BOUNDARY VALUE PROBLEMS OF STATICS OF TWO-TEMPERATURE ELASTIC MIXTURES THEORY FOR A HALF-SPACE

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Abstract

The case of statics of the two-temperature elastic mixtures theory is considered, when partial displacements of the elastic components of the mixture are equal to each other.

We consider boundary value problems of statics of the two-temperature elastic mixture for a half-space, when limiting values of the normal components of displacement, temperature and tangential components of rotation vectors are given on the boundary $x_3 = 0$. Solutions are represented in quadratures.

Key words and phrases: Elastic mixture theory, uniqueness theorem.

AMS subject classification: 74G20, 74E30, 74G05.

1 Introduction

In this paper we develop a new approach to the Dirichlet and Neumann type boundary value problems for the two-temperature elastic mixture theory for a half-space. Solutions are presented in quadratures.

Similar problems are considered in the references J. Barber [1], M. Basheleishvili, L. Bitsadze [2], D. Burchuladze, M. Kharashvili, K. Skhvitaridze [3], E. Constantin, N. Pavel [4], L.Giorgashvili, K. Skhvitaridze, M. Kharashvili [5], L. Giorgashvili, E. Elerdashvili, M. Kharashvili, K. Skhvitaridze [6], R. Kumar, T. Chadha [8], H. Sherief, H.Saleh [10], B. Singh, R. Kumar [11], K. Skhvitaridze, M. Kharashvili [12].

2 Statement of boundary value problems. Uniqueness theorems

When the two partial displacements of two elastic components of the mixture are equal, a homogeneous system of static differential equations of the theory of two-temperature elastic mixtures has the form [7]

$$\mu\Delta u(x) + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u(x) + \mathrm{grad}(\eta_1\vartheta_1(x) + \eta_2\vartheta_2(x)) = 0, \tag{2.1}$$

$$\begin{aligned} &\varkappa_1 \Delta\vartheta_1(x) + \varkappa_2 \Delta\vartheta_2(x) + \alpha(\vartheta_2(x) - \vartheta_1(x)) = 0, \\ &\varkappa_2 \Delta\vartheta_1(x) + \varkappa_3 \Delta\vartheta_2(x) - \alpha(\vartheta_2(x) - \vartheta_1(x)) = 0, \end{aligned} \tag{2.2}$$

where $u = (u_1, u_2, u_3)^\top$ is the displacement vector, $\vartheta_1,\ \vartheta_2$ are temperature functions. $\alpha,\ \varkappa_j,\ \ j = 1, 2, 3$ are physical constants; $\lambda,\ \mu$ are elastic constants; $\eta_1,\ \eta_2$ are coupling constants for which the following inequalities are valid,

$$\mu > 0,\ \ 3\lambda + 2\mu > 0,\ \ \varkappa_1\varkappa_3 - \varkappa_2^2 > 0,\quad \eta_j > 0,\quad j = 1, 2,$$

$\top$ is the transposition, Δ is the three-dimensional Laplace operator.

Denote by Ω^- a half-space $x_3 > 0$, and by $\partial\Omega$ its boundary plane $x_3 = 0$.

Problem (A). Find, in the domain Ω^-, a regular solution $U = (u, \vartheta_1, \vartheta_2)^\top \in C^2(\Omega^-) \bigcap C^1(\overline{\Omega^-})$ of system (2.1)-(2.2) such that on the boundary $\partial\Omega$ one of the following group of boundary conditions is fulfilled:

$$\{n(z) \cdot u(z)\}^- = f_3(z),\quad \{n(z) \times \operatorname{rot} u(z)\}^- = f(z), \tag{2.3}$$

$$\{\vartheta_1(z)\}^- = f_4(z),\quad \{\vartheta_2(z)\}^- = f(z), \tag{2.4}$$

or

$$\left\{\frac{\partial\vartheta_1(z)}{\partial n(z)}\right\}^- = f_4(z),\quad \left\{\frac{\partial\vartheta_2(z)}{\partial n(z)}\right\}^- = f_5(z), \tag{2.5}$$

where $f = (f_1, f_2, F_3)^\top,\ F_3,\ f_j,\ j = 1, 2, 3, 4, 5$ are the functions given on the boundary $\partial\Omega$ $z = (z_1, z_2, 0)$,

