METHOD OF REFINEMENT BY HIGHER ORDER DIFFERENCES FOR ELLIPTIC EQUATIONS WITH BITSADZE - SAMARSKII TYPE NONLOCAL BOUNDARY CONDITIONS

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(Received: 07.01.2016; accepted: 22.06.2016)

Abstract

We consider the Bitsadze–Samarskii type nonlocal boundary value problem for a second order elliptic equation on a rectangle, which is solved by a difference scheme of second-order accuracy. Using this solution, the right-hand side of the difference scheme is corrected. It is shown that the solution of the corrected scheme converges at the rate $O(|h|^s)$ in the discrete L_2-norm provided that the solution of the original problem belongs to the Sobolev space with exponent $s \in [2, 4]$.

Key words and phrases: Nonlocal boundary-value problem, difference scheme, method of correction, convergence rate, Sobolev spaces.

AMS subject classification: 65N06, 65N12.

1 Introduction

In the present work we study the question of solvability of the BitsadzeSamarskii type nonlocal boundary value problem for an elliptic equation by the finite difference method. In order to minimize the amount of calculations it is desirable for the difference scheme to be sufficiently good on coarse meshes, i.e. to have high order accuracy. For obtaining high precision we use a two - stage finite difference method.

At the first stage we solve the difference scheme $\mathcal{L}_h\tilde{U} = \tilde{\varphi}$, which has the second order of approximation. Using the solution $\tilde{U}$ the right-hand side of the difference scheme is corrected, $\mathcal{L}_hU = \tilde{\varphi}+R\tilde{U}$, and solved again on the same mesh.

It is proved that the solution U of the corrected difference scheme converges at rate $O(h^s)$ in the discrete L_2 -norm, when the exact solution belongs to the Sobolev space $W_2^s,\ s \in [2,4]$.

This approach for some boundary value problems posed for Poisson and Laplace equations has been studied in Volkov's papers (see, e.g. [1–3]), where the input data were chosen so as to ensure that the exact solution belongs to the Hölder class $C_{6,\lambda}(\bar{\Omega})$.

For establishing the convergence we use the methodology of obtaining the compatible estimates of convergence rate of difference schemes. This methodology develops from the works of Samarskii, Lazarov and Makarov (see, e.g., [4–6]), and later in the works of other authors (see e.g.[7,8]). For the elliptic problems such estimates have the form

$$\|U-u\|_{W_2^k(\omega)} \le c|h|^{s-k}\|u\|_{W_2^s(\Omega)}, \quad s > k \ge 0,$$

where u is the solution of the original problem, U is the approximate solution, k and s are integer and real numbers, respectively, $W_2^k(\omega)$ and $W_2^s(\Omega)$ are the Sobolev norms on the set of functions with discrete and continuous arguments. Here and below c denotes a positive generic constant, independent of h and u.

The generalization of Bitsadze-Samarski problem [9] was investigated by many authors (see, e.g., [10-13]).

In [11] for a Poisson equation a difference scheme is considered which converges by the rate $O(h^2)$ in the discrete W_2^2-norm to the exact solution from the class $C^4(\bar{\Omega})$.

In [13] difference scheme is considered for a second order elliptic equation and the compatible estimate of convergence rate in discrete W_2^2-norm is obtained.

Results, analogous to those given in the present work, are obtained in [14] for the Dirichlet problem posed for an elliptic equation, and also in [15] for the mixed problem with third kind boundary conditions.

One of the methods for obtaining compact high order approximations is Mehrstellen method ("Mehrstellenverfahren"), defined by Collatz (see [16]). Instead of approximating only the left hand side of the differential equation, he proposes to take several points of the right hand side as well. In the case of the two-dimensional problem, the differential operator is approximated on a 9-point stencil with the fourth order accuracy.

The advantage of the Mehrstellen schemes over ordinary (second order) accuracy schemes on a coarse grid is obvious.

The advantage of our method is:

a) It needs to approximate the differential operator on minimally acceptable stencil (5-point stencil for a two-dimensional problem). Therefore, the condition number of this operator is better as compared with Mehrstellen schemes, which is notable on a fine grid.

b) It is a two-stage method, nevertheless it requires matrix inversion only once (on the second stage we change only the right-hand side of the equation, while the operator is kept unchanged).

c) The method of correction is handy even in the case when construction of high precision schemes is impossible.

2 Statement of the Problem and Notations

Let $\bar{\Omega} = \{(x_1, x_2):\ 0 \le x_\alpha \le 1,\ \alpha = 1, 2\}$ be a unit square with a boundary Γ; $\Gamma_0 = \Gamma \setminus \{(1, x_2) : 0 < x_2 < 1\}$; let ξ_k be fixed points from the interval $(0; 1)$, $0 < \xi_1 < \xi_2 < \cdots \xi_m < 1$. Denote $\xi_0 = 0,\ \xi_{m+1} = 1$.

