SOLUTION OF A NONCLASSICAL PROBLEMS OF STATICS OF TWO-COMPONENT ELASTIC MIXTURES FOR A HALF-SPACE

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Abstract

In this paper we consider boundary value problems of statics of two-component elastic mixtures for a half-space, when the normal components of partial displacement vectors and the tangent components of partial rotation vectors are given on the boundary.Uniqueness theorems of the considered problem are proved. Solutions are represented in quadratures.

Key words and phrases: Elastic mixture theory, uniqueness theorems, tangential components, normal components.
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1 Introduction

In the early 60_s of the last century, C. Truesdell and R. Toupin formulated in [25] the fundamental mechanical principles of a new model of a deformable elastic medium with complex inner structure and thereby laid the foundation for the continual theory of elastic mixtures. In subsequent years this theory was generalized and developed in different directions. Based on kinematic and thermodynamic principles, theories were created for two- and many-component mixtures of such as fluid-fluid (Crochet and Naghdi [9], Atkin [2],Green and Naghdi [13], [14], Green and Steel [12], and solid body-solid body (Crochet and Naghdi [9], Atkin [2], Green and Steel [12], Khoroshun and Soltanov [16], Hill [15]).

In Natrosvili, Jaghmaidze and Svanadze [20], static and dynamic problems on the linear theory of a mixture of two isotropic elastic components are investigated by the method of a potential and singular integral equations. Atkin, Chadvick and Steel [3] and Knops and Steel [17] deal with uniqueness theorems for warious linearized dynamic problems of the theory of anizotropic mixtures.

Questions as to the existence and uniqueness of weak solutions of mixed static linear problems for mixtures of two nonhomogeneous anisotropic components where considered in Aron [1] and Borrelli and Patria [6], in the former work, the problem was studied by the method of functional analysis, while in the latter by the variational method. In Khoroshun and Soltanov's monograph [16], along with theoretical questions, quite interesting concrete problems of thermoelasticity were considered for two-component mixtures.

For a wider overview of the subject (half-space) area of applocations we refer to the references J. Barber [4], M. Basheleishvili, L. Bitsadze [5], D. Burchuladze, M. Kharashvili, K. Skhvitaridze [7], E. Constantin, N. Pavel [8], L.Giorgashvili, K. Skhvitaridze, M. Kharashvili [10], L. Giorgashvili, E. Elerdashvili, M. Kharashvili, K. Skhvitaridze [11], R. Kumar, T. Chadha [18], H. Sherief, H.Saleh [21], B. Singh, R. Kumar [22], K. Skhvitaridze, M. Kharashvili [23], the references therein.

2 Statement of boundary value problems. Uniqueneous theorems

In the three-dimensional linear theory of elastic two-component mixtures, a system of homogeneous differential equations of statics is written in the form [13]

$$a_1\Delta u' + b_1 \operatorname{grad}\operatorname{div} u' + c\Delta u'' + d \operatorname{grad}\operatorname{div} u'' = 0, \tag{2.1}$$

$$c\Delta u' + d \operatorname{grad}\operatorname{div} u' + a_2\Delta u'' + b_2 \operatorname{grad}\operatorname{div} u'' = 0, \tag{2.2}$$

where $u' = (u'_1, u'_2, u'_3)^\top$, $u'' = (u''_1, u''_2, u''_3)^\top$ are partial displacement vectors, $\top$ is the transposition symbol, Δ is three-dimensional Laplace operator,

$$a_1 = \mu_1 - \lambda_5, \quad b_1 = \mu_1 + \lambda_5 + \lambda_1 - \frac{\rho_2}{\rho}\alpha', \quad a_2 = \mu_2 - \lambda_5,$$

$$b_2 = \mu_2 + \lambda_5 + \lambda_2 + \frac{\rho_1}{\rho}\alpha', \quad c = \mu_3 + \lambda_5, \quad \alpha' = \lambda_3 - \lambda_4$$

$$d = \mu_3 + \lambda_3 - \lambda_5 - \frac{\rho_1}{\rho}\alpha', \quad \rho = \rho_1 + \rho_2,$$