$$\frac{\partial}{\partial n(x)} = \sum_{k=1}^{3} n_k \frac{\partial}{\partial x_k},$$

in neighborhood of infinity the vector $U = (u, \vartheta_1, \vartheta_2)$ satisfies the following conditions:

$$\begin{gathered} u(x) = O(1),\ \ \vartheta_j(x) = O(|x|^{-1}),\quad j = 1, 2,\ \ |x| \to \infty, \\ \lim_{R\to\infty} \frac{1}{2\pi R^2} \int\limits_{\Sigma_R} n(x) \cdot u(x) d\Sigma_R = 0, \end{gathered} \tag{2.6}$$

Σ_R is the part of the boundary of the ball $B(O, R) = \{y \in \mathbb{R}^3 : |y| < R\}$ which lies in the domain $x_3 > 0$.

We denote the problems with boundary conditions (2.3), (2.4) and (2.3), (2.5) respectively by $(A.I)$ and $(A.II)$.

Theorem 2.1.T*he problems* $(A.I)$ *and* $(A.II)$ *have at most one solution.*

Proof. The theorem will be proved if we show that corresponding the homogeneous problems ($f = 0,\ \ f_j = 0,\ \ j = 3, 4, 5$) have only the trivial solutions.

Denote by $\Omega_R := \Omega^- \cap B(O, R)$ with $R > 0$. Denote by $\Sigma_R = \partial\Omega_R$ that part of the boundary of the ball $B(O, R)$ which lies in the domain $x_3 > 0$ and by $S(O, R)$ the circle with center at the origin and radius R which lies in the plane $x_3 = 0$.

Using the Stokes formula from system (2.2) we obtain

$$\int\limits_{\partial\Omega_R}\Big[(\varkappa_1\vartheta_1(x)+\varkappa_2\vartheta_2(x))\frac{\partial\vartheta_1(x)}{\partial n(x)}+(\varkappa_2\vartheta_1(x)+\varkappa_3\vartheta_2(x))\frac{\partial\vartheta_2(x)}{\partial n(x)}\Big]ds$$

$$-\int\limits_{\overline{S(O,R)}}\Big[(\varkappa_1\vartheta_1(z)+\varkappa_2\vartheta_2(z))\frac{\partial\vartheta_1(z)}{\partial n(z)}+(\varkappa_2\vartheta_1(z)+\varkappa_3\vartheta_2(z))\frac{\partial\vartheta_2(z)}{\partial n(z)}\Big]ds$$

$$-\int\limits_{\Omega_R}\Big[\varkappa_1|\operatorname{grad}\vartheta_1(x)|^2+2\varkappa_2\operatorname{grad}\vartheta_1(x)\cdot\operatorname{grad}\vartheta_2(x)+\varkappa_3|\operatorname{grad}\vartheta_2(x)|^2$$

$$+\alpha(\vartheta_1(x)-\vartheta_2(x))^2\Big]\,dx=0. \tag{2.7}$$

Passing to the limit in both sides of equality (2.7) as $R\to+\infty$ and taking into consideration the boundary conditions of the homogeneous problems $(A.I)_0$ and $(A.II)_0$ as well as the asymptotic representations (2.6), we obtain

$$\int\limits_{\Omega^-}\Big[\varkappa_1|\operatorname{grad}\vartheta_1(x)|^2+2\varkappa_2\operatorname{grad}\vartheta_1(x)\cdot\operatorname{grad}\vartheta_2(x)+\varkappa_3|\operatorname{grad}\vartheta_2(x)|^2+$$

$$+\alpha(\vartheta_1(x)-\vartheta_2(x))^2\Big]\,dx=0.$$

This relation implies $\vartheta_1(x)=\vartheta_2(x)=C=const$, $x\in\Omega^-$.

