THE METHOD OF I. VEKUA FOR THE NON-SHALLOW SPHERICAL SHELL FOR THE GEOMETRICALLY NONLINEAR THEORY

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(Received: 07.11.15; accepted: 22.12.15)

Abstract

In the present paper we consider the geometrically nonlinear and non-shallow spherical shells, when components of the deformation tensor have nonlinear terms. By means of I. Vekua's method the system of equilibrium equations in two variables is obtained. Using complex variable functions and the method of the small parameter approximate solutions are constructed for $N = 1$ in the hierarchy by I. Vekua. Concrete problem has been solved.

Key words and phrases: Non-shallow shells, geometrically nonlinear theory, small parameter, spherical shells.

AMS subject classification: 74K25.

1 Introduction

There are many different method of passage (reduction) from three- dimensional problems of elasticity to tow-dimensional problems of the theory of shells. I.Vekua had obtained the equations of shallow shells [1],[2]. It means that the interior geometry of the shell does not vary in thickness. This method for non-shallow shells in case of geometrical and physical non-linear theory was generalized by T.Meunargia [3],[4].

2 Equations of Equilibrium of an Elastic Medium

A complete system of equations of the three-dimensional nonlinear theory of elasticity can be written as:

$$\partial_i\sqrt{g}\boldsymbol{\sigma}^i + \sqrt{g}\boldsymbol{\Phi} = 0, \quad \Big(\partial_i = \frac{\partial}{\partial x^i}\Big),$$

$$\boldsymbol{\sigma}^i = \lambda\Big(\boldsymbol{R}^j\partial_j\boldsymbol{U} + \frac{1}{2}\partial^j\boldsymbol{U}\partial_j\boldsymbol{U}\Big)\Big(\boldsymbol{R}^i + \partial^i\boldsymbol{U}\Big)$$
$$+\mu\Big(\boldsymbol{R}^i\partial^j\boldsymbol{U} + \boldsymbol{R}^j\partial^i\boldsymbol{U} + \partial^i\boldsymbol{U}\partial^j\boldsymbol{U}\Big)\Big(\boldsymbol{R}_j + \partial_j\boldsymbol{U}\Big),$$

where x^1, x^2, x^3 are curvilinear coordinates, g is the discriminant of the metric tensor of the space, $\boldsymbol{\Phi}$ is an external force, $\boldsymbol{\sigma}^i$ are contravariant stress vectors, λ and μ are Lame's constants, $\boldsymbol{R}_i$ and $\boldsymbol{R}^i$ are covariant and contravariant base vectors of the space and $\boldsymbol{U}$ is the displacement vector

3 Approximation of Order $N = 1$

The displacement vector $\boldsymbol{U}(x^1, x^2, x^3)$ are expressed by the following formula [1, 2] (approximation $N = 1$)

$$\boldsymbol{U}(x^1, x^2, x^3) = \mathbf{u}(x^1, x^2) + \frac{x^3}{h}\mathbf{v}(x^1, x^2).$$

Here $\mathbf{u}(x^1, x^2)$ and $\mathbf{v}(x^1, x^2)$ are the vector fields on the middle surface $x^3 = 0$, $2h$ is the thickness of the shell, x^3 is a thickness coordinate ($-h \leq x^3 \leq h$), x^1 and x^2 are isometric coordinates on the spherical surface

$$x^1 = \tan\frac{\theta}{2}\cos\varphi, \quad x^2 = \tan\frac{\theta}{2}\sin\varphi,$$

where θ and φ are the geographical coordinates.

Let us construct the solutions of the form [2, 5]

$$u_i = \sum_{k=1}^{\infty} \overset{k}{u}_i \varepsilon^k, \qquad v_i = \sum_{k=1}^{\infty} \overset{k}{v}_i \varepsilon^k, \quad (i = 1, 2, 3),$$

where u_i and v_i are the components of the vectors $\mathbf{u}$ and $\mathbf{v}$ respectively, $\varepsilon = \frac{h}{R_0}$ is a small parameter, R_0 is the radius of the midsurface of the sphere.

