EXACT L_1-CONSERVATIVE FINITE-DIFFERENCE SCHEME FOR THE NEUMANN PROBLEM FOR THE HEAT CONDUCTION EQUATION

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Abstract

This article is devoted to construction of exact L_1-conservative finite-difference schemes for the Neumann problem for the multidimensional heat equation with a quasilinear heat conductivity. The aim of the paper is to construct numerical algorithms which exactly satisfy the law of heat conservation for the differential problem. We use Steklov averaging of the source term and boundary conditions and special approximation of initial data based on simple quadrature formulas. The proposed finite-difference schemes have the usual order of approximation. We show how to implement this algorithms with the use of iteration methods which also exactly satisfy the law of heat conservation for the differential problem on each iteration. We present the exact L_1-conservative modification of the flow scheme for the heat conduction equation.

Key words and phrases: Integral law of heat conservation for quasilinear equation, Neumann problem, exact L_1-conservative finite-difference scheme.

AMS subject classification: 65 M 06, 35 K 20, 35 K 59.

1 Introduction

The success of mathematical modelling often depends on how well the important properties of the original differential model are reflected in the numerical approximation. Difference numerical approximations for which discrete analogues of conservation laws are satisfied are known as conservative (L_1-conservative) (e.g. [3]). Conservative numerical schemes play a fundamental role in applications, and their development for quasilinear problems constitutes an important task in applied numerical mathematics. Such schemes preserve fundamental properties of the continuous model at the discrete level and allow us to carry out computations on coarse grids with minimal computational expenses.

A turning point in this development was the paper [8] where it was shown that nonconservative finite-difference schemes (FDS) diverge in the case of discontinuous coefficients and conservative FDS were constructed for a class of problems with discontinuous coefficients. A natural extension of that early work was the development of the concept of full conservativeness. The full conservativeness requires not only discrete analogues of conservation laws to be fulfilled, but also additional relationships that express a balance of different types of energy [8].

Conservative FDS can be constructed for quasilinear parabolic equations with Neumann boundary conditions. Suppose, a conservative FDS is constructed by some method. The mesh function (or several functions) obtained satisfies the FDS with a certain accuracy. Consequently, even if the FDS is conservative, the solution obtained may not exactly satisfy the conservation laws. The question of how much the magnitude of the resulting energy imbalances depends on the type of FDS, on the accuracy of the iteration process used to solve difference system of equations therefore becomes important. In particular, it is important to know how to assess the imbalance in multidimensional problems where highly accurate calculations are impossible [6]. A way out may be to construct FDS which exactly satisfy the law of heat conservation for the differential problem (exact L_1-conservative algorithms) and iteration processes for which an imbalance does not occur for any iteration accuracy.

The aim of the paper is to construct numerical algorithms which exactly satisfy the law of heat conservation for the differential problem. Let us note that energy imbalances also occur when one uses iteration methods for solving linear system of the iteration process. A flow scheme [6] for the equation of heat conduction for which this problem doesn't exist satisfies the law of heat conservation for the differential problem with the first order in time. We give a modification of this algorithm which exactly satisfies this law for any iteration process for solving system of linear equations of the iteration process.

The article is organized as follows. In Section 2 we give problem statement and construct exact L_1-conservative finite-difference scheme for 1D problem. Section 3 is devoted to the implementation of the iteration process for the solution of the difference problem which also exactly satisfies the law of heat conservation for the differential problem on each iteration. In the next Section 4 we show that M. Shashkov's flow difference scheme [6] satisfies the law of heat conservation for the differential problem with the first order in time and in Section 5 we give a modification of this algorithm which exactly satisfies the conservation law. The last Section 6 is devoted to generalization of the proposed modified flow FDS to multidimensional problems.

