ON SOME QUADRATURE FORMULAS FOR CAUCHY TYPE SINGULAR INTEGRALS WITH JACOBI WEIGHTS

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Abstract

The estimation of quadrature formula residual term for Cauchy type singular integrals is given for arbitrary values of Jacobi weight function.

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In the present paper, questions connected with approximation of singular integrals (in the Cauchy principal value sense) of type

$$S_{p,q}(\varphi;x)=\int_{-1}^{1}\rho(t)\frac{\varphi(t)}{t-x}dt \quad (-1<x<1) \tag{1}$$

with Jacobian weight function $\rho(t)=(1-t)^p(1+t)^q$, $(p,q-1)$ are considered, at that it is meant that $\varphi(t)$ is an arbitrary function from a certain class of functions sufficiently smooth on $[-1,+1]$, for which the singular integral under consideration exists in the Cauchy principal value sense for any values $x\in(-1,1)$.

In the theory of quadrature for usual (regular) integrals it is well known (see, e.g. [1], [2]) that at approximate calculation of integrals of type $\int_a^b \rho(t)\varphi(t)dt$ with given weight functions $\rho(t)$, the possible highest algebraic accuracy rate can be achieved in the cases when the knots of the corresponding quadrature formula represent zeros of polynomials of proper order which are orthogonal on segment $[-1,+1]$ with the given weight.

Hence, similarly to this, below we consider a quadrature formula for integrals of type (1) with $\varphi(t)$ replaced by its interpolating polynomial constructed by the values of $\varphi(t)$ at the Jacobian knots of order n representing zeros of polynomial which is orthogonal on the segment $[-1,+1]$ with the weight $\rho(t)=(1-t)^p(1+t)^q$. On account of this we come to the following quadrature formula for the singular integral under consideration (cf.[3])

$$\int_{-1}^{1}(1-t)^p(1+t)^q\frac{\varphi(t)}{t-x}dt \approx \sum_{k=1}^{n}\frac{J_n^{(p,q)}(x)\gamma_{pq}(x)+\lambda_n^{(p,q)}(x)-A_{kn}J_n^{(p,q)'}(x_{kn})}{(x-x_{kn})J_n^{(p,q)'}(x_{kn})}\varphi(x_{kn}) \tag{2}$$

where $\gamma_{pq}(x)=\int_{-1}^{1}\frac{(1-t)^p(1+t)^q}{t-x}dt$, $\{A_{kn}\}_{k=1}^{n}$ $(A_{kn}>0)$ are coefficients of Gauss quadrature formula, corresponding to the indicated weight, and $\lambda_n^{(p,q)}(x)=\sum_{j=1}^{n}\frac{A_{jn}J_n^{(p,q)}(x)}{x-x_{jn}}$. The integrals of type $\gamma_{p,q}(x)$ should be calculated separately. We will consider some cases when the mentioned integrals are calculated in closed form or approximately within any needed accuracy.

1. $p+q=-1$ or $p+q=0$ (provided $p,q\neq 0$). In the plane of complex variable z, under $F(z)=(z-1)^p(z+1)^q$ we will mean an arbitrarily fixed branch which is holomorphic (under condition 1) on the plane cut along $[-1,+1]$ and attaining from above on $[-1,+1]$ values $(1-t)^p(1+t)^q$. By Cauchy theorem we have

$$\frac{1}{2\pi i}\int_{\Gamma}(t-1)^p(t+1)^q\frac{dt}{t-z}=0,$$

where Γ is the boundary of doubly connected domain, bounded by straight cut $[-1,+1]$ and a closed Jordan curve containing segment $[-1,+1]$ without z inside it. Considering that contour L goes around the segment $[-1,+1]$ anticlockwise we obtain

$$\frac{1}{2\pi i}\int_{\Gamma}(t-1)^p(t+1)^q\frac{dt}{t-z}=-\frac{\sin p\pi}{\pi e^{\pi ip}}\int_{\Gamma}(t-1)^p(t+1)^q\frac{dt}{t-z}.$$

Further, given that z is located outside the domain bounded by contour L and the function $F(z)$ is clearly holomorphic in the indicated domain, via Cauchy theorem we have

$$\frac{1}{2\pi i}\int_{\Gamma}(t-1)^p(t+1)^q\frac{dt}{t-z}=-(z-1)^p(z+1)^q+F(\infty).$$

Therefore choosing the branch of function $F(z)$ so that $F(\infty) = 1$ for $p+q=0$ and besides that assuming $(-1)^p = e^{p\pi i}$

$$\int_{-1}^{1}(1-t)^p(1+t)^q\frac{dt}{t-z} = \begin{cases} \dfrac{\pi}{\sin p\pi}(z-1)^p(z+1)^q, \ p+q=-1; \\ -\dfrac{\pi}{\sin p\pi} + \pi(1-x)^p(1+x)^q\cot p\pi, \ p+q=0 \end{cases}$$

for any $x \in (-1,1)$. Particularly, if $p = q = -\dfrac{1}{2}$ we have $\gamma_{-\frac{1}{2},-\frac{1}{2}}(x) = 0$ $(-1 < x < 1)$.