Using I. Vekua's method and complex variable functions the system of equilibrium equations can be represented in the form

$$4\mu h^2 \frac{\partial}{\partial \bar{z}}\left(\frac{1}{\Lambda}\frac{\partial \overset{k}{u}_+}{\partial \bar{z}}\right) + 2(\lambda+\mu)h^2\frac{\partial \theta^k}{\partial \bar{z}} + 2\lambda h \frac{\partial \overset{k}{v}_+}{\partial \bar{z}} = \overset{k}{X}_+, \tag{1}$$
$$\mu h^2 \nabla^2 \overset{k}{v}_3 - 3\left[\lambda \overset{k}{\theta} + (\lambda + 2\mu)\overset{k}{v}_3\right] = \overset{k}{X}_3,$$

$$4\mu h^2 \frac{\partial}{\partial \bar{z}} \left(\frac{1}{\Lambda} \frac{\partial \overset{k}{v}_+}{\partial \bar{z}} \right) + 2(\lambda+\mu)h^2 \frac{\partial \overset{k}{\Theta}}{\partial \bar{z}} - 3\mu \left(2h \frac{\partial \overset{k}{v}_3}{\partial \bar{z}} + \overset{k}{v}_+ \right) = \overset{k}{Y}_+, \quad (2)$$

$$\mu h \left(\nabla^2 \overset{k}{u}_3 + \overset{k}{\Theta} \right) = \overset{k}{Y}_3,$$

$$(k = 1, 2, ...),$$

where $z = x^1 + ix^2, \Lambda = \dfrac{4R_0^2}{(1+z\bar{z})^2}, \nabla^2 = \dfrac{4}{\Lambda} \dfrac{\partial^2}{\partial z \partial \bar{z}}$ and

$$\overset{k}{u}_+ = \overset{k}{u}_1 + i \overset{k}{u}_2, \quad \overset{k}{v}_+ = \overset{k}{v}_1 + i \overset{k}{v}_2,$$

$$\overset{k}{\theta} = \frac{1}{\Lambda} \left(\frac{\partial \overset{k}{u}_+}{\partial z} + \frac{\partial \overset{k}{\overline{u}}_+}{\partial \bar{z}} \right), \quad \overset{k}{\Theta} = \frac{1}{\Lambda} \left(\frac{\partial \overset{k}{v}_+}{\partial z} + \frac{\partial \overset{k}{\overline{v}}_+}{\partial \bar{z}} \right).$$

Introducing the well-known differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x^1} - i \frac{\partial}{\partial x^2} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x^1} + i \frac{\partial}{\partial x^2} \right).$$

$\overset{k}{X}_+, \overset{k}{Y}_+, \overset{k}{X}_3, \overset{k}{Y}_3$ are the components of external force and well-known quantity, defined by functions $\overset{0}{u}_i, ..., \overset{k-1}{u}_i, \overset{0}{v}_j, ..., \overset{k-1}{v}_j$.

The complex representation of a general solutions of systems (2) end (3) are written in the following form

$$\overset{k}{u}_+ = -\frac{5\lambda+6\mu}{3\lambda+2\mu} \frac{1}{\pi} \iint_D \frac{\Lambda(\zeta,\bar{\zeta})\varphi'(\zeta)d\xi d\eta}{\bar{\zeta}-\bar{z}} + \left(\frac{1}{\pi} \iint_D \frac{\Lambda(\zeta,\bar{\zeta})d\xi d\eta}{\bar{\zeta}-\bar{z}} \right) \overline{\varphi'(z)}$$

$$- \overline{\psi(z)} - \frac{\lambda h}{6(\lambda+\mu)} \frac{\partial \chi(z,\bar{z})}{\partial \bar{z}},$$

$$\overset{k}{v}_3 = \chi(z,\bar{z}) - \frac{2\lambda h}{3\lambda+2\mu} \left(\varphi'(z) + \overline{\varphi'(z)} \right),$$

$$\overset{k}{v}_+ = \frac{2(\lambda+2\mu)h^2}{3\mu} \overline{f''(z)} + \frac{1}{\pi} \iint_D \frac{\Lambda(\zeta,\bar{\zeta})f'(\zeta)d\xi d\eta}{\bar{\zeta}-\bar{z}}$$

$$- \left(\frac{1}{\pi} \iint_D \frac{\Lambda(\zeta,\bar{\zeta})d\xi d\eta}{\bar{\zeta}-\bar{z}} \right) \overline{f'(z)} - 2h\overline{g'(z)} + i \frac{\partial \omega(z,\bar{z})}{\partial \bar{z}},$$