Consider the problem

$$\mathcal{L}u := -\sum_{\alpha,\beta=1}^{2} \frac{\partial}{\partial x_\alpha}\Big(a_{\alpha\beta}\frac{\partial u}{\partial x_\beta}\Big) + a_2\frac{\partial u}{\partial x_2} + a_0 u = f(x),\ x \in \Omega, \tag{2.1}$$

$$u(x) = 0, \quad x \in \Gamma_0, \quad u(1, x_2) = \sum_{k=1}^{m} \alpha_k u(\xi_k, x_2), 0 < x_2 < 1. \tag{2.2}$$

We assume that

$$\begin{gathered} \sum_{\alpha,\beta=1}^{2} a_{\alpha\beta} t_\alpha t_\beta \ge \nu_1(t_1^2 + t_2^2), \quad \nu > 0, \\ a_{\alpha\beta},\ a_2, a_0 = \text{const}, \quad a_0 \ge 0, \quad \varkappa := \sum_{k=1}^{m} |\alpha_k|\sqrt{\xi_k} < 1. \end{gathered} \tag{2.3}$$

It was shown in [12] that, for $f(x) \in L_2(\Omega, r)$, there exists a unique strong solution of problem (2.1), (2.2) in the weighted Sobolev space $W_2^2(\Omega, r)$. Throughout the following, we assume that the function $f(x)$ provides the unique solvability of problem (2.1), (2.2) in the $W_2^s(\Omega),\ 2 \le s \le 4$.

Consider the following grid domains in $\bar{\Omega}$:

$$\bar{\omega}_k = \{x_k = i_k h:\ i_k = 0, 1, \ldots, n,\ h = 1/n\}, \quad \omega_k = \bar{\omega}_k \cap (0, 1),$$

$$\omega_k^+ = \bar{\omega}_\alpha \cap (0, 1], \quad k = 1, 2, \quad \omega = \omega_1 \times \omega_2, \quad \bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2, \quad \gamma_0 = \Gamma_0 \cap \bar{\omega}.$$

We assume that the points ξ_k coincide with grid nodes

$$\xi_k = n_k h, \quad k = 1, 2, \ldots, m,$$

where n_k are nonnegative integers, $0 < n_1 < n_2 < \cdots < n_m < n$. We suppose also that

$$h/2 \le 1 - \xi_m - \nu_1, \quad \nu_1 = \text{const} > 0.$$

For grid functions we define difference quotients in x_k directions as follows

$$V_{x_k} = \big(V^{(+1_k)} - V\big)/h, \quad V_{\bar{x}_k} = \big(V - V^{(-1_k)}\big)/h,$$

where

$$V = V(x), \quad V^{(\pm 1_1)} = V(x_1 \pm h, x_2), \quad V^{(\pm 1_2)} = V(x_1, x_2 \pm h).$$

Denote

$$\begin{gathered} I_\alpha Y := \frac{Y + Y^{(+1_\alpha)}}{2},\ \alpha = 1, 2, \\ \Lambda Y := -Y_{\bar{x}_1 x_1} - Y_{\bar{x}_2 x_2}. \end{gathered}$$

For functions, defined on Ω, we need the following averaging operators:

$$T_1u(x) := \frac{1}{h^2}\int_{x_1-h_1}^{x_1+h_1}(h_1 - |x_1 - t_1|)u(t_1, x_2)\,dt_1,$$

$$S_1u(x) = \frac{1}{h}\int_{x_1}^{x_1+h} u(t_1, x_2)\,dt_1.$$

Analogously is defined the operator T_2. Note that these operators commute and

$$T_k\frac{\partial^2 u}{\partial x_k^2} = u_{\bar{x}_k x_k}, \quad k = 1, 2, \quad T_1\frac{\partial u}{\partial x_\alpha} = (S_\alpha u)_{\bar{x}_\alpha}.$$

Define the following weight functions

$$r(x_1) = 1 - x_1, \quad \rho(x_1) = 1 - x_1 - \sum_{k=1}^{m} \varkappa\sigma_k\chi(\xi_k - x_1),$$

where

$$\sigma_k = \frac{|\alpha_k|}{\sqrt{\xi_k}}, \quad \chi(t) = \begin{cases} t, \text{ if } t \geq 0, \\ 0, \text{ if } t < 0. \end{cases}$$

Let

$$\bar{r} = \big(r + r^{(-1_1)}\big)/2, \quad \overline{\rho} = \big(\rho + \rho^{(-1_1)}\big)/2.$$

Notice that the following inequality

$$(1 - \varkappa^2)r(x_1) \leq \rho(x_1) \leq r(x_1) \tag{2.4}$$

holds.

Indeed, the right-hand side inequality is obvious. The left-hand side of the inequality can be verified as follows:

$$\rho(x_1) = 1 - x_1 - \varkappa\sum_{k=j+1}^{m}\sigma_k(\xi_k - x_1) \geq \big(1 - \varkappa\sum_{k=j+1}^{m}\sigma_k\xi_k\big)(1 - x_1) \geq$$

$$\geq (1 - \varkappa^2)(1 - x_1), \quad x_1 \in (\xi_j, \xi_{j+1}).$$

Remark. Introduction of auxiliary (equivalent to r) weight function ρ gives possibility to state the positive definiteness of the difference scheme operator. A weighted inner product and induced by it norm were used firstly in the work of *D. Gordeziani* [10, pp. 10–14] to prove the uniqueness of a classical solution of the nonlocal boundary value problems.