2. $p+q=1$. $(1-t)^p(1+t)^q$ is taken as the boundary value of function $(1-z)^p(1+z)^q$ which is holomorphic outside the cut $[-1,+1]$ and has an expression $(1-z)^p(1+z)^q = e^{-\pi ip}\{z-2p+1+O(z^{-1})\}$ for big values of $|z|$. With the help of the Cauchy theorem we get

$$\int_{-1}^{1}(1-t)^p(1+t)^q\frac{dt}{t-z} = \frac{2\pi i}{1-e^{-2\pi ip}}\{(1-z)^p(1+z)^q - e^{-\pi ip}(z-2p+1)\}$$

$$(z \in [-1,1]).$$

From this using Sokhotskii-Plemelj formula we get

$$\gamma_{pq}(x) = \pi\left\{(1-x)^p(1+x)^q\cot p\pi - \frac{x-2p+1}{\sin p\pi}\right\}.$$

In several cases when the integrals $\gamma_{pq}(x)$ cannot be calculated in a closed form they can be computed approximately with some or other accuracy [4].

Coming back to formula (1), let us note that in paper [3], a question on accuracy estimation of singular integrals of type (1) was studied using the indicated above quadrature formula (2). Namely, in [3] it is stated that for functions $\varphi(t) \in H_r(\alpha)$ $(0 < \alpha \le 1)$, which have a derivative of order r $(r \ge 1)$ on the segment $[-1,+1]$ and satisfy the Hölder condition on this segment with index α $(0 < \alpha \le 1)$, provided $x \in (-1,+1)$ and $p, q \ge \dfrac{1}{2}$ (for any values of x, p, q) the following is valid

$$S_{p,q}(\varphi; x) - S_n^{(p,q)} = O\left(\frac{\ln n}{n^{r+\alpha}}\right) \quad (n > 1), \tag{3}$$

at this the estimate is uniform with respect to on any segment belonging to $(-1,+1)$. The further reasoning refers to proof that the estimate of type (3) is true under condition $p, q \ge -1$.

Proof of the corresponding statement however turns out to be significantly tedious. In our considerations, it is based on evidence of some lemmas which are provided by short indication to the corresponding approach of their proof.

Everywhere further, we will assume that the above mentioned Jacobi polynomials $J_n^{(p,q)}(x)$ are normed under condition $J_n^{(p,q)}(1) = C_{n+p}^n$.

Proposition 1. *For any $p, q \geq -1$ and $x \in (-1, +1)$ along each segment contained in $(-1, +1)$ the following estimate $\lambda_n^{(p,q)}(x) = O\left(n^{-\frac{1}{2}}\right)$ is true, where $\lambda_n^{(p,q)}(x)$ stands for expression of type*

$$\begin{aligned}&\int_{-1}^{1}(1-t)^p(1+t)^q\frac{J_n^{(p,q)}(t)-J_n^{(p,q)}(x)}{t-x}dt\\&=J_n^{(p,q)}(x)\gamma_{pq}(x)+\int_{-1}^{+1}\frac{(1-t)^p(1+t)^qJ_n^{(p,q)}}{t-x}dt.\end{aligned}\tag{4}$$

Proof. Since x is fixed in $(-1, +1)$, according to the known estimate ([5], sec. 8.21) the first term in the right side of (4) is $O\left(n^{-\frac{1}{2}}\right)$. The second term (integral) in (4) can be brought to the form

$$2^{p+q}\int_{0}^{\pi}\left(\sin\frac{\vartheta}{2}\right)^{2p}\left(\cos\frac{\vartheta}{2}\right)^{2q}\frac{J_n^{(p,q)}(\cos\vartheta)}{\cos\vartheta-\cos\vartheta_0}\sin\vartheta d\vartheta.\tag{5}$$