2 Exact L_1-conservative finite-difference scheme

In the domain

$$\bar{\Omega}_T = \{(x,t) : 0 \leq x \leq l, 0 \leq t \leq T\},$$

consider the Neumann initial-boundary value problem (IBVP) for the quasi-linear parabolic equation:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(k\frac{\partial u}{\partial x}\Big)+f(x,t), \quad (x,t) \in \Omega_T, \tag{1}$$

$$u(x,0) = u_0(x), \quad x \in [0,l], \tag{2}$$

$$-k\frac{\partial u}{\partial x}\Big|_{x=0} = \varphi(t), \quad k\frac{\partial u}{\partial x}\Big|_{x=l} = \psi(t), \quad t \in (0,T], \tag{3}$$

where

$$0 < k_1 \leq k(x,t,u) \leq k_2, \quad k_1, k_2 = const.$$

The quantity of heat Q contained in the system at time t is determined by the relation

$$Q(t) = \int\limits_0^l u(x,t)dx. \tag{4}$$

Integrating (1) by x in the interval $[0,l]$ from equation (4) and boundary conditions (3) we obtain the ordinary differential equation (ODE)

$$\frac{dQ}{dt} = \psi(t) + \varphi(t) + \int\limits_0^l f(x,t)dx, \tag{5}$$

which expresses the differential law of conservation of heat. Finally integrating (5) by t in the interval $[0,t]$ we get

$$Q(t) = Q(0) + \int\limits_0^t \left(\psi(t) + \varphi(t) + \int\limits_0^l f(x,t)dx\right) dt, \tag{6}$$

which expresses the integral law of conservation of heat. Let us introduce the next grids:

$$\bar{\omega}_\tau = \{t_{n+1} = t_n + \tau_n, \quad \tau_n > 0, \quad n = \overline{0, N_T - 1}, \quad t_0 = 0, \quad t_{N_T} = T\},$$

$$\omega_\tau = \bar{\omega}_\tau \backslash \{t_{N_T} = T\}, \quad \omega_h = \{x_i = ih, \quad h = l/N, \quad i = \overline{1, N-1}\},$$
$$\bar{\omega}_h = \omega_h \cup \{0, l\},$$

and Steklov averaging operators [4]:

$$S_x p(x) = \frac{1}{h} \int\limits_x^{x+h} p(\xi) d\xi, \quad S_t^n q(t) = \frac{1}{\tau_n} \int\limits_t^{t+\tau_n} q(\eta) d\eta,$$

$$S_{tx}^n s(x,t) = S_t^n S_x s(x,t), \quad x \in \omega_h, \quad t \in \omega_\tau.$$

For approximation of problem (1)–(3) consider the next FDS with Steklov averaging of the source term and boundary conditions and special approximation of the initial data

$$y_{t,i}^n = (a_i^{n,(\sigma)} y_{\bar{x},i}^{n+1})_x + (S_{tx}^n f)_i^n, \quad (x,t) \in \omega, \quad \omega = \omega_h \times \omega_\tau, \tag{7}$$

$$y(x,0) = u_0(x) \frac{\int\limits_0^l u_0(\xi) d\xi}{\frac{h}{2} u_{0,0} + \sum\limits_{i=1}^{N-1} h u_{0,i} + \frac{h}{2} u_{0,N}}, \quad x \in \bar{\omega}_h. \tag{8}$$

$$\begin{aligned} &- a_1^{n,(\sigma)} y_{x,0}^{n+1} + \frac{h}{2} y_{t,0} = (S_t^n \varphi)(t_n), \quad t \in \omega_\tau, \\ &a_N^{n,(\sigma)} y_{\bar{x},N}^{n+1} + \frac{h}{2} y_{t,N} = (S_t^n \psi)(t_n) \quad t \in \omega_\tau. \end{aligned} \tag{9}$$

Here and below we use standard notation of the theory of FDS [3, 4]

$$y_i^n = y(x_i, t_n), \quad y_{t,i}^n = \frac{y_i^{n+1} - y_i^n}{\tau_n}, \quad y_{\bar{x},i}^n = \frac{y_i^n - y_{i-1}^n}{h}, \quad y_{x,i}^n = \frac{y_{i+1}^n - y_i^n}{h},$$

$$a_i^n = a(x_i, t^n, y(x_i, t_n)) = 0.5(k(x_{i-1}, t_n, y(x_{i-1}, t_n)) + k(x_i, t_n, y(x_i, t_n))),$$
$$a_i^{n,(\sigma)} = \sigma a_i^{n+1} + (1-\sigma) a_i^n, \quad \sigma \in [0,1].$$

It's easy to check that on smooth solutions this FDS has the order of accuracy $O(h^2 + \tau^*)$, $\tau^* = \max\limits_{0 \le n \le N_T - 1} \tau_n$ [3]. Let us introduce the grid quantity of heat

$$Q_h^n = \frac{h}{2} y_0^n + \sum_{i=1}^{N-1} h y_i^n + \frac{h}{2} y_N^n,$$

which approximates the quantity of heat $Q(t_n)$ with the order of $O(h^2)$.