We define the following inner products and discrete norms:

$$(Y, V) = \sum_{\omega} h^2Y(x)V(x), \quad \|V\| = (V, V)^{1/2},$$

$$(Y, V)_r = \sum_{\omega} h^2r(x_1)Y(x)V(x), \quad \|V\|_r = \|V\|_{L_2(\omega,r)} = (V, V)_r^{1/2},$$

$$\|Y]|_r^2 = \sum_{\omega_1^+\times\omega_2} h^2\bar{r}Y^2, \quad \|Y|]_r^2 = \sum_{\omega_1\times\omega_2^+} h^2 rY^2, \quad \|Y]]_r^2 = \sum_{\omega_1^+\times\omega_2^+} h^2\bar{r}Y^2,$$

$$|Y|^2_{1,\omega,r} = \|Y_{\bar{x}_1}]|_r^2 + \|Y_{\bar{x}_2}|]_r^2, \quad \|Y\|^2_{1,\omega,r} = |Y|^2_{1,\omega,r} + \|Y\|_r^2,$$

$$|Y|^2_{2,\omega,r} = \|Y_{\bar{x}_1x_1}\|_r^2 + \|Y_{\bar{x}_2x_2}\|_r^2 + 2\|Y_{\bar{x}_1\bar{x}_2}]]_r^2, \quad \|Y\|^2_{W_2^2(\omega,r)} = |Y|^2_{2,\omega,r} + \|Y\|^2_{1,\omega,r}.$$

Inner product and norm, involving ρ in index will make similar to the expression with index r sense.

Denote by $\overset{o}{H} = \overset{o}{H}(\bar{\omega})$ the set of grid functions $V(x)$, given on $\bar{\omega}$ and satisfying conditions

$$V(x) = 0,\ x \in \gamma_0, \quad V(1,x_2) = \sum_{k=1}^{m} \alpha_k V(\xi_k, x_2), \quad x_2 \in \omega_2. \tag{2.5}$$

3 Finite Difference Method

At the *first stage*, we approximate problem (2.1), (2.2) with the finite-difference scheme

$$\mathcal{L}_h\tilde{U} = \tilde{\varphi}(x), \quad x \in \omega, \quad \tilde{U} \in \overset{\circ}{H}, \tag{3.1}$$

where

$$\mathcal{L}_hY := -a_{11}Y_{\bar{x}_1x_1} - a_{22}Y_{\bar{x}_2x_2} - 2a_{12}Y_{\overset{\circ}{x}_1\overset{\circ}{x}_2} + a_2Y_{\overset{\circ}{x}_2} + a_0Y$$

and $\tilde{\varphi} = T_1T_2f$ is the average of function f.

Theorem 3.1. *The finite-difference scheme (3.1) is uniquely solvable.*

Indeed, using Lemma 4.2 and the result

$$|Y|^2_{1,\omega,\rho} \geq 8\|Y\|^2_\rho,$$

implied from the following inequality

$$\sum_{\omega_2^+} hY^2_{\bar{x}_2} \geq 8\sum_{\omega_2} hY^2,$$

we have

$$(\mathcal{L}_hY, Y)_\rho \geq (8\nu_1/9)\|Y\|^2_{1,\omega,\rho}, \quad Y \in \overset{\circ}{H}\ . \tag{**}$$

Therefore the operator $\mathcal{L}_h$ is positive definite in $\overset{\circ}{H}$, which confirms the validity of the Theorem 3.1.

Theorem 3.2. *Let the solution u of problem (2.1), (2.2) belong to the space $W_2^s(\Omega)$, $s \geq 2$. Then the convergence rate of the finite difference scheme (3.1) in the discrete W_2^2-norm is defined by the estimate*

$$\|\tilde{U} - u\|_{W_2^2(\omega,r)} \leq ch^{s-2}\|u\|_{W_2^s(\Omega)}, \quad 2 \leq s \leq 4.$$

The proof of the Theorem 3.2 with minor differences is similar to that given in [13].

Let

$$\mathcal{P}_\alpha \mathcal{Y} := \begin{cases} Y_{\bar{x}_\alpha x_\alpha x_\alpha} - \dfrac{h_\alpha}{2} Y_{\bar{x}_\alpha x_\alpha x_\alpha x_\alpha}, & x_\alpha = h_\alpha, \\ Y_{\bar{x}_\alpha x_\alpha \overset{\circ}{x}_\alpha}, & x_\alpha \in \omega_\alpha \setminus \{h_\alpha, l_\alpha - h_\alpha\}, \\ Y_{\bar{x}_\alpha x_\alpha \bar{x}_\alpha} + \dfrac{h_\alpha}{2} Y_{\bar{x}_\alpha x_\alpha \bar{x}_\alpha \bar{x}_\alpha}, & x_\alpha = l_\alpha - h_\alpha. \end{cases}$$