Since $\vartheta_j(x)\to 0$ as $|x|\to\infty$, we have $C=0$, i. e. $\vartheta_j(x)=0$, $j=1,2$, $x\in\Omega^-$. Thus for the vector $u(x)$ we obtain the following problem

$$\mu\Delta u(x)+(\lambda+\mu)\operatorname{grad}\operatorname{div}u(x)=0,\quad x\in\Omega^-,$$
$$\{u(z)\cdot n(z)\}^-=0,\quad \{n(z)\times\operatorname{rot}u(z)\}^-=0.$$

These problem has only a trivial solution (for details see [9]). Thus $U(x)=0$, $x\in\Omega^-$. □

3 Solution of the boundary value problems

Taking into consideration in the boundary conditions (2.3)-(2.4) that $n(z)=(0,0,1)^\top$, then these boundary conditions can be rewritten as follows:

$$\{u_3(z)\}^-=f_3(z),\quad \left\{\frac{\partial u_j(z)}{\partial x_3}\right\}^-=\frac{\partial f_3(z)}{\partial z_j}-f_j(z),\quad j=1,2,\ z\in\partial\Omega, \tag{3.1}$$

$$\left\{\frac{\partial\vartheta_1(z)}{\partial x_3}\right\}^-=f_4(z),\quad \left\{\frac{\partial\vartheta_2(z)}{\partial x_3}\right\}^-=f_5(z),\quad z\in\partial\Omega, \tag{3.2}$$

where

$$\left\{\frac{\partial v(z)}{\partial x_3}\right\}^- = \lim_{\Omega^-\ni\varkappa\to z\in\partial\Omega}\frac{\partial v(x)}{\partial x_3}.$$

In view of the conditions (3.2) we get

$$\begin{aligned}&\left\{(\varkappa_1+\varkappa_2)\frac{\partial\vartheta_1(z)}{\partial x_3}+(\varkappa_2+\varkappa_3)\frac{\partial\vartheta_2(z)}{\partial x_3}\right\}^-\\&=(\varkappa_1+\varkappa_2)f_4(z)+(\varkappa_2+\varkappa_3)f_5(z),\end{aligned}\tag{3.3}$$

$$\left\{\frac{\partial\vartheta_1(z)}{\partial x_3}-\frac{\partial\vartheta_2(z)}{\partial x_3}\right\}^- = f_4(z)-f_5(z).\tag{3.4}$$

By simple transformations, from system (2.2) we find

$$\Delta[(\varkappa_1+\varkappa_2)\vartheta_1(x)+(\varkappa_2+\varkappa_3)\vartheta_2(x)]=0,\quad x\in\Omega^-,\tag{3.5}$$

$$(\Delta-\lambda_1^2)(\vartheta_1(x)-\vartheta_2(x))=0.\tag{3.6}$$

where $\lambda_1^2=\alpha d_2/d_1$, $d_1=\varkappa_1\varkappa_3-\varkappa_2^2$, $d_2=\varkappa_1+2\varkappa_2+\varkappa_3$.

The Neumann boundary value problems (3.5), (**??**) and (3.6), (3.4) have the following solutions [13]

$$\begin{aligned}&(\varkappa_1+\varkappa_2)\vartheta_1(x)+(\varkappa_2+\varkappa_3)\vartheta_2(x)\\&=-\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}\frac{1}{r}[(\varkappa_1+\varkappa_2)f_4(y)+(\varkappa_2+\varkappa_3)f_5(y)]dy_1\,dy_2,\\&\vartheta_1(x)-\vartheta_2(x)=-\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}\frac{e^{-\lambda_1 r}}{r}(f_4(y)-f_5(y))dy_1\,dy_2.\end{aligned}$$

From this equalities we derive

$$\begin{aligned}\vartheta_1(x)&=-\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}\left[\frac{1}{r}f_4(y)+\frac{\varkappa_2+\varkappa_3}{d_2}\frac{e^{-\lambda_1 r}-1}{r}(f_4(y)-f_5(y))\right]dy_1\,dy_2,\\\vartheta_2(x)&=-\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}\left[\frac{1}{r}f_5(y)-\frac{\varkappa_1+\varkappa_2}{d_2}\frac{e^{-\lambda_1 r}-1}{r}(f_4(y)-f_5(y))\right]dy_1\,dy_2,\end{aligned}\tag{3.7}$$

where $r=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+x_3^2}$.

