ON THE NUMERICAL SOLUTION OF CONTACT PROBLEM FOR POISSON'S AND KIRCHHOFF EQUATION SYSTEM

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Abstract

In this paper stress-deformed state for some "bridge-form" multystructures studied having difficult geometry. Particularly the boundary-contacted problem is considered. Two rectangle (particularly a square) form membranes are connected by a string; We consider classic linear boundary problems for membranes (Poisson's equation), but for string nonlinear Kirchhoff type integro-differential equation. The account program in MATLAB is created and numerical experiments are made.

Key words and phrases: Poisson's equation, Kirchhoff type nonlinear integro-differential equation, finite-difference method.

AMS subject classification: 34B05, 34B15, 65M06, 65N06.

1 Introdaction

The stress - deformed condition for some "bridge - form" multystructures with difficult geometry (two rectangular membranes is connected by the string, (see fig.1)) is studied using numerical methods (finite-difference methods). Membrane bending is represented by the Poisson's equation (see, for example [1]). The equation of the string by Kirchhoff type non-linear integro-differential equation (see, for example [2]). The function of a membranes bending in central points is found by direct numerical methods, and the iterative method for definition of numerical values of function of a bend of a string for the approached decision of nonlinear equation Kirchhoff type.

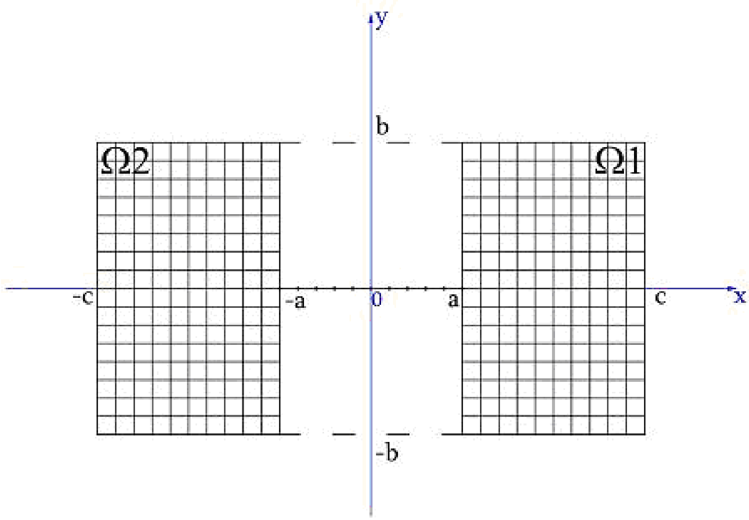

Fig.1

2 Statement of the Problem

It is possible to dismantle the above boundary - contact problem in three separate tasks:

a). Boundary value problem for the right membranes

$$\Delta w_1(x,y)=f_1(x,y),\ (x,y)\in \Omega_1=\{(x,y): a\le x\le c,\ -b\le y\le b\}, \quad (2.1)$$

$$w_1(x,\pm b)=0, \quad w_1(c,y)=0, \quad a\le x\le c, \quad -b\le y\le b, \quad (2.2)$$

$$\frac{\partial w_1(x,y)}{\partial x}\Big|_{x=a}=0, \quad -b\le y\le b. \quad (2.3)$$

b). Boundary value problem for the left membranes

$$\Delta w_2(x,y)=f_2(x,y),\ (x,y)\in \Omega_2=\{(x,y): -c\le x\le -a, -b\le y\le b\}, \quad (2.4)$$

$$w_2(x,\pm b)=0, \quad w_2(-c,y)=0, \quad -c\le x\le -a, \quad -b\le y\le b, \quad (2.5)$$

$$\frac{\partial w_2(x,y)}{\partial x}\Big|_{x=-a}=0, \quad -b\le y\le b. \quad (2.6)$$

c). Boundary value problem for a string

$$\left[m_0+m_1\int_{-a}^{+a}\left(w_3'(t)\right)^2dt\right]w_3''(x)=f_3(x), \quad -a\le y\le a, \quad (2.7)$$

$$w_3(-a)=a_2, \quad w_3(a)=a_1, \quad (2.8)$$

where $a_1\approx w_1(a,0), \ \ a_2\approx w_2(-a,0), \ \ m_0,m_1>0.$

3 The Algorithm

In order to solve of this boundary value problem we use the finite - difference method. Let's consider the case of the square is $c-a=2b$; Ω_1 and Ω_2 squares to make a regular square grid step $h_1=h_2=h$, $(n_1=n_2=n)$,

$$h_1=\frac{c-a}{n_1}=h_2=\frac{2b}{n_2}=h,\ \ x_i=a+ih_1,\ i=0,1,2,\cdots,n_1$$

or

$$x_i=-c+ih_1,\ i=0,1,2,\cdots,n_1,\quad y_j=-b+jh_2,\ j=0,1,2,\cdots,n_2.$$

The part of a string $[-a,a]$ section is divided $2n_3$ by step h_3,

$$h_3=a/n_3,\ \ x_i=-a+ih_3,\quad i=0,1,2,\cdots,2n_3.$$

Let's replace differential operators the finite - difference analog. It is changed (2.1), (2.4) equations of the second order differential operators by the template difference five point margin of error $O(h^2)$;

Let's replace the first order differential operators (2.3), (2.6) by method A : two-point template to change the error $O(h)$ and by method B : three - point template to change the error $O(h^2)$.