It is easy to verify that

$$\mathcal{P}_\alpha \mathcal{Y} = (\mathcal{Q}_\alpha \mathcal{V})_{\bar{\S}_\alpha}, \ \S_\alpha \in \omega_\alpha,$$

where

$$\mathcal{Q}_\alpha \mathcal{Y} := \begin{cases} (I_\alpha Y)_{x_\alpha x_\alpha} - h_\alpha Y_{x_\alpha x_\alpha x_\alpha} + \dfrac{h_\alpha^2}{2} Y_{x_\alpha x_\alpha x_\alpha x_\alpha}, & x_\alpha = 0, \\ (I_\alpha Y)_{\bar{x}_\alpha x_\alpha}, & x_\alpha \in \omega_\alpha \setminus \{l_\alpha - h_\alpha\}, \\ (I_\alpha Y)_{\bar{x}_\alpha \bar{x}_\alpha} + h_\alpha Y_{\bar{x}_\alpha \bar{x}_\alpha x_\alpha} + \dfrac{h_\alpha^2}{2} Y_{\bar{x}_\alpha \bar{x}_\alpha \bar{x}_\alpha x_\alpha}, & x_\alpha = l_\alpha - h_\alpha, \end{cases}$$

At the *second (refinement) stage*, we use the earlier-found solution $\tilde{U}$ of the finite difference scheme (3.1), define the correcting addend $\mathcal{R}\tilde{\mathcal{U}}$ and solve the difference scheme

$$\mathcal{L}_h U = \varphi, \quad x \in \omega, \quad U \in \overset{\circ}{H}, \quad \varphi = \tilde{\varphi} + \mathcal{R}\tilde{\mathcal{U}}, \tag{3.2}$$

where

$$\mathcal{R}Y := \frac{h^2}{12}(a_{11} + a_{22}) Y_{\bar{x}_1 x_1 \bar{x}_2 x_2} - \frac{h^2}{6} a_{12} (\mathcal{P}_1 Y_{\overset{\circ}{x}_2} + \mathcal{P}_2 Y_{\overset{\circ}{x}_1}) + \\ + \frac{h^2}{12} a_2 (\mathcal{P}_2 Y - Y_{\overset{\circ}{x}_2 \bar{x}_1 x_1}) - \frac{h^2}{12} a_0 (Y_{\bar{x}_1 x_1} + Y_{\bar{x}_2 x_2}). \tag{3.3}$$

Theorem 3.3. *Let the solution U of problem (2.1), (2.2) belong to the Sobolrv space $W_2^s(\Omega)$, $s \geq 2$. Then the convergence rate of the corrected difference scheme (3.2) in the discrete L_2-norm is defined by the estimate*

$$\|U - u\|_{L_2(\omega, r)} \leq c h^s \|u\|_{W_2^s(\Omega)}, \quad 2 \leq s \leq 4.$$

The validity of the Theorem 3.3 is implied from (4.20), by estimating the addends of its right hand side by well-known methodology [6], based on the generalized Bramble-Hilbert Lemma [17], [18].

4 Auxiliary Estimates

Lemma 4.1. [13] *For each function $Y(x)$, defined on mesh $\bar{\omega}$, which equals zero on $x_1 = 0$ and satisfies the nonlocal condition from (2.5) the following inequalities*

$$-\sum_{\omega_1} h\rho Y_{\bar{x}_1 x_1} Y \geq \sum_{\omega_1^+} h\bar{\rho} Y_{\bar{x}_1}^2, \tag{4.1}$$

$$\sum_{\omega_1} hrY^2 \leq 4\sum_{\omega_1^+} h\bar{r}(Y_{\bar{x}_1})^2 \tag{4.2}$$

hold.

Lemma 4.2. *For any grid function $Y \in \overset{\circ}{H}$ the equality*

$$(\mathcal{L}_h Y, Y)_\rho \geq \nu_1 |Y|^2_{1.\omega,\rho} \tag{4.3}$$

holds.

Proof. It is not hard to verify that

$$-\sum_{\omega} h^2\rho Y_{\overset{\circ}{x}_1\overset{\circ}{x}_2} Y = \frac{1}{4}\sum_{\omega_1^+\times\omega_2^+} h^2\rho Y_{\bar{x}_1}Y_{\bar{x}_2} + \frac{1}{4}\sum_{\omega_1^+\times\omega_2^-} h^2\rho Y_{\bar{x}_1}Y_{x_2} +$$

$$+\frac{1}{4}\sum_{\omega_1^-\times\omega_2^+} h^2\rho Y_{x_1}Y_{\bar{x}_2} + \frac{1}{4}\sum_{\omega_1^-\times\omega_2^-} h^2\rho Y_{x_1}Y_{x_2}, \tag{4.4}$$

$$-\sum_{\omega} h^2\rho Y_{\bar{x}_2 x_2} Y = \frac{1}{2}\sum_{\omega_1\times\omega_2^+} h^2\rho (Y_{\bar{x}_2})^2 + \frac{1}{2}\sum_{\omega_1\times\omega_2^-} h^2\rho (Y_{x_2})^2, \tag{4.5}$$