Theorem 2.1. *FDS (7)–(9) is exact L_1-conservative, i.e.*

$$Q_h(t_n) \equiv Q(t_n), \quad n = \overline{0, N_T}. \tag{10}$$

Proof. According to initial condition (8) we immediately obtain

$$Q_h(0) = Q(0).$$

Multiplying the difference equation (7) by h and summing the result over the internal nodes ω_h, we obtain

$$(y_{t,n}, 1) = ((a^{n,(\sigma)} y_{\bar{x}}^{n+1})_x, 1) + (S_{tx}^n f, 1). \tag{11}$$

Using difference Green formula [3, 4] and approximation of boundary conditions (9) we get

$$((a^{(\sigma)} \hat{y}_{\bar{x}})_x, 1) = (a_N^{(\sigma)} \hat{y}_{\bar{x},N}) - (a_1^{(\sigma)} \hat{y}_{x,0}) = S_t^n \psi + S_t^n \varphi - \frac{h}{2} y_{t,N} - \frac{h}{2} y_{t,0}.$$

Thus for all n

$$Q_{h,t}^n = S_t^n \left(\psi(t) + \varphi(t) + \int_0^l f(x,t)dx \right). \tag{12}$$

The last formula is the EDS [1] for (6). Thus we obtain that which expresses exact L_1-conservativeness.

3 Exact conservative iteration process

In the case $\sigma = 0$, FDS (7) –(9) is called linearised and its solution is found by explicit formulas (Thomas algorithm [3]). Otherwise, we need to use iteration processes. Our aim is to implement the iteration process for the solution of the difference problem which also exactly satisfies the law of heat conservation for the differential problem on each iteration. For convenience consider the case $\sigma = 1$ and the next iteration process [3]:

$$\frac{\overset{(s+1)}{y_i^{n+1}} - y_i^n}{\tau_n} = (\overset{(s)}{a_i^{n+1}} \overset{(s+1)}{y_i^{n+1}}{}_{\bar{x}})_x + (S_{tx}^n f)_i^n, \quad \overset{(0)}{y_i^{n+1}} = y_i^n, \quad (x,t) \in \omega, \tag{13}$$

with the boundary conditions

$$\begin{aligned} - \overset{(s)}{a_1^{n+1}} \overset{(s+1)}{y_0^{n+1}}{}_x + \frac{h}{2} \frac{\overset{(s+1)}{y_0^{n+1}} - y_0^n}{\tau_n} &= (S_t^n \varphi)^n, \quad t \in \omega_\tau \\ \overset{(s)}{a_N^{n+1}} \overset{(s+1)}{y_N^{n+1}}{}_{\bar{x}} + \frac{h}{2} \frac{\overset{(s+1)}{y_N^{n+1}} - y_N^n}{\tau_n} &= (S_t^n \psi)^n, \quad t \in \omega_\tau. \end{aligned} \tag{14}$$

The important property of this iteration process is the consistent approximation of boundary conditions and flux in the iteration process (13)–(14). Define the iteration quantity of heat as

$$\overset{(s)}{Q}{}_h^{n+1} = \frac{h}{2}\overset{(s)}{y}{}_0^{n+1} + \sum_{i=1}^{N-1} h\overset{(s)}{y}{}_i^{n+1} + \frac{h}{2}\overset{(s)}{y}{}_N^{n+1}.$$

Repeating the proof of Theorem 2.1 one can prove

Theorem 3.1. *The iteration process* (13)–(14) *exactly satisfies the law of heat conservation for the differential problem* (7)–(9) *on each iteration, i.e.*

$$\overset{(s)}{Q}{}_h^{n} \equiv Q(t_n), \quad s = 0, 1, \ldots, \quad n = \overline{0, N_T}.$$

4 Flow finite-difference scheme

The proposed FDS (7)–(9) has the order of accuracy $O(h^2 + \tau^*)$, $\tau^* = \max\limits_{0 \le n \le N_T - 1} \tau_n$, because we use the trapezoid quadrature formula

$$\int_0^l u_0(\xi)d\xi \approx \frac{h}{2}u_{0,0} + \sum_{i=1}^{N-1} hu_{0,i} + \frac{h}{2}u_{0,N}$$

of second order of approximation and also the approximation of boundary conditions (9) has the order of accuracy $O(\tau^* + h^2)$.

