-1,n)}{U} +\overset{(m-3,n)}{U_+}\cdots,\right) \\
+\left(\overset{(m-1,n)}{U} +\overset{(m-3,n)}{U_+}\cdots,\right)\Big] + \overset{(m,n)}{F_+} =0,
\end{gathered} \tag{11}
$$

$$
\mu\left(\nabla^2 \overset{(m,n)}{U_3} + \overset{(m,n)}{\theta'}\right) - \frac{2m+1}{h}\Big[\lambda\left(\overset{(m-1,n)}{\theta'} + \overset{(m+3,n)}{\theta'} +\cdots,\right)
$$

$$
+(\lambda+2\mu)\left(\overset{(m-1,n)}{U_3} + \overset{(m-3,n)}{U_3} +\cdots,\right)\Big] + \overset{(m,n)}{F_3} =0,
$$

where (below it will be omit the upper index n)

$$
\overset{(m)}{U_+} = \overset{(m)}{U_1} +i\overset{(m)}{U_2}, \quad \overset{(m)}{\Theta} = \Lambda^{-1}\left(\partial_z \overset{(m)}{U_+} +\partial_{\bar z}\overset{(m)}{\bar U_+}\right),
$$

$$
\overset{(m)}{U_+'} = \frac{2m+1}{h}\left(\overset{(m+1)}{U_i} + \overset{(m+3)}{U_i} +\cdots\right), \quad \nabla^2 = \frac{4}{\Lambda}\frac{\partial^2}{\partial z\partial\bar z}
$$

The beharmonic solution of the homogeneous system (4.4) we can find the form

$$
\overset{(m)}{U_+} = \partial_{\bar z}\left(\overset{(m)}{V_1} +i\overset{(m)}{V_2}\right) + \left(\frac{1}{\pi}\iint\limits_S \frac{\overline{\varphi_0'(\xi)} - k_1\varphi_0'(\xi)}{\bar\xi-\bar z}ds_s - \overline{\psi_0'\xi}\right)\overset{m}{\delta_0} +k_2\overline{\varphi_0''z}\,\overset{m}{\delta_2}
$$

$$
-\frac{1}{\pi}\left(\iint\limits_S \frac{\varphi_1'(\xi)+\bar\varphi_1'(\xi)}{\bar\xi-\bar z}ds_\xi + \eta_1\overline{\varphi_1''}(z) - 2\overline{\varphi_1'(z)}\right)\sigma_1^m + \eta_2\overline{\varphi_1''}(z)\sigma_3^m, \tag{12}
$$

$$
\overset{(m)}{U_3} = \overset{(m)}{U_3} - \left\{\frac{1}{\pi}\iint\limits_S [\varphi_1'(\xi)+\varphi_1'(\xi)]\ln|\xi-z|ds_\xi - (\psi_1(z)+\overline{\psi_1(z)})\}\right\}\sigma_0^m
$$

$$
-\frac{3}{2}k_2[(\varphi_0'(z)+\overline{\varphi_0'(z)})]\sigma_1^m + (\varphi_1'(z)+\overline{\varphi_1'(z)}\sigma_2^m)],
$$

$$
(m=0,1,\cdots,N)
$$

$$\overset{0}{V}_1+\overset{0}{V}_2=0,\quad \overset{0}{U}_3=\psi_1(z)+\overline{\psi_1(z)},\ \ if\ \ N=0$$

where $V_i(i=1.2.3)$ are unknown metaharmonic functions, $\varphi_0,\varphi_0,\psi_0,\psi_1$ are analytic functions of Z,δ_i^j - Kroneeker delta, $ds_\xi=\Lambda(\xi,\bar{\xi})d\xi d\eta,\xi=\xi+i\eta$, then

$$k_1=\begin{cases}\dfrac{\lambda+3\mu}{\lambda+\mu},\ N=0\\ \dfrac{5\lambda+6\mu}{3\lambda+2\mu},\ N\neq 0,\end{cases}\quad \eta_1=\begin{cases}\dfrac{\lambda+\mu}{\mu},\ N=1\\ 4\dfrac{\lambda+\mu}{\lambda+2\mu},\ N=2,\\ \dfrac{23\lambda+24\mu}{5(\lambda+2\mu)},\ N=3.\end{cases}\ \nu=\begin{cases}k_2=\dfrac{4}{3}\dfrac{\lambda}{3\lambda+2\mu},\\ \eta_2=\dfrac{4}{15}\dfrac{3\lambda+4\mu}{\lambda+2\mu}.\end{cases}$$