If we substitute equalities (3.7) into (2.1), we obtain

$$\begin{aligned}
&\mu\Delta u(x)+(\lambda+\mu)\operatorname{grad}\operatorname{div}u(x)\\
&=-\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}\operatorname{grad}\Big[\frac{1}{r}(\eta_1 f_4(y)\eta_2 f_5(y))+(\eta_1(\varkappa_2+\varkappa_3)\\
&+-\eta_2(\varkappa_1+\varkappa_2))\frac{e^{-\lambda_1 r}-1}{r}(f_4(y)-f_5(y))\Big]\,dy_1\,dy_2,
\end{aligned}\tag{3.8}$$

A general solution of system (3.12) has the form

$$u(x)=u_0(x)+\widetilde{u}(x),\quad x\in\Omega^-,\tag{3.9}$$

where $u_0(x)$ is a solution of the homogeneous system

$$\mu\Delta u(x)+(\lambda+\mu)\operatorname{grad}\operatorname{div}u(x)=0,\quad x\in\Omega^-,\tag{3.10}$$

satisfying the boundary conditions (3.1) on the boundary $\partial\Omega$.

Vector $\widetilde{u}(x)$ is a particular solution of system (3.12) satisfying the homogeneous boundary conditions $(3.1)_0$ on $\partial\Omega$.

The solution $u_0(x)$ of the problem (3.10), (3.1) can be represented in the form

$$u_0(x)=-\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}K^{(1)}(x,y)f'(y)dy_1\,dy_2,\tag{3.11}$$

where $f'=(f_1,f_2,f_3)^\top$,

$$K^{(1)}(x,y)=\Big[K^{(1)}_{lj}(x,y)\Big]_{3\times3},$$

$$K^{(1)}_{lj}(x,y)=(1-\delta_{l3})(1-\delta_{3j})\left(-\frac{1}{r}\delta_{lj}+a\frac{\partial^2 r}{\partial x_l\partial x_j}\right)$$

$$+(1-\delta_{l3})\delta_{3j}\frac{\partial}{\partial x_l}\frac{1}{r}+a(1-\delta_{3j})\delta_{l3}x_3\frac{\partial}{\partial x_j}\frac{1}{r}+\delta_{l3}\delta_{3j}\frac{\partial}{\partial x_3}\frac{1}{r},\quad a=\frac{\lambda+\mu}{2(\lambda+2\mu)}.$$

The solution $\widetilde{u}(x)$ of the problem (3.10), $(3.1)_0$ can be represented in the form

$$\begin{aligned}
\widetilde{u}(x)=&-\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}\operatorname{grad}\Big[r(\alpha_1 f_4(y)+\alpha_2 f_5(y))\\
&+\alpha_3\frac{e^{-\lambda_1 r}-1}{r}(f_4(y)-f_5(y))\Big]dy_1dy_2,
\end{aligned}\tag{3.12}$$

where

$$\alpha_1=\frac{(\eta_1+\eta_2)(\varkappa_1+\varkappa_2)}{2(\lambda+2\mu)d_2},\quad \alpha_2=\frac{(\eta_1+\eta_2)(\varkappa_1+\varkappa_2)}{2(\lambda+2\mu)d_2},$$

$$\alpha_3 = \frac{\eta_1(\varkappa_2+\varkappa_3)-\eta_2(\varkappa_1+\varkappa_2)}{(\lambda+2\mu)d_2\lambda_1^2}.$$

Substituting the expresions of the vectors $u_0(x)$ and $\widetilde{u}(x)$ into (3.9) and taking into account expressions of functions $\vartheta_1(x)$ and $\vartheta_2(x)$, finally we get

$$U(x) = -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} K^{(1)}(x,y)f(y)dy_1\, dy_2, \tag{3.13}$$

where $U=(u,\vartheta_1,\vartheta_2)^\top$, $f=(f_1,f_2,f_3,f_4,f_5)^\top$,

$$K(x,y) = \begin{bmatrix} K^{(1)}(x,y) & K^{(2)}(x,y) \\ K^{(3)}(x,y) & K^{(4)}(x,y) \end{bmatrix}_{5\times5},$$

$$K^{(2)}(x,y) = \left[K_{lj}^{(2)}(x,y)\right]_{3\times2}$$

$$K^{(4)}(x,y) = \left[K^{(4)}(x,y)\right]_{2\times2}, \quad K^{(3)}(x,y) = [0]_{2\times3}.$$