Stating further $\delta > 0$ so that $[\vartheta_0 - \delta, \vartheta_0 + \delta] \subset (0, \pi)$, we will partition the considered integral (5) into three integrals I_1, I_2, I_3 along $[0, \vartheta_0 - \delta]$, $[\vartheta_0 - \delta, \vartheta_0 + \delta]$, $[\vartheta_0 + \delta, \pi]$ respectively. Taking into account that for any fixed constant $c > 0$ and for a sufficiently large n relation $cn^{-1} < \vartheta_0 - \delta$ is true, on the basis of estimates $J_n^{(p,q)}(x)$ on segments $[0, cn^{-1}]$, $[cn^{-1}, \vartheta_0 - \delta]$, ([5], sec. 7.32), also due to the fact that $(\cos\vartheta - \cos\vartheta_0)$ for $0 \leq \vartheta \leq \vartheta_0 - \delta$ is bounded from below by a fixed positive number, we find $I_1 = O\left(n^{-\frac{1}{2}}\right)$ $(0 < \vartheta_0 < \pi)$. The estimate $I_3 = O\left(n^{-\frac{1}{2}}\right)$ $(0 < \vartheta_0 < \pi)$ can be stated with the help of similar consideration. As far as the singular integral

$$\int_{\vartheta_0-\delta}^{\vartheta_0+\delta}\left(\sin\frac{\vartheta}{2}\right)^{2p}\left(\cos\frac{\vartheta}{2}\right)^{2q}\frac{J_n^{(p,q)}(\cos\vartheta)}{\cos\vartheta-\cos\vartheta_0}\sin\vartheta d\vartheta$$

is concerned, transform it to the form

$$\begin{aligned}&J_n^{(p,q)}\cos\vartheta_0\int_{\vartheta_0-\delta}^{\vartheta_0+\delta}\left(\sin\frac{\vartheta}{2}\right)^{2p}\left(\cos\frac{\vartheta}{2}\right)^{2q}\frac{\sin\vartheta d\vartheta}{\cos\vartheta-\cos\vartheta_0}\\&+\int_{\vartheta_0-\delta}^{\vartheta_0+\delta}\left(\sin\frac{\vartheta}{2}\right)^{2p}\left(\cos\frac{\vartheta}{2}\right)^{2q}\frac{J_n^{(p,q)}(\cos\vartheta)-J_n^{(p,q)}(\cos\vartheta_0)}{\cos\vartheta-\cos\vartheta_0}\sin\vartheta d\vartheta.\end{aligned}\tag{6}$$

As far as the point ϑ_0 is stated in the indicated way in $(0,\pi)$, in accordance to the known estimate of $I_n^{(p,q)}$ ([5], sec.8.21) the first term in the right side of (6) is evidently $O\left(n^{-\frac{1}{2}}\right)$.Proceeding to estimation of the second term in the mentioned expression, according to the known formulas ([5], sec. 8.8) we come to the following

$$\begin{gathered}\frac{J_n^{(p,q)}(\cos\vartheta)-J_n^{(p,q)}(\cos\vartheta_0)}{\cos\vartheta-\cos\vartheta_0}=n^{-\frac{1}{2}}k(\vartheta)\frac{\cos(N\vartheta+\gamma)-\cos(N\vartheta_0+\gamma)}{\cos\vartheta-\cos\vartheta_0}\\ +O\left(n^{-\frac{1}{2}}\right)\quad(\vartheta_0-\delta<\vartheta<\vartheta_0+\delta),\end{gathered}\tag{7}$$

where

$$N=n+\frac{p+q+1}{2},\quad \gamma=-\left(p+\frac{1}{2}\right)\frac{\pi}{2},$$

$$k(\vartheta)=\frac{1}{\sqrt{\pi}}\sin\left(\frac{\vartheta}{2}\right)^{-p-\frac{1}{2}}\cos\left(\frac{\vartheta}{2}\right)^{-q-\frac{1}{2}},$$

at that the estimate of the remainder in (7) is uniform on segment $[\vartheta_0-\delta,\vartheta_0+\delta]$. On this basis we have only to show that

$$\int\limits_{\vartheta_0-\delta}^{\vartheta_0+\delta}F(\vartheta)\frac{\cos(N\vartheta+\gamma)-\cos(N\vartheta_0+\gamma)}{\cos\vartheta-\cos\vartheta_0}\sin\vartheta d\vartheta=O(1)\ \ (0<\vartheta_0<\vartheta),\tag{8}$$

where $F(\vartheta)$ stands for $\sin\left(\frac{\vartheta}{2}\right)^{p-\frac{1}{2}}\cos\left(\frac{\vartheta}{2}\right)^{q-\frac{1}{2}}$. Further, using identity $\frac{\vartheta+\vartheta_0}{2}=\frac{\vartheta-\vartheta_0}{2}+\vartheta_0$, the expression in the left side of (8) can be represented in the form of sum

$$\begin{gathered}\sin(N\vartheta_0+\gamma)\int\limits_{\vartheta_0-\delta}^{\vartheta_0+\delta}F(\vartheta)\frac{\sin N\frac{\vartheta-\vartheta_0}{2}\cos N\frac{\vartheta-\vartheta_0}{2}}{\sin\frac{\vartheta+\vartheta_0}{2}\sin\frac{\vartheta-\vartheta_0}{2}}\sin\vartheta d\vartheta\\ \cos(N\vartheta_0+\gamma)\int\limits_{\vartheta_0-\delta}^{\vartheta_0+\delta}F(\vartheta)\frac{\sin^2 N\frac{\vartheta-\vartheta_0}{2}\cos N\frac{\vartheta-\vartheta_0}{2}}{\sin\frac{\vartheta+\vartheta_0}{2}\sin\frac{\vartheta-\vartheta_0}{2}}\sin\vartheta d\vartheta.\end{gathered}\tag{9}$$