Note that for approximation of order $N=0$, when $\lambda(z,\bar{z})=1$, the expression for $\overset{(0)}{U}_+$, coincides with well-known representation of Kolosov-Muskhelishvili for plane deformation (see[1])

$$\overset{(0)}{U}_+=U_+=\frac{\lambda+3\mu}{\lambda+\mu}\varphi(z)-z\overline{\varphi'(z)}-\overline{\varphi(z)}.$$

Further

Case $N=1$

$$\overset{(0)}{U}_+=-\frac{\lambda h}{6(\lambda+\mu)}\partial_{\bar{z}}\omega+\frac{1}{\pi}\iint\limits_S\frac{\overline{\varphi'(\xi)}-\kappa'\varphi'(\xi)}{\bar{\xi}-\bar{z}}ds_\xi-\overline{\psi(z)},$$

$$\overset{(1)}{U}_3=i\partial_z\chi+\frac{1}{\pi}\iint\limits_S\frac{\overline{\varphi'(\xi)}-\varphi'(\xi)}{\bar{\xi}-\bar{z}}ds_\xi+\frac{2(\lambda+2\mu)}{3\mu}\overline{\varphi'(\xi)}-2h\overline{\varphi'(z)}$$

$$\overset{(0)}{U}_3=\psi(z)+\overline{\psi(z)}-\frac{1}{\pi}\iint\limits_S[\phi'(\xi)+\overline{\phi'(\xi)}]\ln|\xi-z|ds_\xi,$$

$$\overset{(1)}{U}_3=\omega(z,\bar{z})+\frac{2\lambda h}{3\lambda+2\mu}[\varphi'(z)+\overline{\varphi'(z)}],$$

where

$$\nabla^2\chi-\frac{3}{h^2}\chi=0,\quad \nabla^2\omega-\frac{3(\lambda+\mu}{(\lambda+2\mu)h}\omega=0,$$

Case $N=2$

$$\overset{(0)}{U}_+=\frac{1}{\pi}\iint\limits_S\frac{\overline{\varphi'(\xi)}-\kappa\varphi'(\xi)}{\bar{\xi}-\bar{z}}dS_\xi-\overline{\psi(z)}-\frac{2\lambda}{\lambda+2\mu}\sum_{k=1}^{2}\frac{1}{\alpha_k}\partial_{\bar{z}}V_k,$$

$$\overset{(1)}{U}_+ = i\partial_{\bar{z}}\chi + \frac{4}{3}\frac{\lambda+\mu}{\mu}\overline{\varphi"(z)} - \frac{1}{\pi}\iint\limits_S \frac{\varphi'(\xi)+\overline{\varphi'(\xi)}}{\bar{\xi}-\bar{z}}ds_\xi - 2h\overline{\varphi"(z)} - \frac{\lambda}{10(\lambda+\mu)}\partial_{\bar{z}}\omega,$$

$$\overset{(2)}{U}_+ = \frac{2}{3}\left(i\partial_{\bar{z}}\omega + \sum_{k=1}^{8}\frac{\alpha_3-k}{\alpha_k}\partial_{\bar{z}}V_k + \frac{2\lambda}{3\lambda+2\mu}\overline{\varphi"(z)}\right),$$

$$\overset{(0)}{U}_3 = \Psi(z) + \overline{\Psi(z)} - \frac{1}{\pi}\iint\limits_S \overline{\phi'(\xi)} + \phi(\xi)\ln|(\xi)-z|ds_\xi + \frac{\lambda}{2(\lambda+\mu)}\omega$$

$$\overset{(1)}{U}_3 = V_1 + V_2 - \frac{2\lambda}{3\lambda+2\mu}(\varphi'(z)\overline{\varphi'(z)}),$$

$$\overset{(2)}{U}_3 = w - \frac{2\lambda}{3\lambda+2\mu}(\varphi'(z)\overline{\varphi'(z)}),$$

where

$$\nabla^2 V_k = \alpha_k V_k, \quad \alpha_k^2 - \frac{12(\lambda+\mu)}{\lambda+2\mu}\alpha_k + \frac{180\mu(\lambda+\mu)}{(\lambda+2\mu)^2} = 0, \quad (k=1,2)$$

$$\nabla^2 w = \frac{60(\lambda+\mu)}{\lambda+2\mu}w, \quad \nabla^2\chi = 3\chi, \quad \nabla^2\omega = 15\omega.$$

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