Let's change (2.7) a string equation lookup function by the $O(h_3^2)$ - order derivatives of the second order derivative by the three - point template.

In order to solve of this given nonlinear difference problem we use the iterative method.

Let's accept following marking for grid functions
$w_{1,i,j}\equiv w1_{i,j}\approx w_1(x_i,y_j)$, $w_{2,i,j}\equiv w2_{i,j}\approx w_2(x_i,y_j)$, $w_{3,i}\equiv w3_i\approx w_3(x_i)$, $f_{1,i,j}\equiv f1_{i,j}\approx f_1(x_i,y_j)$, $f_{2,i,j}\equiv f2_{i,j}=f_2(x_i,y_j)$, $f_{3,i}\equiv f3_i=f_3(x_i)$.

Method A. In case of (2.1)-(2.3) problem we have will the task of following a tree - block diagonal system of equation

$$\begin{pmatrix} A & E & \Theta & \Theta & \cdot & \cdot & \cdot & \Theta & \Theta \\ E & B & E & \Theta & \cdot & \cdot & \cdot & \Theta & \Theta \\ \Theta & E & B & E & \Theta & \cdot & \cdot & \Theta & \Theta \\ \Theta & \Theta & E & B & E & \Theta & \cdot & \Theta & \Theta \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Theta & \Theta & \cdot & \cdot & \Theta & E & B & E & \Theta \\ \Theta & \Theta & \cdot & \cdot & \cdot & \Theta & E & B & E \\ \Theta & \Theta & \Theta & \cdot & \cdot & \cdot & \Theta & E & B \end{pmatrix} \begin{pmatrix} W1,1 \\ W1,2 \\ W1,3 \\ W1,4 \\ \cdot \\ \cdot \\ W1,n-3 \\ W1,n-2 \\ W1,n-1 \end{pmatrix} = \begin{pmatrix} F1,1 \\ F1,2 \\ F1,3 \\ F1,4 \\ \cdot \\ \cdot \\ F1,n-3 \\ F1,n-2 \\ F1,n-1 \end{pmatrix}$$

and in case of (2.4) - (2.6) problem we will have follow task

$$\begin{pmatrix} B & E & \Theta & \Theta & \cdot & \cdot & \cdot & \Theta & \Theta \\ E & B & E & \Theta & \cdot & \cdot & \cdot & \Theta & \Theta \\ \Theta & E & B & E & \Theta & \cdot & \cdot & \Theta & \Theta \\ \Theta & \Theta & E & B & E & \Theta & \cdot & \Theta & \Theta \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Theta & \Theta & \cdot & \cdot & \Theta & E & B & E & \Theta \\ \Theta & \Theta & \cdot & \cdot & \cdot & \Theta & E & B & E \\ \Theta & \Theta & \Theta & \cdot & \cdot & \cdot & \Theta & E & A \end{pmatrix} \begin{pmatrix} W2,1 \\ W2,2 \\ W2,3 \\ W2,4 \\ \cdot \\ \cdot \\ W2,n-3 \\ W2,n-2 \\ W2,n-1 \end{pmatrix} = \begin{pmatrix} F2,1 \\ F2,2 \\ F2,3 \\ F2,4 \\ \cdot \\ \cdot \\ F2,n-3 \\ F2,n-2 \\ F2,n-1 \end{pmatrix},$$

where

$$A = \begin{pmatrix} -3 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 1 & -3 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 1 & -3 & 1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & -3 & 1 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & 1 & -3 & 1 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & -3 \end{pmatrix}$$

$$B = \begin{pmatrix} -4 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 1 & -4 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & -4 & 1 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & -4 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 1 \end{pmatrix}$$

$$\Theta = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \end{pmatrix}$$

$$W1,1 = \begin{pmatrix} w1_{1,1} \\ w1_{1,2} \\ w1_{1,3} \\ w1_{1,4} \\ \cdot \\ \cdot \\ w1_{1,n-3} \\ w1_{1,n-2} \\ w1_{1,n-1} \end{pmatrix}, W1,2 = \begin{pmatrix} w1_{2,1} \\ w1_{2,2} \\ w1_{2,3} \\ w1_{2,4} \\ \cdot \\ \cdot \\ w1_{2,n-3} \\ w1_{2,n-2} \\ w1_{2,n-1} \end{pmatrix}, \cdots, W1,n-1 = \begin{pmatrix} w1_{n-1,1} \\ w1_{n-1,2} \\ w1_{n-1,3} \\ w1_{n-1,4} \\ \cdot \\ \cdot \\ w1_{n-1,n-3} \\ w1_{n-1,n-2} \\ w1_{n-1,n-1} \end{pmatrix},$$