Using the identity $\sin\vartheta=\sin\frac{\vartheta+\vartheta_0}{2}+2\sin\frac{\vartheta-\vartheta_0}{4}\cos\frac{3\vartheta+\vartheta_0}{4}$, the first integral in (9) can be presented in the form

$$\frac{1}{2}\int\limits_{\vartheta_0-\delta}^{\vartheta_0+\delta}\frac{\sin N(\vartheta-\vartheta_0)}{\sin\frac{\vartheta-\vartheta_0}{2}}d\vartheta+O(1).\tag{10}$$

Further, allowing n to be big enough to satisfy $\left(\vartheta_0 - \dfrac{1}{n}, \vartheta_0 + \dfrac{1}{n}\right) \subset (\vartheta_0 - \delta, \vartheta_0 + \delta)$, split the integral in (10) into sum of integrals on segments $\left[\vartheta_0 - \delta, \vartheta_0 - \dfrac{1}{n}\right]$, $\left[\vartheta_0 - \dfrac{1}{n}, \vartheta_0 + \dfrac{1}{n}\right]$, $\left[\vartheta_0 + \dfrac{1}{n}, \vartheta_0 + \delta\right]$. At this, for the integral along the segment $\left[\vartheta_0 - \dfrac{1}{n}, \vartheta_0 + \dfrac{1}{n}\right]$ via $|\sin x| \le |x|$ we get estimate $\dfrac{2N}{n} = O(1)$. Regarding the integral along $\left[\vartheta_0 - \delta, \vartheta_0 - \dfrac{1}{n}\right]$, with the help of integration by parts it is reduced to the following expression

$$\frac{F(\vartheta_0 - \frac{1}{n})\cos\frac{N}{n}}{N\sin\frac{1}{2n}} - \frac{F(\vartheta_0 - \frac{1}{\delta})\cos N\delta}{N\sin\frac{\delta}{n}}$$

$$-\frac{1}{2N}\int\limits_{\vartheta_0-\delta}^{\vartheta_0-\frac{1}{n}} F(\vartheta)\frac{\cos N(\vartheta-\vartheta_0)\cos N\frac{\vartheta-\vartheta_0}{2}}{\left[\sin\frac{\vartheta-\vartheta_0}{2}\right]^2}d\vartheta + \frac{1}{N}\int\limits_{\vartheta_0-\delta}^{\vartheta_0-\frac{1}{n}} F'(\vartheta)\frac{\cos(\vartheta-\vartheta_0)d\vartheta}{\sin\frac{\vartheta-\vartheta_0}{2}}.$$

Taking into account $\int\limits_{x_{\nu-1\,n}}^{x_{\nu\,n}} J_n^{(p,q)''}(t)dt = -\int\limits_{\vartheta_{\nu-1}}^{\vartheta_\nu} J_n^{(p,q)''}(\cos\vartheta)\sin\vartheta dt$, it is clear that

$$\frac{F(\vartheta - \frac{1}{n})\cos N\delta}{N\sin\frac{1}{2n}} - \frac{F(\vartheta - \frac{1}{\delta})\cos N}{N\sin\frac{\delta}{2}} = O(1),$$

also

$$\frac{1}{N}\int\limits_{\vartheta_0-\delta}^{\vartheta_0-\frac{1}{n}} F'(\vartheta)\frac{\cos N(\vartheta-\vartheta_0)d\vartheta}{\sin\frac{\vartheta-\vartheta_0}{2}} = \frac{O(1)}{N}\max_{\vartheta\in\left[\vartheta_0-\delta,\vartheta-\frac{1}{n}\right]}\frac{1}{\vartheta_0-\vartheta} = O(1).$$

Besides,

$$\frac{1}{2N}\int\limits_{\vartheta_0-\delta}^{\vartheta_0-\frac{1}{n}}\frac{\cos N(\vartheta-\vartheta_0)\cos(\vartheta-\vartheta_0)}{\left[\sin\frac{\vartheta-\vartheta_0}{2}\right]^2}d\vartheta$$

$$= \frac{O(1)}{N}\int\limits_{\vartheta_0-\delta}^{\vartheta_0+\frac{1}{n}}\frac{d\vartheta}{(\vartheta-\vartheta_0)^2} = \frac{O(1)}{N}\left(n + \frac{1}{\delta}\right) = O(1).$$

The presented statements convince us in the correctness of Proposition 1.