ρ_1, ρ_2 are partial densities of the mixture; $\lambda_1, \lambda_2 \ldots \lambda_5, \mu_1, \mu_2, \mu_3$ are the elastic module characterizing the mechanical properties of the mixture, which satisfy the conditions [20]

$$\mu_1 > 0, \ \mu_1\mu_2 - \mu_3^2 > 0, \quad \lambda_5 < 0, \quad \lambda_1 + \frac{2}{3}\mu_1 - \frac{\rho_2}{\rho}\alpha' > 0,$$

$$\left(\lambda_1 + \frac{2}{3}\mu_1 - \frac{\rho_2}{\rho}\alpha'\right)\left(\lambda_2 + \frac{2}{3}\mu_2 + \frac{\rho_1}{\rho}\alpha'\right) > \left(\lambda_3 + \frac{2}{3}\mu_3 - \frac{\rho_1}{\rho}\alpha'\right)^2.$$

From these inequalities it follows that [20]

$$d_1 := a_1 a_2 - c^2 > 0, \;\; d_2 := (a_1 + b_1)(a_2 + b_2) - (c + d)^2 > 0, \tag{2.3}$$

$$a_1 > 0, \;\; a_1 + b_1 > 0.$$

The stress vector is written in the from [20]

$$T(\partial, n)U = [P^{(1)}(\partial, n)U, \;\; P^{(2)}(\partial, n)U]^\top,$$

where

$$\begin{aligned} P^{(1)}(\partial, n)U &= T^{(1)}(\partial, n)u' + T^{(2)}(\partial, n)u'', \\ P^{(2)}(\partial, n)U &= T^{(3)}(\partial, n)u' + T^{(4)}(\partial, n)u'', \\ T^{(1)}(\partial, n)u' &= 2\mu_1 \frac{\partial u'}{\partial n} + (\lambda_1 - \frac{\rho_2}{\rho}\alpha')n \operatorname{div} u' + (\mu_1 + \lambda_5)[n \times \operatorname{rot} u'], \\ T^{(2)}(\partial, n)u'' &= 2\mu_3 \frac{\partial u''}{\partial n} + (\lambda_3 - \frac{\rho_1}{\rho}\alpha')n \operatorname{div} u'' + (\mu_3 - \lambda_5)[n \times \operatorname{rot} u''], \\ T^{(3)}(\partial, n)u' &= 2\mu_3 \frac{\partial u'}{\partial n} + (\lambda_3 - \frac{\rho_1}{\rho}\alpha')n \operatorname{div} u' + (\mu_3 - \lambda_5)[n \times \operatorname{rot} u'], \\ T^{(4)}(\partial, n)u'' &= 2\mu_2 \frac{\partial u''}{\partial n} + (\lambda_2 + \frac{\rho_1}{\rho}\alpha')n \operatorname{div} u'' + (\mu_2 + \lambda_5)[n \times \operatorname{rot} u''], \end{aligned}$$

$n = (n_1, \; n_2, \; n_3)^\top$is a unit vector, $\frac{\partial}{\partial n} = \sum_{j=1}^{3} n_j \frac{\partial}{\partial x_j}$is a derivative with respect to the vector n, the symbol $[a \times b]$denote the vector products of two vectors in $\mathbb{R}^3$.

Denote by Ω^- a half-space $x_3 > 0$ and let $\partial\Omega$ be the plane $x_3 = 0$.