$$F1,1 = \begin{pmatrix} f1_{1,1} \\ f1_{1,2} \\ f1_{1,3} \\ f1_{1,4} \\ \cdot \\ \cdot \\ f1_{1,n-3} \\ f1_{1,n-2} \\ f1_{1,n-1} \end{pmatrix}, F1,2 = \begin{pmatrix} f1_{2,1} \\ f1_{2,2} \\ f1_{2,3} \\ f1_{2,4} \\ \cdot \\ \cdot \\ f1_{2,n-3} \\ f1_{2,n-2} \\ f1_{2,n-1} \end{pmatrix}, \cdots, F1, n-1 = \begin{pmatrix} f1_{n-1,1} \\ f1_{n-1,2} \\ f1_{n-1,3} \\ f1_{n-1,4} \\ \cdot \\ \cdot \\ f1_{n-1,n-3} \\ f1_{n-1,n-2} \\ f1_{n-1,n-1} \end{pmatrix},$$

$$W2,1 = \begin{pmatrix} w2_{1,1} \\ w2_{1,2} \\ w2_{1,3} \\ w2_{1,4} \\ \cdot \\ \cdot \\ w2_{1,n-3} \\ w2_{1,n-2} \\ w2_{1,n-1} \end{pmatrix}, W2,2 = \begin{pmatrix} w2_{2,1} \\ w2_{2,2} \\ w2_{2,3} \\ w2_{2,4} \\ \cdot \\ \cdot \\ w2_{2,n-3} \\ w2_{2,n-2} \\ w2_{2,n-1} \end{pmatrix}, \cdots, W2,n-1 = \begin{pmatrix} w2_{n-1,1} \\ w2_{n-1,2} \\ w2_{n-1,3} \\ w2_{n-1,4} \\ \cdot \\ \cdot \\ w2_{n-1,n-3} \\ w2_{n-1,n-2} \\ w2_{n-1,n-1} \end{pmatrix},$$

$$F2,1=\begin{pmatrix} f2_{1,1} \\ f2_{1,2} \\ f2_{1,3} \\ f2_{1,4} \\ \cdot \\ \cdot \\ f2_{1,n-3} \\ f2_{1,n-2} \\ f2_{1,n-1} \end{pmatrix}, F2,2=\begin{pmatrix} f2_{2,1} \\ f2_{2,2} \\ f2_{2,3} \\ f2_{2,4} \\ \cdot \\ \cdot \\ f2_{2,n-3} \\ f2_{2,n-2} \\ f2_{2,n-1} \end{pmatrix}, \cdots, F2,n-1=\begin{pmatrix} f2_{n-1,1} \\ f2_{n-1,2} \\ f2_{n-1,3} \\ f2_{n-1,4} \\ \cdot \\ \cdot \\ f2_{n-1,n-3} \\ f2_{n-1,n-2} \\ f2_{n-1,n-1} \end{pmatrix},$$

Method B. Let's replace the first order differential equations (2.3), (2.6) by three-point template to change the error $O(h^2)$:

$$\frac{\partial w_1(x,y)}{\partial x}\Big|_{x=a} = \frac{-1.5W1_{0,j} + 2W1_{1,j} - 0.5W1_{2,j}}{h} + O(h^2),$$

$$\frac{\partial w_2(x,y)}{\partial x}\Big|_{x=-a} = \frac{+1.5W2_{n,j} - 2W2_{n-1,j} + 0.5W2_{n-2,j}}{h} + O(h^2),$$

$$j = 1, 2, \cdots, n-1.$$

In case of (2.1) - (2.3) problem matrix form of algebraic equation system will have such view

$$\begin{pmatrix} C & D & \Theta & \Theta & \cdot & \cdot & \cdot & \Theta & \Theta \\ E & B & E & \Theta & \cdot & \cdot & \cdot & \Theta & \Theta \\ \Theta & E & B & E & \Theta & \cdot & \cdot & \Theta & \Theta \\ \Theta & \Theta & E & B & E & \Theta & \cdot & \Theta & \Theta \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Theta & \Theta & \cdot & \cdot & \Theta & E & B & E & \Theta \\ \Theta & \Theta & \cdot & \cdot & \cdot & \Theta & E & B & E \\ \Theta & \Theta & \Theta & \cdot & \cdot & \cdot & \Theta & E & B \end{pmatrix} \begin{pmatrix} W1,1 \\ W1,2 \\ W1,3 \\ W1,4 \\ \cdot \\ \cdot \\ W1,n-3 \\ W1,n-2 \\ W1,n-1 \end{pmatrix} = \begin{pmatrix} F1,1 \\ F1,2 \\ F1,3 \\ F1,4 \\ \cdot \\ \cdot \\ F1,n-3 \\ F1,n-2 \\ F1,n-1 \end{pmatrix}$$