$K^{(1)}(x,y)$ is defined in (3.11)

$$K_{lj}^{(2)}(x,y) = (\delta_{1j}+\delta_{2j})\alpha_j\frac{\partial r}{\partial x_l} + \alpha_3(\delta_{1j}-\delta_{2j})\frac{e^{-\lambda_1 r}-1}{r}\frac{\partial r}{\partial x_l},$$
$$K_{lj}^{(4)}(x,y) = \delta_{lj}\frac{1}{r} + \frac{1}{d_2}(\delta_{1j}-\delta_{2j})\left(\delta_{1l}(\varkappa_2+\varkappa_3)-\delta_{2l}(\varkappa_1+\varkappa_2)\right)\frac{e^{-\lambda_1 r}-1}{r}.$$

From (3.13) we can calculate the stress vector $P(\partial,n)U(x)$,

$$P(\partial,n)U(x) = -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} L(x,y)f(y)dy_1\, dy_2, \tag{3.14}$$

where

$$L(x,y) = [L^{(1)}(x,y)L^{(2)}(x,y)],$$

$$L^{(1)}(x,y) = [L_{kj}^{(1)}(x,y)]_{3\times3}, \quad L^{(2)}(x,y) = [L_{kj}^{(1)}(x,y)]_{3\times2},$$

$$L_{kj}^{(1)}(x,y) = (1-\delta_{k3})(1-\delta_{3j})\left(-\mu\delta_{kj}\frac{\partial}{\partial x_3}\frac{1}{r} + 2\mu a x_3\frac{\partial^2}{\partial x_k\partial x_j}\frac{1}{r}\right)$$
$$+2\mu(1-\delta_{k3})\delta_{3j}\frac{\partial^2}{\partial x_k\partial x_3}\frac{1}{r} + (1-\delta_{3j})\delta_{k3}\left(\frac{\mu^2}{\lambda+2\mu}\frac{\partial}{\partial x_j}\frac{1}{r}\right.$$
$$\left.+2\mu a x_3\frac{\partial^2}{\partial x_j\partial x_3}\frac{1}{r}\right) + 2\mu\delta_{k3}\delta_{3j}\frac{\partial^2}{\partial x_3^2}\frac{1}{r}, \quad k,j=1,2,3,$$

$$L_{kj}^{(2)}(x,y) = (\delta_{1j}+\delta_{2j})\left\{\left[(2(\lambda+\mu)\alpha_j-\eta_j)\frac{1}{r}+2\mu\alpha_j x_3\frac{\partial}{\partial x_3}\frac{1}{r}\right]\delta_{k3}\right.$$
$$+2\mu\alpha_j(1-\delta_{k3})x_3\frac{\partial}{\partial x_k}\frac{1}{r}\right\}+(\delta_{1j}-\delta_{2j})\alpha\left\{\left[\lambda\lambda_1^2\frac{1}{r}\right.\right.$$
$$\left.\left.+2\mu\left(\frac{\partial^2}{\partial x_3^2}-\lambda_1^2\right)\frac{e^{-\lambda_1 r}-1}{r}\right]\delta_{k3}+2\mu(1-\delta_{k3})\frac{\partial^2}{\partial x_k\partial x_3}\frac{e^{-\lambda_1 r}-1}{r}\right\}.$$

If the boundary vector-functions satisfy the conditions

$$f_j(z)\in C^{0,\alpha}(\partial\Omega),\quad j=1,2,4,5,\quad f_3(z)\in C^{1,\beta}(\partial\Omega),\quad 0<\beta<1,$$

$$|f_j(z)|<\frac{B}{1+|z|^2},\quad j=1,2,4,5,\quad |f_3(z)|<\frac{B}{1+|z|},\quad z\in\partial\Omega,\quad B=const,$$

then the vector $U(x)$ represented by formula (3.13) is a regular solution of the problem $(A.II)$ which satisfies the following decay conditions at infinity

$$u_j(x)=O\left(|x|^{-1}\ln|x|\right),\quad \vartheta_j(x)=O\left(|x|^{-1}\ln|x|\right),\quad j=1,2,$$
$$u_3(x)=O\left(|x|^{-1}\right),\quad \frac{\partial}{\partial x_k}u_j(x)=O\left(|x|^{-2}\right),\quad \frac{\partial}{\partial x_k}\vartheta_j(x)=O\left(|x|^{-2}\right),$$
$$\frac{\partial}{\partial x_k}u_3(x)=O\left(|x|^{-2}\ln|x|\right),\quad k=1,2,\quad j=1,2.$$

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