Proposition 2. *For any* $p, q > -1$ *and* $x \in (-1, +1)$, *estimate* $a_{kn}(x) = O(\ln n)\,(n > 1)$, $k = 1, 2, \ldots, n$ *is true.*

Proof. Along with this it is uniform on any segment $[a,b] \subset (-1,1)$. In the proof of this statement it is sufficient to put $0 \leqslant x \leqslant 1$. For the rest values the considerations are similar. Assuming $\nu \neq k$ with the help of expression $\lambda_n^{(p,q)}(x) = \sum_{j=1}^{n} \frac{A_{jn} J_n^{(p,q)}(x)}{x - x_{jn}}$ we can write

$$a_{kn}(x_{\nu n}) = \left[\frac{A_{\nu n}}{J_n^{(p,q)'}(x_{kn})} - \frac{A_{\nu n}}{J_n^{(p,q)'}(x_{\nu n})} \right] \frac{J_n^{(p,q)'}(x_{\nu n})}{x_{\nu n} - x_{kn}}. \tag{11}$$

To be definite, we consider in (11) k and ν such that $k \neq \nu$ and besides that $0 < \vartheta_k \leqslant \frac{\pi}{2}$, $0 < \vartheta_\nu \leqslant \frac{\pi}{2}$. Further, as is known ([5], sec 8.9), the following [1] holds: $J_n^{(p,q)'}(\cos(\vartheta_k)) \sim n^{\frac{1}{2}} \vartheta_k^{-p-\frac{3}{2}}$. Also, $\vartheta_k = \frac{1}{n}[k\pi + O(1)]$, where $O(1)$ is bounded uniformly for all values $k = 1, 2, \dots, n \, (n = 1, 2, \dots)$.

We will as well recall a known estimate ([5], sec. 15.3) $A_\nu n \sim \nu^{2p+1} n^{-2p-2}$ $(0 < \vartheta_\nu \leqslant \pi - \delta)$. For $0 < \delta < \pi$ we can obtain $\frac{A_{\nu n}}{J_n^{(p,q)'}(\cos\vartheta_k)} \sim$ $\nu^{2p+1} n^{-3p-3} k^{p+\frac{1}{2}} \sin(\vartheta_k)^{-1}$. Additionally, for any ν and k $(\nu, k = 1, 2, \dots, n)$ the following is true:

$$\frac{A_{\nu n}}{J_n^{(p,q)'}(x_{kn})} = O(n^{-\frac{3}{2}}) \sin\vartheta_k. \tag{12}$$

Continuing consideration, we have

$$\frac{1}{n} \frac{\sin\vartheta_k}{|x_{\nu n} - x_{kn}|} \leqslant \frac{\sin\left(\frac{\vartheta_\nu + \vartheta_k}{2}\right) \cos\left(\frac{\vartheta_\nu - \vartheta_k}{2}\right)}{n \sin\left(\frac{\vartheta_\nu + \vartheta_k}{2}\right) \left|\sin\left(\frac{\vartheta_\nu - \vartheta_k}{2}\right)\right|}, (\nu \neq k). \tag{13}$$

Using in (13) $\sin x \geqslant \frac{2}{\pi} x \quad \left(0 \leqslant x \leqslant \frac{\pi}{2}\right)$, we also have

$$\frac{\sin\vartheta_k}{|x_{n\nu} - x_{k\nu}|} \leqslant \frac{\pi}{|\vartheta_\nu - \vartheta_k|}, \, \nu \neq k \tag{14}$$

Now we will state the lower estimate of expression $\vartheta_\nu - \vartheta_k$ basing on the fact that for any $(\nu \neq k)$ relation $|\vartheta_\nu - \vartheta_k| \geqslant \vartheta_\nu - \vartheta_{\nu-1}$ holds. Considering

$$\left| J_n^{(p,q)'}(x_{\nu-1n}) - J_n^{(p,q)'}(x_{\nu n}) \right| = \left| \int_{x_{\nu n-1}}^{x_{\nu n}} J_n^{(p,q)''}(t) dt \right|$$