$$\overset{k}{u}_3 = g(z) + \overline{g(z)} - \frac{1}{\pi h} \iint_D \Lambda(\zeta,\bar{\zeta}) \left[f'(z) + \overline{f'(z)} \right] \ln|\zeta - z| d\xi d\eta,$$

where $\zeta=\xi+i\eta$, $\varphi(z)$,$\psi(z)$,$f(z)$ and $g(z)$ are any analytic functions of z, $\chi(z,\bar z)$ and $\omega(z,\bar z)$ are the general solutions of the following Helmholtz's equations, respectively:

$$\Delta\chi-\kappa^2\chi=0 \quad \left(\kappa^2=\frac{3(\lambda+\mu)}{\lambda+2\mu}h^2\right),$$
$$\Delta\omega-\gamma^2\omega=0 \quad \left(\gamma^2=\frac{3}{h^2}\right).$$

D is the domain of the plane Ox^1x^2 onto which the midsurface S of the shell is mapped topologically.

Here we present a general scheme of solution of boundary problems when the domain D is the circular ring with radius R_1 and R_2 [6, 7, 8, 9].

The second boundary problem (in displacements) for any k takes the form

$$\begin{aligned}\overset{k}{u}_{+}=&-\frac{5\lambda+6\mu}{3\lambda+2\mu}\frac{1}{\pi}\int\limits_D\int\frac{\Lambda(\zeta,\bar\zeta)\varphi'(\zeta)d\xi d\eta}{\bar\zeta-\bar z}\\&+\left(\frac{1}{\pi}\int\limits_D\int\frac{\Lambda(\zeta,\bar\zeta)d\xi d\eta}{\bar\zeta-\bar z}\right)\overline{\varphi'(z)}-\overline{\psi(z)}-\frac{\lambda h}{6(\lambda+\mu)}\frac{\partial\chi(z,\bar z)}{\partial\bar z}\\&=\begin{cases}\overset{(k)}{G}{}'_{+}, & |z|=R_1,\\ \overset{(k)}{G}{}''_{+}, & |z|=R_2,\end{cases}\end{aligned}\tag{3}$$

$$\overset{k}{v}_{3}=\chi(z,\bar z)-\frac{2\lambda h}{3\lambda+2\mu}\left(\varphi'(z)+\overline{\varphi'(z)}\right)=\begin{cases}\overset{(k)}{G}{}'_{3}, & |z|=R_1,\\ \overset{(k)}{G}{}''_{3}, & |z|=R_2,\end{cases}\tag{4}$$

$$\begin{aligned}\overset{k}{v}_{+}=&\frac{2(\lambda+2\mu)h^2}{3\mu}\overline{f''(z)}+\frac{1}{\pi}\int\limits_D\int\frac{\Lambda(\zeta,\bar\zeta)f'(\zeta)d\xi d\eta}{\bar\zeta-\bar z}\\&-\left(\frac{1}{\pi}\int\limits_D\int\frac{\Lambda(\zeta,\bar\zeta)d\xi d\eta}{\bar\zeta-\bar z}\right)\overline{f'(z)}-2h\overline{g'(z)}+i\frac{\partial\omega(z,\bar z)}{\partial\bar z}\\&=\begin{cases}\overset{(k)}{Q}{}'_{+}, & |z|=R_1,\\ \overset{(k)}{Q}{}''_{+}, & |z|=R_2,\end{cases}\end{aligned}\tag{5}$$

$$\begin{aligned}\overset{k}{u}_{3}=&g(z)+\overline{g(z)}-\tfrac{1}{\pi h}\int\limits_D\int\Lambda(\zeta,\bar\zeta)\left[f'(z)+\overline{f'(z)}\right]\ln|\zeta-z|d\xi d\eta\\=&\begin{cases}\overset{(k)}{Q}{}'_{3}, & |z|=R_1,\\ \overset{(k)}{Q}{}''_{3}, & |z|=R_2,\end{cases}\end{aligned}\tag{6}$$

where $\overset{(k)}{G}{}'_{+}$, $\overset{(k)}{G}{}''_{+}$, $\overset{(k)}{G}{}'_{3}$, $\overset{(k)}{G}{}''_{3}$, $\overset{(k)}{Q}{}'_{+}$, $\overset{(k)}{Q}{}''_{+}$, $\overset{(k)}{Q}{}'_{3}$ and $\overset{(k)}{Q}{}''_{3}$ are the known values.