$$-\sum_{\omega} h^2\rho Y_{\bar{x}_1 x_1} Y \geq \sum_{\omega_1^+\times\omega_2} h^2\bar{\rho}(Y_{\bar{x}_1})^2 =$$

$$= \frac{1}{2}\sum_{\omega_1^+\times\omega_2} h^2\rho (Y_{\bar{x}_1})^2 + \frac{1}{2}\sum_{\omega_1^-\times\omega_2} h^2\rho (Y_{x_1})^2, \tag{4.6}$$

$$(Y_{\overset{\circ}{x}_2}, Y)_\rho = 0. \tag{4.7}$$

Hence

$$(\mathcal{L}_h Y, Y)_\rho \geq \frac{1}{4}\sum_{\omega_1^+\times\omega_2^+} h^2\rho\big(a_{11}(Y_{\bar{x}_1})^2 + 2a_{12}Y_{\bar{x}_1}Y_{\bar{x}_2} + a_{22}(Y_{\bar{x}_2})^2\big) +$$

$$+\frac{1}{4}\sum_{\omega_1^+\times\omega_2^-} h^2\rho\big(a_{11}(Y_{\bar{x}_1})^2 + 2a_{12}Y_{\bar{x}_1}Y_{x_2} + a_{22}(Y_{x_2})^2\big) +$$

$$+\frac{1}{4}\sum_{\omega_1^-\times\omega_2^+} h^2\rho\big(a_{11}(Y_{x_1})^2 + 2a_{12}Y_{x_1}Y_{\bar{x}_2} + a_{22}(Y_{\bar{x}_2})^2\big) +$$

$$+\frac{1}{4}\sum_{\omega_1^-\times\omega_2^-} h^2\rho\big(a_{11}(Y_{x_1})^2 + 2a_{12}Y_{x_1}Y_{x_2} + a_{22}(Y_{x_2})^2\big)$$

and by the condition of ellipticity we have:

$$(\mathcal{L}_h Y, Y)_\rho \geq \frac{\nu_1}{4}\Big[\sum_{\omega_1^+\times\omega_2^+} h^2\rho\big((Y_{\bar{x}_1})^2 + (Y_{\bar{x}_2})^2\big) + \sum_{\omega_1^+\times\omega_2^-} h^2\rho\big((Y_{\bar{x}_1})^2 + (Y_{x_2})^2\big) +$$

$$+\sum_{\omega_1^-\times\omega_2^+} h^2\rho\big((Y_{x_1})^2 + (Y_{\bar{x}_2})^2\big) + \sum_{\omega_1^-\times\omega_2^-} h^2\rho\big((Y_{x_1})^2 + (Y_{x_2})^2\big)\Big],$$

which proves (4.3).

Lemma 4.3. *For any grid function* $Y \in \overset{\circ}{H}$ *the following inequality*

$$c_1\|\mathcal{L}_h Y\|_\rho \geq \nu |Y|_{2,\rho,\omega}, \quad c_1 = 2(1 + |a_2|/\nu) \tag{4.8}$$

holds.

Proof. We have

$$(\mathcal{L}_h Y, \Lambda Y)_\rho = J_1 + J_2 + J_3, \tag{4.9}$$

where

$$J_1 := \sum_\omega h^2 \rho \Big(a_{11} Y^2_{\bar{x}_1 x_1} + 2a_{12} Y_{\overset{\circ}{x}_1 \overset{\circ}{x}_2} Y_{\bar{x}_1 x_1} + a_{22} Y^2_{\overset{\circ}{x}_1 \overset{\circ}{x}_2} \Big) +$$

$$+ \sum_\omega h^2 \rho \Big(a_{11} Y^2_{\overset{\circ}{x}_1 \overset{\circ}{x}_2} + 2a_{12} Y_{\overset{\circ}{x}_1 \overset{\circ}{x}_2} Y_{\bar{x}_2 x_2} + a_{22} Y^2_{\bar{x}_2 x_2} \Big),$$

$$J_2 := (a_{11} + a_{22}) \sum_\omega h^2 \rho \Big(Y_{\bar{x}_1 x_1} Y_{\bar{x}_2 x_2} - Y^2_{\overset{\circ}{x}_1 \overset{\circ}{x}_2} \Big),$$

$$J_3 := a_2 (Y_{\overset{\circ}{x}_2}, \Lambda Y)_\rho + a_0 (Y, \Lambda Y)_\rho.$$

According to the condition of ellipticity we receive

$$J_1 \geq \nu_1 \big(\|Y_{\bar{x}_1 x_1}\|^2_\rho + \|Y_{\bar{x}_2 x_2}\|^2_\rho + 2\|Y_{\overset{\circ}{x}_1 \overset{\circ}{x}_2}\|^2_\rho \big). \tag{4.10}$$