[1] As is usual $\tau_\nu \sim \mu_\nu \, \tau_\nu \neq 0 \mu_\nu \neq 0$ means that the modulus of ratio of these values is bounded from below and above by constants independent of n

$$= \left| \int_{\vartheta_{\nu-1}}^{\vartheta_\nu} J_n^{(p,q)''}(\cos\vartheta)\sin\vartheta d\vartheta \right|,$$

for calculation of $J_n^{(p,q)''}(\cos\vartheta)$, we use known equalities ([5], sec. 4.21): $J_n^{(p,q)'}(t) = \frac{1}{2}(n+p+q+1)J_n^{(p+1,q+1)}(t)$, taking into account at this (see [5], sec. 7.32) validness of estimate $J_n^{(p+2,q+2)}(\cos\vartheta) = \vartheta^{-p-\frac{5}{2}}O\left(n^{-\frac{1}{2}}\right)$ $\left(0 < \vartheta \leqslant \frac{\pi}{2}\right)$ $(p, q > -1)$ under condition $p+2 > -\dfrac{1}{2}$, from these estimates we obtain

$$\int_{x_{\nu-1n}}^{x_{\nu n}} J_n^{(p,q)''}(t)dt = O(n^{\frac{3}{2}}) \int_{\vartheta_{\nu-1}}^{\vartheta_\nu} \vartheta^{-p-\frac{5}{2}}\sin\vartheta d\vartheta.$$

Next, since usually $p + \frac{3}{2} > 0$, we can write

$$\int_{\vartheta_{\nu-1}}^{\vartheta_\nu} \vartheta^{-p-\frac{5}{2}}\sin\vartheta d\vartheta \leqslant \int_{\vartheta_{\nu-1}}^{\vartheta_\nu} \vartheta^{-p-\frac{3}{2}}d\vartheta \leqslant \vartheta_{\nu-1}^{-p-\frac{3}{2}}(\vartheta_\nu - \vartheta_{\nu-1}).$$

Consequently, there exists a constant $C_0 > 0$ such that

$$\vartheta_\nu - \vartheta_{\nu-1} \geqslant C_0 n^{-\frac{3}{2}}\vartheta_\nu^{p+\frac{3}{2}} \left| \int_{x_{\nu n}}^{x_{\nu-1\,n}} J_n^{(p,q)''}(t)dt \right|.$$

Moreover, in spite of $J_n^{(p,q)'}(x_{\nu-1\,n})$ and $J_n^{(p,q)'}(x_{\nu\,n})$ have different signs, the following holds:

$$\left| J_n^{(p,q)'}(x_{\nu\,n}) - J_n^{(p,q)'}(x_{\nu-1\,n}) \right| > \left| J_n^{(p,q)'}(x_{\nu-1\,n}) \right|.$$

Besides this it is clear that (cf. [5], sec. 8.9), $|J_n^{(p,q)'}(\cos\vartheta_{\nu-1})| \sim (\nu-1)^{-p-\frac{3}{2}}\, n^{p+2}$. Collating the indicated equalities yields $\vartheta_\nu - \vartheta_{\nu-1} \geqslant C_1 n^{-1}$ $(\nu = 1, 2, \dots)$, where C_1 is a constant independent of ν and n. Onward, on the basis of the above mentioned asymptotic representation of ϑ_k and inequality $\dfrac{|\nu-k|\,\pi}{n} \leqslant |\vartheta_\nu - \vartheta_k| + \left| \dfrac{\nu-k}{n}\pi - \vartheta + \vartheta_k \right|$, after some computations we find $|\vartheta_\nu - \vartheta_k| \geqslant \dfrac{C_1\pi|\nu-k|}{(C_1+C_2)n}$ $(C_1, C_2 = const)$ for any k and ν. Also, remembering (14), we obtain that $\dfrac{1}{n}\dfrac{\sin\vartheta_k}{x_{\nu n} - x_{kn}} = O(1)$ $(\nu \neq k)$. Combination of (12) and (14) leads us to the following

$$\begin{aligned} \frac{A_{\nu\,n}}{J_n^{(p,q)'}(x_{kn})(x_{\nu n} - x_{kn})} &= O(n^{-\frac{1}{2}}), \\ \frac{A_{k\,n}}{J_n^{(p,q)'}(x_{\nu n})(x_{\nu n} - x_{kn})} &= O(n^{-\frac{1}{2}}), \end{aligned}$$

uniformly with respect to ν and k (under condition $\nu \neq k$). From this and another known relation (cf. [5], sec. 8.9): $\left|J_n^{(p,q')}\cos\vartheta_\nu\right| \sim \nu^{-p-\frac{3}{2}}n^{p+2}$ $(0 \leqslant \vartheta \leqslant \frac{\pi}{2})$ it follows that $a_{kn}(x_{\nu n}) = O\left(\nu^{-p-\frac{3}{2}}n^{p+\frac{3}{2}}\right)$ $(-1 < x_{\nu n} \leqslant 0)$. Analogous estimate takes place also for ν such that $0 \leqslant x_{\nu n} < 1$. Beyond that we can make sure in validity of the next equality

$$a_{\nu n}(x_{\nu n}) = \frac{1}{2}A_{\nu n}\frac{J_n^{(p,q)''}(x_{\nu n})}{J_n^{(p,q)'}(x_{\nu n})} + \sum_{\sigma=1}^{n}\frac{A_{\sigma n}}{x_{\nu n} - x_{\sigma n}}\,(\nu = 1, 2, \ldots, n),$$