Moreover as we have shown the exact solution of the FDS exactly satisfies the L_1-conservation law. Unfortunately, as for the approximate solution of the linear FDS (especially in multidimensional problems) is found by iteration method the conservation law may not be valid. In [6, 7] it is shown that for constant coefficient $k \equiv 1$ the following estimate for the dis-balance for the quantity of heat is obtained for the Seidel iteration method

$$\|\overset{(s)}{Q}{}_h^{n} - Q_h^n\| \le \frac{\varepsilon\tau(n-1)}{h^2}.$$

So one should choose ε from the condition above, which can be very rather strong. Alternatively in the paper mentioned above how to choose iteration process without disbalance are presented. Unfortunately these conditions are valid only for the constant coefficient $k \equiv 1$.

That is why M. Shashkov [7] proposed so called flow scheme for heat equation. Let us first rewrite (1) in the form of the system:

$$\frac{\partial u}{\partial t} = \frac{\partial w}{\partial x} + f(x,t), \quad w = k\frac{\partial u}{\partial x}, \quad (x,t) \in \Omega_T, \tag{15}$$

where w is flux of heat. Boundary conditions (3) take the form

$$-w\Big|_{x=0} = \varphi(t), \quad w\Big|_{x=l} = \psi(t), \quad t \in (0,T]. \tag{16}$$

Shashkov's flow difference scheme approximating (15), (16), (2) for the case $f(x,t) \equiv 0, \quad k \equiv 1$, is

$$\bar{u}^n_{ht,i} = (w_h)^{n+1}_{x,i}, \quad i = \overline{0, N-1}, \quad w^{n+1}_{h,i} = \bar{u}^{n+1}_{h\bar{x},i}, \quad i = \overline{1, N-1}, \tag{17}$$

$$-w^{n+1}_{h,0} = \varphi(t_{n+1}), \quad w^{n+1}_{h,N} = \psi(t_{n+1}), \quad n = \overline{0, N_T - 1}. \tag{18}$$

$$\bar{u}^0_{h,i} = u_0\left(x_{i+\frac{1}{2}}\right), \quad i = \overline{0, N-1}. \tag{19}$$

Here the temperature $\bar{u}_h$ refers to the centers of the cells, and the flow w_h refer to the surrounding nodes. FDS (17)–(19) is also of order $O(\tau^* + h^2)$. Moreover it has no disbalances and for difference quantity of heat defined by

$$Q^{n,*}_h = \sum_{i=0}^{N-1} h\bar{u}^n_{hi}, \tag{20}$$

it satisfies [7] the following difference analogue of (6) for $f(x,t) \equiv 0$:

$$Q^{n,*}_{h,t} = \psi(t_n) + \varphi(t_n).$$

So flux FDS is L_1-conservative.

5 Exact L_1-conservative flow algorithm

As the relation

$$Q^{n,*}_{h,t} = \psi(t_n) + \varphi(t_n).$$

is an explicit Euler method for the differential equation

$$\frac{dQ}{dt} = \psi + \varphi, \tag{21}$$

it means that Shashkov's flow FDS (17)–(19) satisfies the differential conservation law of the first order.

Our aim is to modify this scheme to make it exact L_1-conservative. Consider system (15)–(16) and appropriate difference scheme with Steklov averaging of the source term and boundary conditions and special approximation of initial data.

$$\begin{gathered} \bar{u}^n_{ht,i} = (w_h)^{n+1}_{x,i} + (S^n_{tx} f)^n_i, \quad i = \overline{0, N-1}, \\ w^{n+1}_{h,i} = a^n_i \bar{u}^{n+1}_{h\bar{x},i}, \quad i = \overline{1, N-1}, \end{gathered} \tag{22}$$

$$-w_{h,0}^{n+1} = (S_t^n \varphi)^n, \quad w_{h,N}^{n+1} = (S_t^n \psi)^n, \quad n = \overline{0, N_T - 1}, \tag{23}$$

$$\bar{u}_{h,i}^0 = \bar{u}_{0,i} \frac{\int\limits_0^l u_0(\xi) d\xi}{\sum\limits_{i=0}^{N-1} h \bar{u}_{0,i}}, \quad i = \overline{0, N-1}. \tag{24}$$

Let us note that $Q_h^{n,*}$ approximates $Q(t)$ with the second order as it is midpoint quadrature formula. Analogously to Theorems 2.1, 3.1 we can prove

Theorem 5.1. *FDS* (22)–(24) *is flow exact L_1-conservative, i.e.*

$$Q_h^{n,*} \equiv Q(t_n), \quad n = \overline{0, N_T}. \tag{25}$$

6 Multidimensional generalization

In this Section the exact L_1-conservative flow finite-difference schemes are constructed for the Neumann problem for multidimensional heat equation in an isotropic medium.