Problem (N). Find, in the domain Ω^-, a regular solution $U \in C^2(\Omega^-) \cap C^1(\overline{\Omega^-})$ of system (2.1)-(2.2) such that on the boundary $\partial\Omega$ one of the following boundary conditions is fulfilled:

$$\begin{aligned} \{n(y) \cdot u'(y)\}^- = f_3'(y), \quad \{n(y) \times rot u'(y)\}^- = f'(y), \quad y \in \partial\Omega, \\ \{n(y) \cdot u''(y)\}^- = f_3''(y), \quad \{n(y) \times rot u''(y)\}^- = f''(y), \quad y \in \partial\Omega, \end{aligned} \tag{2.4}$$

in the neighborhood of a point at infinity the vector $U(x)$ satisfies the following conditions:

$$\begin{aligned} &a)\, U_j(x) = o(1), \qquad \partial_k U_j(x) = o(|x|^{-1}), \\ &\qquad k = 1, 2, \;\; j = 1, 2, ..., 6, \;\; x_3 = 0, \;\; |x| \to \infty, \\ &b)\, U_j(x) = O(|x|^{-1}), \qquad \partial_k U_j(x) = O(|x|^{-2}), \\ &\qquad k = 1, 2, 3, \;\; j = 1, 2, ..., 6, \;\; x_3 > 0, \;\; |x| \to \infty, \end{aligned} \tag{2.5}$$

where $F' = (f_1', f_2', F_3')^\top$, $F'' = (f_1'', f_2'', F_3'')^\top$, $f_j', \;\; f_j'', \;\; j = 1, 2, 3, \;\; F_3', \;\; F_3''$ are the function given on the boundary, $n(y)$ is the internal normal unit vector passing at a point $y \in \partial\Omega$ in the domain Ω^-, $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, 0)$.

Theorem 2.1. *If problem* (**N**) *have solutions, these solutions are unique.*

Proof. The theorem will be proved if we show that the homogeneous problems ($F' = 0$, $F'' = 0$, $f_3' = 0$, $f_3'' = 0$) have only the trivial solution.

Denote by $\Omega_R := \Omega^- \cap B(O,R)$, where $B(O,R)$ is the ball with center at the origin and radius R. Denote by $\partial\Omega_R$ that part of the boundary of the ball $B(O,R)$ which lies in the domain $x_3 > 0$, by $S(O,R)$ the circle with center at the origin and radius R which lies on the plane $x_3 = 0$. Let us introduce the matrix differential operator $A(\partial)$

$$A(\partial) := \begin{bmatrix} A^{(1)}(\partial) & A^{(2)}(\partial) \\ A^{(3)}(\partial) & A^{(4)}(\partial) \end{bmatrix}_{6\times 6}, \quad A^{(l)}(\partial) := \left[A_{kj}^{(l)}(\partial)\right]_{3\times 3}, \quad l = 1,2,3,4,$$

$$\begin{aligned} A_{kj}^{(1)}(\partial) :=& a_1\delta_{kj}\Delta + b_1\partial_k\partial_j, \\ A_{kj}^{(l)}(\partial) :=& c\delta_{kj}\Delta + d\partial_k\partial_j, \quad l = 2,3, \\ A_{kj}^{(4)}(\partial) :=& a_2\delta_{kj}\Delta + b_2\partial_k\partial_j, \end{aligned}$$

where δ_{kj} is the Kronecker's symbol $\partial_j = \frac{\partial}{\partial x_j}$, $j = 1,2,3$, $\partial = (\partial_1, \partial_2, \partial_3)$.

Using these notations, we rewrite system (2.1)-(2.2) as $A(\partial)U(x) = 0$.

Let us consider the scalar derivative

$$\begin{aligned} U \cdot A(\partial)U = &(a_1u' + cu'') \cdot \Delta u' + (cu' + a_2u'') \cdot \Delta u'' \\ &+ (b_1u' + du'') \cdot \operatorname{grad}\operatorname{div} u' + (du' + b_2u'') \cdot \operatorname{grad}\operatorname{div} u''. \end{aligned} \tag{2.6}$$