By partial summing up and using (4.1) we obtain

$$\sum_\omega h^2 \rho Y_{\bar{x}_1 x_1} Y_{\bar{x}_2 x_2} = - \sum_{\omega_1^+ \times \omega_2} h^2 \rho Y_{\bar{x}_1 x_1 \bar{x}_2} Y_{\bar{x}_2} \geq \sum_{\omega_1^+ \times \omega_2^+} h^2 \bar{\rho} Y^2_{\bar{x}_1 \bar{x}_2}. \tag{4.11}$$

On the other hand,

$$\|Y_{\overset{\circ}{x}_1 \overset{\circ}{x}_2}\|^2_\rho \leq (1/16) \|Y_{\bar{x}_1 \bar{x}_2} + Y_{\bar{x}_1 x_2} + Y_{x_1 \bar{x}_2} + Y_{x_1 x_2}\|^2_\rho \leq \|Y_{\bar{x}_1 \bar{x}_2}\|^2_{\bar{\rho}}. \tag{4.12}$$

By (4.11),(4.12) we have

$$J_2 \geq 0. \tag{4.13}$$

Besides,

$$J_3 \geq -|a_2| (Y_{\overset{\circ}{x}_2}, \Lambda Y)_\rho \geq -|a_2| \|Y_{\bar{x}_2}\|_\rho \, \|\Lambda Y\|_\rho. \tag{4.14}$$

Using estimates (4.10), (4.13), (4.14) from (4.9) we obtain

$$\nu \big(\|Y_{\bar{x}_1 x_1}\|^2_\rho + \|Y_{\bar{x}_2 x_2}\|^2_\rho \big) \leq \|\mathcal{L}_h Y\|_\rho \|\Lambda Y\|_\rho + |a_2| \, \|Y_{\bar{x}_2}\|_\rho \|\Lambda Y\|_\rho. \tag{4.15}$$

Notice, that

$$\nu \|Y_{\bar{x}_2}\|_\rho \leq \|\mathcal{L}_h Y\|_\rho.$$

Indeed, this is implied from the relationship

$$\|Y_{\bar{x}_2}\|_\rho \leq (1/\nu)(\mathcal{L}_h Y, Y)_\rho \leq (1/\nu)\|\mathcal{L}_h Y\|_\rho \|Y\|_\rho \leq (1/\nu)\|\mathcal{L}_h Y\|_\rho \|Y_{\bar{x}_2}\|_\rho.$$

Therefore

$$\nu \big(\|Y_{\bar{x}_1 x_1}\|^2_\rho + \|Y_{\bar{x}_2 x_2}\|^2_\rho \big) \leq (1 + |a_2|/\nu) \|\mathcal{L}_h Y\|_\rho \|\Lambda Y\|_\rho. \tag{4.16}$$

Further, from (4.11) we have

$$2\|Y_{\bar{x}_1\bar{x}_2}\|_\rho^2 \le \big(\|Y_{\bar{x}_1x_1}\|_\rho^2 + \|Y_{\bar{x}_2x_2}\|_\rho^2\big),$$

which with (4.16) gives

$$\nu\big(\|Y_{\bar{x}_1x_1}\|_\rho^2 + 2\|Y_{\bar{x}_1\bar{x}_2}\|_\rho^2 + \|Y_{\bar{x}_2x_2}\|_\rho^2\big) \le 2(1+|a_2|/\nu)\|\mathcal{L}_hY\|_\rho\|\Lambda Y\|_\rho.$$

Hence the validity of (4.8) is obvious. Lemma 4.3 is proved.

Let $\tilde{Z} = \tilde{U} - u$ and $Z = U - u$ represent the errors of schemes (3.1) and (3.2) respectively.

From (3.2) it follows that

$$\mathcal{L}_hZ = \mathcal{L}_hU - \mathcal{L}_hu = \tilde{\varphi} + \mathcal{R}\tilde{U} - \mathcal{L}_hu = T_1T_2\mathcal{L}u + \mathcal{R}u - \mathcal{L}_hu + \mathcal{R}\tilde{Z}. \qquad (4.17)$$

Using the equation

$$T_1T_2\mathcal{L}u = -\sum_{\alpha,\beta=1}^{2} a_{\alpha\beta}T_1T_2\big(\frac{\partial^2u}{\partial x_\alpha\partial x_\beta}\big) + a_2T_1T_2\frac{\partial u}{\partial x_2} + a_0T_1T_2u$$

and the expressions of the operators $\mathcal{L}_h$, $\mathcal{R}$, we have

$$T_1T_2\mathcal{L}u + \mathcal{R}u - \mathcal{L}_hu = a_{11}\psi_{11} + a_{22}\psi_{22} + 2a_{12}\psi_{12} + a_2\psi_2 + a_0\psi_0, \qquad (4.18)$$

where

$$\psi_{\alpha\alpha} := u_{\bar{x}_\alpha x_\alpha} + \frac{h^2}{12}u_{\bar{x}_1x_1\bar{x}_2x_2} - T_1T_2\frac{\partial^2u}{\partial x_\alpha^2}, \quad \alpha = 1,2,$$