besides $J_n^{(p,q)''}(\cos\vartheta_\nu) = \vartheta_\nu^{-p-\frac{5}{2}}O\left(n^{\frac{3}{2}}\right)\left(0 < \vartheta_\nu \leqslant \frac{\pi}{2}\right)$ for any $p, q > -1$. From here again by virtue of performance $\vartheta_\nu = \frac{1}{n}[\nu\pi + O(1)]$ we get $J_n^{(p,q)''}(\cos\vartheta_\nu) = O\left(\nu^{-p-\frac{5}{2}}n^{p+4}\right)\left(0 < \vartheta_\nu \leqslant \frac{\pi}{2}; p, q > -1\right)$. For the lower bound of expression $\left|J_n^{(p,q)'}(x_{\nu n})\right|$ we will recall again estimate (used above) from ([5], sec. 8.9) that as a result gives

$$\frac{A_{\nu n}J_n^{(p,q)''}(x_{\nu n})}{J_n^{(p,q)'}(x_{\nu n})} = O(\nu^{2p}n^{-2p}).$$

Taking into account these formulas and some next simple transformations, with the help of asymptotic equality $\begin{pmatrix} n+p \\ n \end{pmatrix} = \frac{n^p}{\Gamma(p+1)}\left[1 + O(\frac{1}{n})\right]$, where Γ is Euler function [6], we get $a_{kn}(x_{kn}) = O(k^{2p}n^{-2p}\ln n)$. Moreover, $a_{kn}(x_{kn}) = O(\ln n)\,(n > 1; k = 1, 2, \ldots, n)$. Evidently, the expression $a_{kn}(x)$ can be represented by the Lagrange interpolating polynomial:

$$a_{kn}(x) = \sum_{i=1}^{n}\frac{J_n^{(p,q)}(x)}{(x - x_{ln})J_n^{(p,q)'}(x_{ln})}a_{kn}(x_{ln})$$

For further considerations we split this sum into $\sum_1^{m-2} + \sum_{m-1}^{n}$. Both sums are estimated similarly. Putting for definiteness $0 \leqslant x < 1$, we outline shortly some details related to this question, considering, e.g., the sum $\sum_1^{m-2}$. Meaning as above,$\vartheta_j \leqslant \frac{\pi}{2}$ $(\vartheta_j = \arccos x_{jn}; j = 1, 2, \ldots, n)$ and noting that under condition $x - x_{kn} = O(n^{-1})$, the following estimate ([5], sec. 14.4) is true

$$\frac{J_n^{(p,q)}(x)}{(x - x_{kn})J_n^{(p,q)'}(x_{kn})} = O(1),$$

we may get the relation

$$\frac{J_n^{(p,q)}(x)}{\cos\vartheta - \cos\vartheta_0} = n^{-\frac{1}{2}}K(\vartheta)\frac{\cos(N\vartheta + \gamma) - \cos(N\vartheta_\nu + \gamma)}{\cos\vartheta - \cos\vartheta_\nu} + r_n(\vartheta),$$

$$\gamma = -\frac{(p+\frac{1}{2})\pi}{2},\ K(\vartheta) = \frac{1}{\pi}\left(\sin\frac{\vartheta}{2}\right)^{-p-\frac{1}{2}}\left(\cos\frac{\vartheta}{2}\right)^{-q-\frac{1}{2}}.$$

At that $r_n(\vartheta) = O(n^{-\frac{1}{2}})$. Besides, if in the indicated presentation under assumption that for the given m the point x is located between x_{mn} and x_{m+1n}, then for $k = m$ after some estimations we obtain

$$\frac{J_n^{(p,q)}(x)a_{mn}(x_{m-1n})}{(x-x_{mn})J_n^{(p,q)'}(x_{mn})} = O(1),$$

$$\frac{J_n^{(p,q)}(x)a_{mn}(x_{mn})}{(x-x_{mn})J_n^{(p,q)'}(x_{mn})} = O(\ln n)\ (n>1).$$

Similar estimates are obtained for $k = m-1$ and along with that for the rest values of x. Next,using the known estimate ([5], sec. 89) $J_n^{(p,q)}(\cos\vartheta_{\nu-1}) \sim (\nu-1)^{-p-\frac{3}{2}}n^{p+2}$ and estimate $|\vartheta_\nu - \vartheta_{\nu-1}| \geqslant \dfrac{C_0}{n}$ $(\nu = 1,2,\ldots,n)$ where C_0 is a constant independent of ν and n, we come to validity of Proposition 2. □

Proposition 3. *For any $p, q > -1$ and $x \in (-1,1)$ the equality*

$$\sum_{k=1}^{n}\left|\frac{J_n^{(p,q)}(x)\gamma_{p,q}(x) + \lambda_n^{(p,q)}(x) - A_{kn}J_n^{(p,q)'}(x_{kn})}{(x-x_{kn})J_n^{(p,q)'}(x_{kn})}\right| = 0(\ln n)\,]\ (n>1) \quad (15)$$

is true. The estimate is uniform on any segment that belongs to (-1,1).