We describe the heat state of a solid body, which has the volume

$$\Omega^m = \Omega_1 \times \Omega_2 \times \dots \Omega_m, \quad \Omega_\alpha = \{x_\alpha : 0 \le x_\alpha \le l_\alpha\}, \quad \alpha = \overline{1, m},$$

beginning with the initial time $t = 0$ up to a final time $t = T$, $T > 0$. Let $\Omega_T^m = \{(x, t) \mid x \in \Omega^m, \ 0 < t \le T\}$ and let $\Gamma = \{(x, t) \mid x \in \partial\Omega^m, \ 0 < t \le T\}$ be the lateral surface of Ω.

The propagation of heat in an isotropic medium is described by the parabolic equation [5]

$$\frac{\partial u}{\partial t} = \sum_{\alpha=1}^m \frac{\partial}{\partial x_\alpha} \left(k_\alpha(x, t, u) \frac{\partial u}{\partial x_\alpha} \right) + f(x, t), \quad (x, t) \in \Omega_T, \tag{26}$$

where we assume that

$$0 < k_1 \le k_\alpha(x, t, u) \le k_2, \quad k_1, k_2 = const, \quad \alpha = \overline{1, m}.$$

Let n denote the external with respect to the domain Ω^m, normal to the boundary $\partial\Omega^m$ and $\cos(n, x_\alpha)$, $\alpha = \overline{1, m}$, be the direction cosines of the external normal. Then the flow is specified by the equation

$$\frac{\partial u}{\partial \nu} = \sum_{\alpha=1}^m k_\alpha \frac{\partial u}{\partial x_\alpha} \cos(n, x_\alpha).$$

Equation (26) is complemented by initial and Neumann boundary conditions:

$$\begin{aligned} &u(x,0) = u_0(x), \quad x \in \Omega^m, \\ &\frac{\partial u}{\partial \nu} = \varphi(x,t), \quad (x,t) \in \Gamma. \end{aligned} \tag{27}$$

The quantity of heat Q contained in the system at time t is determined by the relation

$$Q^m(t) = \iint\limits_{\Omega^m} u(x,t) d\Omega^m. \tag{28}$$

Let

$$\Omega^{m,\alpha} = \Omega^m / \Omega_\alpha, \quad \alpha = \overline{1,m},$$

$$\varphi^{\alpha,0} = \varphi(x,t)\,|_{x_\alpha=0}\,, \quad \varphi^{\alpha,l} = \varphi(x,t)\,|_{x_\alpha=l_\alpha}\,, \quad x \in \Omega^{m,\alpha}.$$

Integrating (26) by x in Ω^m from equation (28) and boundary conditions (27) we obtain ODE

$$\frac{dQ^m}{dt} = \sum_{\alpha=1}^{p} \iint\limits_{\Omega^{m,\alpha}} (\varphi^{\alpha,0} + \varphi^{\alpha,l}) d\Omega^{m,\alpha} + \iint\limits_{\Omega^m} f(x,t) d\Omega^m, \tag{29}$$

which expresses the differential law of conservation of heat. Finally integrating (29) by t in the interval $[0,t]$ we get

$$Q^m(t) = Q^m(0) + \int_0^t \left(\sum_{\alpha=1}^{m} \iint\limits_{\Omega^{m,\alpha}} (\varphi^{\alpha,0} + \varphi^{\alpha,l}) d\Omega^{m,\alpha} + \iint\limits_{\Omega^m} f(x,t) d\Omega^m \right) dt, \tag{30}$$

which expresses the integral law of conservation of heat. Let us introduce the next grids

$$\omega^m_{\alpha,h} = \{x_{\alpha,i_\alpha} = i_\alpha h_\alpha, h_\alpha = l_\alpha / N_\alpha, i = \overline{1, N_\alpha - 1}\}, \quad \alpha = \overline{1,m},$$