$$\psi_{12} := u_{\overset{\circ}{x}_1\overset{\circ}{x}_2} - \frac{h^2}{12}(\mathcal{P}_1u_{\overset{\circ}{x}_2} + \mathcal{P}_2u_{\overset{\circ}{x}_1}) - T_1T_2\frac{\partial^2u}{\partial x_1\partial x_2},$$

$$\psi_2 := T_1T_2\frac{\partial u}{\partial x_2} - u_{\overset{\circ}{x}_2} + \frac{h^2}{12}(\mathcal{P}_2u - u_{\overset{\circ}{x}_2\bar{x}_1x_1}),$$

$$\eta_0 := T_1T_2u - u - \frac{h^2}{12}(u_{\bar{x}_1x_1} + u_{\bar{x}_2x_2}).$$

After properly transforming the addends on the right-hand side of (4.18) and using them in (4.17), we get

$$\mathcal{L}_hZ = \mathcal{R}\tilde{Z} + \psi \qquad (4.19)$$

where

$$\psi := a_{11}(\eta_{11})_{\bar{x}_1x_1} + a_{22}(\eta_{22})_{\bar{x}_2x_2} + 2a_{12}(\eta_{12})_{\bar{x}_1\bar{x}_2} + a_2(\eta_2)_{\bar{x}_2} + a_0\eta_0,$$

and

$$\eta_{\alpha\alpha} := u + \frac{h^2}{12}u_{\bar{x}_\beta x_\beta} - T_\beta u, \beta = 3-\alpha, \quad \alpha = 1,2,$$

$$\eta_{12} := I_1I_2u - \frac{h^2}{12}(\mathcal{Q}_1I_2u + \mathcal{Q}_2I_1u) - S_1^+S_2^+u,$$

$$\eta_2 := T_1S^+u - I_2u + \frac{h^2}{12}(\mathcal{Q}_2u - I_2u_{\bar{x}_1x_1}).$$

From (4.19) we have

$$\|Z\|_\rho \le \|\mathcal{L}^{-1}\psi\|_\rho + \|\mathcal{L}^{-1}\mathcal{R}\tilde{Z}\|_\rho.$$

Lemma 4.4. *For the solution of problem (4.19) the following a priori estimate*

$$\|Z\|_\rho \le c(J(u) + h^2\|\tilde{Z}\|_{W_2^2(\omega,\rho)}) \tag{4.20}$$

holds, where

$$J(u) = \|\eta_{11}\|_\rho + \|\eta_{22}\|_\rho + \|\eta_{12}\|_\rho + \|\eta_2\| + \|\eta_0\|_\rho.$$

The estimate (4.20) is obtained from (4.19) by use the result implied from the Lemma 4.3 (compare with [19]):

$$\|\mathcal{L}^{-\infty}\mathcal{Y}\|_\rho \le \rfloor\|\ast^{-\infty}\mathcal{Y}\|_\rho.$$

5 Numerical experiments

The operators $\mathcal{P}_\alpha\mathcal{Y}$, and, therefore, the correcting operator $\mathcal{R}\mathcal{Y}$, are calculated differently on the strictly internal and near-boundary nodes of the mesh ω. Particularly,

$$(\mathcal{P}_1Y)_{1,j} = \frac{1}{2h^3}(6Y_{3,j} - 12Y_{2,j} + 10Y_{1,j} - 3Y_{0,j} - Y_{4,j})$$

and

$$(\mathcal{P}_1Y)_{n-1,j} = \frac{1}{2h^3}(3Y_{n,j} - 10Y_{n-1,j} + 12Y_{n-2,j} - 6Y_{n-3,j} + Y_{n-4,j}).$$

Introducing fictitious nodes beyond the mesh $\bar{\omega}$ and denoting

$$Y_{-1,j} = Y_{4,j} - 5(Y_{3,j} - 2Y_{2,j} + 2Y_{1,j} - Y_{0,j}),$$

$$Y_{n+1,j} = Y_{n-4,j} + 5(Y_{n,j} - 2Y_{n-1,j} + 2Y_{n-2,j} - Y_{n-3,j}),$$

we have $\mathcal{P}_1Y = Y_{\bar{x}_1 x_1 \mathring{x}_1}$, $x_1 \in \omega_1$.

Analogously, denoting

$$Y_{i,-1} = Y_{i,4} - 5(Y_{i,3} - 2Y_{i,2} + 2Y_{i,1} - Y_{i,0}),$$

$$Y_{i,n+1} = Y_{i,n-4} + 5(Y_{i,n} - 2Y_{i,n-1} + 2Y_{i,n-2} - Y_{i,n-3}),$$

we have $\mathcal{P}_2Y = Y_{\bar{x}_2 x_2 \mathring{x}_2}$, $x_2 \in \omega_2$.