and in case of (2.4)-(2.6) problem we will have follow task

$$\begin{pmatrix} B & E & \Theta & \Theta & \cdot & \cdot & \cdot & \Theta & \Theta \\ E & B & E & \Theta & \cdot & \cdot & \cdot & \Theta & \Theta \\ \Theta & E & B & E & \Theta & \cdot & \cdot & \Theta & \Theta \\ \Theta & \Theta & E & B & E & \Theta & \cdot & \Theta & \Theta \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \Theta & \Theta & \cdot & \cdot & \Theta & E & B & E & \Theta \\ \Theta & \Theta & \cdot & \cdot & \cdot & \Theta & E & B & E \\ \Theta & \Theta & \Theta & \cdot & \cdot & \cdot & \Theta & E & C \end{pmatrix} \begin{pmatrix} W2,1 \\ W2,2 \\ W2,3 \\ W2,4 \\ \cdot \\ \cdot \\ W2,n-3 \\ W2,n-2 \\ W2,n-1 \end{pmatrix} = \begin{pmatrix} F2,1 \\ F2,2 \\ F2,3 \\ F2,4 \\ \cdot \\ \cdot \\ F2,n-3 \\ F2,n-2 \\ F2,n-1 \end{pmatrix},$$

where

$$C = \begin{pmatrix} -8/3 & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 1 & -8/3 & 1 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 1 & -8/3 & 1 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 1 & -8/3 & 1 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & 1 & -8/3 & 1 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & -8/3 & 1 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 1 & -8/3 \end{pmatrix}$$

$$D = \begin{pmatrix} 2/3 & 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 2/3 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 2/3 & 0 & 0 & \cdot & \cdot & 0 & 0 \\ 0 & 0 & 0 & 2/3 & 0 & 0 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & \cdot & 0 & 0 & 2/3 & 0 & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 2/3 \end{pmatrix}$$

In order to solve of the (2.7)-(2.8) nonlinear system of equations let's use the iterative method combined with factorization methods:

$$w3_{i+1}^{(k+1)} - 2w3_i^{(k+1)} + w3_{i-1}^{(k+1)} = h_3^2 f3_i / \left(m_0 + m_1 tkf(w3^{(k)}) \right) \equiv F3_i^{(k)},$$

$$i = 1, 2, \cdots, 2n-1; \quad k = 0, 1, 2, \cdots;$$

$$tkf(w3^{(k)}) = 0.5 \left(\frac{w3_1^{(k)} - w3_0^{(k)}}{h} \right)^2 + \left(\frac{w3_2^{(k)} - w3_0^{(k)}}{2h} \right)^2 + \cdots +$$

$$+ \left(\frac{w3_{2n}^{(k)} - w3_{2n-2}^{(k)}}{2h} \right)^2 + 0.5 \left(\frac{w3_{2n}^{(k)} - w3_{2n-1}^{(k)}}{h} \right)^2;$$

$w3_i^{(0)}$, $i = 0, 1, 2, \cdots, 2n$ is the initial approach.

Remark: we can take as initial approach

$$w3_0^{(0)} = a_2, \;\; w3_1^{(0)} = 0, \;\; w3_2^{(0)} = 0, \; \cdots, \; w3_{2n-1}^{(0)} = 0, \;\; w3_{2n}^{(0)} = a_1;$$

For a finding in central points of required functions of a deflection it is received the following tree - diagonal system of the algebraic equations:

$$
\begin{aligned}
&-2w3_1^{(k+1)} + w3_2^{(k+1)} = F3_1^{(k)} - a_2,\\
&w3_1^{(k+1)} - 2w3_2^{(k+1)} + w3_3^{(k+1)} = F3_2^{(k)},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&w3_{2n-3}^{(k+1)} - 2w3_{2n-2}^{(k+1)} + w3_{2n-1}^{(k+1)} = F3_{2n-2}^{(k)},\\
&w3_{2n-2}^{(k+1)} - 2w3_{2n-1}^{(k+1)} = F3_{2n-1}^{(k)} - a_1,
\end{aligned}
$$

$k = 0, 1, 2, \cdots$.

The method of factorization is stabile, as $W3_1$ coefficients through $W3_2$ (in first equation), and $W3_{2n-1}$ through $W3_{2n-2}$ (in last equation) are equal 0.5.

It is created system of programs in MATLAB on the basis of the above-stated algorithm which is intended for a wide range of consumers.

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