$$\bar{\omega}^m_{\alpha,h} = \omega^m_{\alpha,h} \cup \{0, l_\alpha\}\,, \quad \omega^{-m}_{\alpha,h} = \omega^m_{\alpha,h} \cup \{0\}\,, \quad \alpha = \overline{1,m},$$

$$\omega^{-m,\alpha}_h = \omega^{-m}_{1,h} \times \ldots \times \omega^{-m}_{\alpha-1,h} \times \omega^m_{\alpha,h} \times \omega^{-m}_{\alpha+1,h} \times \ldots \times \omega^{-m}_{m,h}, \quad \alpha = \overline{1,m},$$

$$\bar{\omega}^m_h = \bar{\omega}^m_{1,h} \times \bar{\omega}^m_{2,h} \times \ldots \times \bar{\omega}^m_{m,h}, \quad \omega^{-m}_h = \omega^{-m}_{1,h} \times \omega^{-m}_{2,h} \times \ldots \times \omega^{-m}_{m,h},$$

$$\bar{\omega}^m = \bar{\omega}^m_h \times \bar{\omega}_\tau, \quad \omega^{-m} = \omega^{-m}_h \times \omega_\tau, \quad \partial\omega^m = \bar{\omega}^m \cap \Gamma.$$

Introduce the following notation:

$$S_{x_\alpha} p(x) = \frac{1}{h_\alpha} \int_{x_\alpha}^{x_\alpha + h_\alpha} p(x) dx_\alpha, \quad \alpha = \overline{1,m}, \quad (x,t) \in \omega^{-m},$$

$$S_{tx}^{m,n}q(x,t) = S_t^n S_{x_1} S_{x_2} \dots S_{x_m} q(x,t), \quad (x,t) \in \omega^{-m}.$$

On the grid $\bar{\omega}^m$ consider the FDS

$$\begin{gathered} \bar{u}_{h,t}^n = \sum_{\alpha=1}^{m} (w_{h,\alpha}^{n+1})_{x_\alpha} + S_{tx}^{m,n} f, \quad (x,t) \in \omega^{-m}, \\ w_{h,\alpha}^{n+1} = a_\alpha \bar{u}_{h,\bar{x}_\alpha}^{n+1}, \quad (x,t) \in \omega^{-m,\alpha} = \omega_h^{-m,\alpha} \times \omega_\tau, \quad \alpha = \overline{1,m}, \end{gathered} \tag{31}$$

$$a_\alpha = 0.5(k_{\alpha,i_\alpha - 1} + k_{\alpha,i_\alpha}), \quad \alpha = \overline{1,m}.$$

Here the temperature $\bar{u}_h$ refers to the centres of the cells, and the flows $w_{h,\alpha}, \quad \alpha = \overline{1,m}$, refer to the surrounding nodes. For convenience we use the following notation for the scalar products:

$$(w,n) = \sum_{\alpha=1}^{m} w_\alpha \cos(n, x_\alpha), \quad (u,1)_{L_1^m} = \iint\limits_{\Omega^m} u d\Omega^m,$$

$$(u,1)_{l_1^m} = \sum_{\substack{0 \le i_\alpha \le N_\alpha - 1 \\ \alpha = \overline{1,m}}} \left(u_h \prod_{\alpha=1}^{m} h_\alpha \right).$$

Initial and Neumann boundary conditions (27) are approximated by:

$$\bar{u}_h^0 = \bar{u}_0 \frac{(u_0,1)_{L_1^m}}{(\bar{u}_0,1)_{l_1^m}}, \quad x \in \omega_h^{-m}, \quad (w,n)\big|_{(x,t)\in\partial\omega^m} = S_{tx}^{m,n}\varphi. \tag{32}$$

Let us define the difference quantity of heat by $Q_h^{m,n,*} = (u_h,1)_{l_1^m}$. This relation approximates (28) with the order $O((h^*)^2), \quad h^* = \max\limits_{1\le\alpha\le m} h_\alpha$. Analogously to Theorems 2.1, 3.1, 5.1 we can prove

Theorem 6.1. *FDS (31)–(32) is flow exact L_1-conservative, i.e.*

$$Q_h^{m,n,*} \equiv Q^m(t_n), \quad n = \overline{0, N_T}. \tag{33}$$

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