Now, we present some numerical results to demonstrate the convergence order of the proposed method. The experimental order of convergence in the discrete $L_2(\omega, r)$ and $L_2(\omega)$ norms are computed by formulas

$$Ord(Y) = \log_2 \frac{\|Y_h - u\|_r}{\|Y_{h/2} - u\|_r},\ Ord(Y) = \log_2 \frac{\|Y_h - u\|}{\|Y_{h/2} - u\|},$$

where u is the exact solution of the original problem, while Y_h denotes the solution of the difference scheme on the grid with step h.

Below, in the examples the symbols $\tilde{U}$, U denote solutions of the difference schemes (3.1) and (3.2), respectively.

Example 1. Consider the problem

$$\begin{gathered}-\Delta u+3\frac{\partial u}{\partial x_1}+2\frac{\partial u}{\partial x_2}+\pi^2u=f,\ \ x\in(0,1)^2,\ u\big|_{\Gamma_0}=0,\\ f(x)=2\pi\sin\left(\frac{2\pi x_1}{3}+\pi x_2\right)+\frac{22\pi^2}{9}\sin\left(\frac{2\pi x_1}{3}\right)\sin(\pi x_2).\end{gathered}\tag{5.1}$$

with nonlocal conditions

$$u(1,x_2)=u(0.5,x_2),\quad 0<x_2<1,$$

The right-hand side of the scheme is calculated by the formula

$$\varphi(x)=T_1T_2f=\lambda_1^2\lambda_2^2f,\quad \lambda_1=\frac{3}{\pi h}\sin\left(\frac{\pi h}{3}\right),\quad \lambda_2=\frac{2}{\pi h}\sin\left(\frac{\pi h}{2}\right).$$

The exact solution $u(x)=\sin(\frac{2\pi x_1}{3})\sin(\pi x_2)$ belongs to the space W_2^4 and therefore, for the refined scheme we expect the fourth order of convergence.

The results of calculations are given in Tables 1, 2.

Example 2. Consider the problem (5.1) with nonlocal conditions

$$u(1,x_2)=\alpha_1u(\xi_1,x_2))+\alpha_2u(\xi_2,x_2),$$

where

$$\xi_1=1/4,\ \xi_2=1/2,\ \alpha_1=\sqrt{3}/2,\ \alpha_2=1/2.$$

The exact solution is the same as in Example 1.

The results of calculations are given in Tables 3, 4.

The results of numerical experiments justify the expected order of convergence of the method.

Table 1. Experimental order of convergence in $L_2(\omega,r)$-norm for Example 1

h	$\|\tilde{U}_h-u\|_r$	$\|U_h-u\|_r$	$Ord(\tilde{U})$	$Ord(U)$
1/8	3.60780396 e−03	4.89513443 e−05		
			1.9969	4.0028
1/16	9.03874819 e−04	3.05348012 e−06		
			1.9993	3.9917
1/32	2.26082170 e−04	1.91946316 e−07		
			1.9998	3.9970
1/64	5.65275241 e−05	1.20212101 e−08		
			2.0000	3.9993
1/128	1.41323156 e−05	7.51699740 e−10		

Table 2. Experimental order of convergence in $L_2(\omega)$-norm for Example 1

h	$\|\tilde{U}_h - u\|$	$\|U_h - u\|$	$Ord(\tilde{U})$	$Ord(U)$
1/8	5.98636695 $e-03$	8.04010806 $e-05$		
1/16	1.53111837 $e-03$	5.16844875 $e-06$	1.9671	3.9594
1/32	3.86778167 $e-04$	3.29285759 $e-07$	1.9850	3.9723
1/64	9.71760586 $e-05$	2.07435515 $e-08$	1.9928	3.9886
1/128	2.43530927 $e-05$	1.30058111 $e-09$	1.9965	3.9954

Table 3. Experimental order of convergence in $L_2(\omega, r)$-norm for Example 2

h	$\|\tilde{U}_h - u\|_r$	$\|U_h - u\|_r$	$Ord(\tilde{U})$	$Ord(U)$
1/8	3.59991251 $e-03$	4.90450435 $e-05$		
1/16	9.01826711 $e-04$	3.05349689 $e-06$	1.9970	4.0056
1/32	2.25565432 $e-04$	1.91845159 $e-07$	1.9993	3.9924
1/64	5.63980450 $e-05$	1.20132445 $e-08$	1.9998	3.9972
1/128	1.40999274 $e-05$	7.51176468 $e-10$	2.0000	3.9993

Table 4. Experimental order of convergence in $L_2(\omega)$-norm for Example 2

h	$\|\tilde{U}_h - u\|$	$\|U_h - u\|$	$Ord(\tilde{U})$	$Ord(U)$
1/8	5.96418197 $e-03$	8.06214740 $e-05$		
1/16	1.52442950 $e-03$	5.16162822 $e-06$	1.9681	3.9653
1/32	3.84953454 $e-04$	3.28334711 $e-07$	1.9855	3.9746
1/64	9.67002688 $e-05$	2.06718320 $e-08$	1.9931	3.9894
1/128	2.42316616 $e-05$	1.29579931 $e-09$	1.9966	3.9958

Acknowledgment

The work was supported by the Shota Rustaveli National Science Foundation (Grant FR/406/5-106/12).

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