Next $\varphi'(z)$ and $\psi(z)$ are expanded in power series of the type

$$\begin{gathered}\varphi'(z)=\sum_{-\infty}^{\infty} a_n z^n, \quad \psi(z)=\sum_{-\infty}^{\infty} b_n z^n,\\ \chi(z,\bar z)=\sum_{-\infty}^{\infty}\left(\alpha_n I_n(\kappa r)+\beta_n K_n(\kappa r)\right)e^{in\vartheta},\end{gathered} \tag{7}$$

where $I_n(\kappa r)$ and $K_n(\kappa r)$ are Bessel's modificed functions, the expression $\overset{(k)}{G}{}'_{+}$, $\overset{(k)}{G}{}''_{+}$, $\overset{(k)}{G}{}'_{3}$ and $\overset{(k)}{G}{}''_{3}$ in the form of a complex Fourier series

$$\begin{gathered}\overset{(k)}{G}{}'_{+}=\sum_{-\infty}^{\infty} A'_n e^{in\vartheta}, \quad \overset{(k)}{G}{}'_{3}=\sum_{-\infty}^{\infty} B'_n e^{in\vartheta},\\ \overset{(k)}{G}{}''_{+}=\sum_{-\infty}^{\infty} A''_n e^{in\vartheta}, \quad \overset{(k)}{G}{}''_{3}=\sum_{-\infty}^{\infty} B''_n e^{in\vartheta}.\end{gathered} \tag{8}$$

By substituting (7) and (8) into (3) and (4) we obtain the system of algebraic equations:

$$\begin{aligned}
&-\frac{\lambda\kappa h}{12(\lambda+\mu)}\left(I_{n-1}(\kappa R_1)\alpha_n-K_{n-1}(\kappa R_1)\beta_n\right)\\
&-\frac{2(5\lambda+6\mu)}{3\lambda+2\mu}R_1^{n-1}\delta_{-n}\bar a_{-n}-b_{n-1}=\bar A'_{-n+1},\\
&-\frac{\lambda\kappa h}{12(\lambda+\mu)}\left(I_{n-1}(\kappa R_2)\alpha_n-K_{n-1}(\kappa R_2)\beta_n\right)\\
&-\frac{2\delta_0}{R_2^{2n-1}}\bar a_{-n}-b_{n-1}=\bar A''_{-n+1},\\
&-\frac{\lambda\kappa h}{12(\lambda+\mu)}\left(I_{n+1}(\kappa R_1)\alpha_n-K_{n+1}(\kappa R_1)\beta_n\right)-\frac{\bar b_{-n-1}}{R_1^{n+1}}=A'_{n+1},\\
&-\frac{\lambda\kappa h}{12(\lambda+\mu)}\left(I_{n+1}(\kappa R_2)\alpha_n-K_{n+1}(\kappa R_2)\beta_n\right)\\
&+\left(\frac{2(5\lambda+6\mu)}{3\lambda+2\mu}R_2^{-n-1}\delta_n-2R_2^{n-1}\delta_0\right)\bar a_n-\bar b_{-n-1}=A''_{n+1},\\
&I_n(\kappa R_1)\alpha_n+K_n(\kappa R_1)\beta_n-\frac{2\lambda h}{3\lambda+2\mu}\left(R_1^n a_n+R_1^{-n}\bar a_{-n}\right)=B'_n,\\
&I_n(\kappa R_2)\alpha_n+K_n(\kappa R_2)\beta_n-\frac{2\lambda h}{3\lambda+2\mu}\left(R_2^n a_n+R_2^{-n}\bar a_{-n}\right)=B''_n,
\end{aligned} \tag{9}$$

where $\delta_n=\int\limits_{R_1}^{R_2}\varrho^{2n+1}\Lambda(\varrho)d\varrho$.