Proof. In order to prove Proposition 3, denote after $a_{kn}(x)$ expression in $\sum_{k=1}^{n}$ of formula (15). We have

$$|a_{kn}(x)| \leqslant \frac{\left|\lambda_n^{(p,q)}(x)\right| + \left|\gamma_{p,q}(x)J_n^{(p,q)}(x)\right|}{\left|(x-x_{kn})J_n^{(p,q)'}(x_{kn})\right|} + \frac{A_{kn}}{|x-x_{kn}|}.$$

We need to estimate $\sum\limits_{k=1}^{n}|a_{kn}(x)|$ on the basis of the preceding inequality. With this aim, meaning $m = \overline{1,n\text{-}1}$, split the concerned sum into $\sum_1^m + \sum_{m+1}^n$. Both of them are estimated similarly and, for the sake of definiteness, we will show shortly estimation of the sum $\sum_1^m$. Regarding $\sum\limits_{k=1}^{m}|a_{kn}(x)| = \sum\limits_{k=1}^{m-2}|a_{kn}(x)| + |a_{km-1}(x)| + |a_{km-2}(x)|$, due to Proposition 2, we have $|a_{km-1}(x)| + |a_{kn}(x)| = O(\ln n)$. Further, taking into account the previous estimate for $|a_{kn}(x)|$, also relations ([5], sec. 14.4)

$$J_n^{(p,q)}(x) = O(n^{-\frac{1}{2}})\,(-1<x<1),$$

$$\sum_{k=1}^{m-2} \frac{1}{\left|(x-x_{kn})J_n^{(p,q)'}(x_{kn})\right|} = O(n^{-\frac{1}{2}}\ln n)(x_{mn} \leqslant x \leqslant x_{m+1n}),$$

applying some transformations and estimates (see, namely, [5], sec. 15.3), also $A_k = O(n^{-1})(k = \overline{1,\text{n}})$ along with Proposition 1, we can ascertain validity of relation

$$\sum_{k=1}^{n} |a_{kn}(x)| = O(\ln n)\,(-1 < x < 1),$$

which means (15). □

Let $P_{n-1}(t)$ be a polynomial of order $n-1\,(n>1)$, such that for functions $\varphi(t) \in H_r(\alpha)$ (see the formulation of the statement (3)) the following takes place

$$\|\varphi(t) - P_{n-1}(t)\|_C \leqslant \frac{A}{n^{r+1}}\ (A = const > 0).$$

Putting further $\eta = \dfrac{1+|x|}{2}$ for given $x \in (-1,+1)$, denote the expression

$$\frac{\sup\limits_{t\in[-\eta,\eta]} |\varphi(t) - \varphi(x)|}{|t-x|^{\beta}}$$

by $M_\eta(\varphi;\beta)$, where $\beta > 0$ is an arbitrary number, less than α $(0 < \alpha \leqslant 1)$.

Proposition 4. *For any function $\varphi(t) \in H_r(\alpha)$, the following is true*

$$M_\eta\,(\varphi - P_{n-1};\beta) \leqslant \frac{A}{n^{r+\alpha-\beta}}\,, (-1 < x < 1).$$

Proof. We outline the main moments in proof of Proposition 4. Assuming n so small, that $x+h \in [-\eta,\eta]$, denote $\Delta_{xh} = |\varphi(x+h) - P_{n-1}(x+h) - [\varphi(x) - P_{n-1}(x)]\,|$. Two cases are possible: $|h| > n^{-1}$ and $|h| \leqslant n^{-1}$. In the first case we have

$$\Delta_{xh} \leqslant 2\max|\varphi(t) - \varphi_{n-1}(t)| \leqslant \frac{2An^{-\beta}}{n^{r+\alpha-\beta}} < \frac{2A|h|^{\beta}}{n^{r+\alpha-\beta}}.$$

For $|h| \leqslant n^{-1}$ we can write

$$\varphi(t) - P_{n-1}(x) = \sum_{k=1}^{n} V_k(x),\ V_k(x) = P_{2^k\,n}(x) - P_{2^{k-1}\,n}(x)$$

similarly to [7]. We separate two cases $r = 0$ and $r \geqslant 1$. Consideration of the appropriate cases providing the previous reasoning and Proposition 2 (see, also, [8]), leads us to validity of the needed statement. $\square$

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