Let us introduce the functions $f'(z)$, $g(z)$and $\omega(z,\bar z)$, $\overset{(k)}{Q}{}'_{+}$, $\overset{(k)}{Q}{}''_{+}$, $\overset{(k)}{Q}{}'_{3}$,

$\overset{(k)}{Q}{}''_3$ by the series

$$\begin{aligned}
&f'(z)=\sum_{-\infty}^{\infty}c_nz^n,\quad g(z)=\sum_{-\infty}^{\infty}d_nz^n,\\
&\omega(z,\bar z)=\sum_{-\infty}^{\infty}\left(\alpha'_nI_n(\gamma r)+\beta'_nK_n(\gamma r)\right)e^{in\vartheta},\\
&\overset{k}{Q}{}'_+=\sum_{-\infty}^{\infty}N'_ne^{in\vartheta},\quad \overset{k}{Q}{}'_3=\sum_{-\infty}^{\infty}M'_ne^{in\vartheta},\\
&\overset{k}{Q}{}''_+=\sum_{-\infty}^{\infty}N''_ne^{in\vartheta},\quad \overset{k}{Q}{}''_3=\sum_{-\infty}^{\infty}M''_ne^{in\vartheta}.
\end{aligned}\tag{10}$$

We now find the coefficients c_n, d_n, α'_n and β'_n from following system of algebraic equations:

$$\begin{aligned}
&-\frac{2(\lambda+2\mu)h^2n}{3\mu R_1^{n+1}}\bar c_{-n}-\frac{2hn}{R_1^{n+1}}\bar d_{-n}\\
&+\frac{i\gamma}{2}\left(\alpha'_nI_{n+1}(\gamma R_1)-\beta'_nK_{n+1}(\gamma R_1)\right)=N'_{n+1},\ \ n\geq 0\\
&-\frac{2(\lambda+2\mu)h^2n}{3\mu R_2^{n+1}}\bar c_{-n}-\frac{2\delta_n}{R_2^{n+1}}c_n+\frac{2\delta_0}{R_2^{n+1}}\bar c_{-n}+\frac{2hn}{R_2^{n+1}}\bar d_{-n}\\
&+\frac{i\gamma}{2}\left(\alpha'_nI_{n+1}(\gamma R_2)-\beta'_nK_{n+1}(\gamma R_2)\right)=N''_{n+1},\ \ n\geq 0\\
&\frac{2(\lambda+2\mu)h^2n}{3\mu R_1^{n-1}}c_n+2R_1^{n-1}\delta_{-n}\bar c_{-n}-2hnR_1^{n-1}d_n\\
&+\frac{i\gamma}{2}\left(\alpha'_nI_{n-1}(\gamma R_1)-\beta'_nK_{n-1}(\gamma R_1)\right)=\bar N'_{-n+1},\ \ n\geq 1\\
&\frac{2(\lambda+2\mu)h^2n}{3\mu R_2^{n-1}}c_n+\frac{2\delta_0}{R_2^{n-1}}c_n-2hnR_2^{n-1}d_n\\
&+\frac{i\gamma}{2}\left(\alpha'_nI_{n-1}(\gamma R_2)-\beta'_nK_{n-1}(\gamma R_2)\right)=\bar N''_{-n+1},\ \ n\geq 1\\
&R_1^nd_n+R_1^{-n}\bar d_{-n}-\frac{R_1^n}{nh}\delta_0c_n-\frac{R_1^n}{nh}\delta_{-n}\bar c_{-n}=M'_n,\ \ n=\pm1\pm2...\\
&R_2^nd_n+R_2^{-n}\bar d_{-n}-\frac{1}{nhR_2^n}\delta_nc_n-\frac{1}{nhR_2^n}\delta_0\bar c_{-n}=M''_n,\ \ n=\pm1...\\
&d_0+\bar d_0-\frac{2}{h}\int_{R_1}^{R_2}\varrho\Lambda(\varrho)\ln\varrho d\varrho[c_0+\bar c_0]=M'_0,\\
&d_0+\bar d_0-\frac{2}{h}\ln R_2\int_{R_1}^{R_2}\varrho\Lambda(\varrho)d\varrho[c_0+\bar c_0]=M''_0.
\end{aligned}\tag{11}$$

Acknowledgment

The designated project has been fulfilled by financial support of the Shota Rustaveli National Science Foundation (Grant SRNSF/FR /358/5-109/14).

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