Assume that $u = (u_1, \ u_2, \ u_3)^\top$ and $v = (v_1, \ v_2, \ v_3)^\top$ are three-component vectors. Then, after performing some transformations, we obtain

$$u \cdot \Delta v = \operatorname{div}(u \operatorname{div} v) + \operatorname{div}[u \times \operatorname{rot} v] - \operatorname{div} u \operatorname{div} v - \operatorname{rot} u \cdot \operatorname{rot} v,$$
$$u \cdot \operatorname{grad}\operatorname{div} v = \operatorname{div}(u \operatorname{div} v) - \operatorname{div} u \operatorname{div} v.$$

Substituting these equalities into (2.6), we have

$$\begin{aligned} U \cdot A(\partial)U = \operatorname{div}\big\{&\big[(a_1 + b_1)u' + (c+d)u''\big]\operatorname{div} u' + \big[(c+d)u' \\ &+ (a_2 + b_2)u''\big]\operatorname{div} u'' + a_1[u' \times \operatorname{rot} u'] + c[u' \times \operatorname{rot} u''] \\ &+ c[u'' \times \operatorname{rot} u'] + a_2[u'' \times \operatorname{rot} u'']\big\} - E(U,U), \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} E(U,U) &= \frac{1}{a_1+b_1} \\ &\times \left[((a_1+b_1)\operatorname{div} u' + (c+d)\operatorname{div} u'')^2 + d_2(\operatorname{div} u'')^2\right] \\ &+ \frac{1}{a_1}\left[(a_1 \operatorname{rot} u' + c \operatorname{rot} u'')^2 + d_1(\operatorname{rot} u'')^2\right]. \end{aligned} \tag{2.8}$$

Applying the Gauss-Ostrogradski theorem, from (2.7), we obtain

$$\begin{aligned} \int\limits_{\Omega_R} U(x)\cdot A(\partial)U(x)dx &= -\int\limits_{\partial\Omega_R} U(x)\cdot P(\partial,n)U(x)ds \\ -\int\limits_{S(O,R)} \{U(y)\}^- \cdot \{P(\partial,n)U(y)\}^- ds &- \int\limits_{\Omega_R} E(U,U)dx, \end{aligned} \tag{2.9}$$

where

$$\begin{gathered} U\cdot P(\partial,n)U = (n\cdot u')[(a_1+b_1)\operatorname{div} u' + (c+d)\operatorname{div} u''] \\ +(n\cdot u'')[(c+d)\operatorname{div} u' + (a_2+b_2)\operatorname{div} u''] - (a_1 u' + c u'')\cdot[n\times \operatorname{rot} u'] \\ -(cu' + a_2 u'')\cdot[n\times\operatorname{rot} u'']. \end{gathered}$$

Here we have used the identity

$$n\cdot[u\times \operatorname{rot} v] = -u\cdot[n\times \operatorname{rot} v].$$

Applying the boundary conditions of problem (**N**), we obtain

$$\{U(y)\}^-\cdot\{P(\partial,n)U(y)\}^- = 0, \quad y\in S(O,R).$$

Using this equality in (2.9), we have

$$\int\limits_{\Omega_R} E(U,U)dx + \int\limits_{\partial\Omega_R} U(x)\cdot P(\partial,n)U(x)ds = 0. \tag{2.10}$$

Passing to the limit on both sides of equality (2.10) as $R\to+\infty$ and taking into consideration the asymptotic representations (2.5), we obtain

$$\int\limits_{\Omega^-} E(U,U)dx = 0. \tag{2.11}$$

According to inequalities (2.3) we have $E(U,U)\geq 0$, $x\in\Omega^-$. By virtue of this fact, (2.11) implies

$$E(U,U)=0, \quad x\in\Omega^-.$$

Hence, taking into account (2.8), we obtain

$$\operatorname{div} u'(x)=0, \;\; \operatorname{div} u''(x)=0, \;\; \operatorname{rot} u'(x)=0, \;\; \operatorname{rot} u''(x)=0, \;\; x\in\Omega^-.$$

A solution of this system has the form

$$u'(x)=\operatorname{grad}\Psi_1(x), \quad u''(x)=\operatorname{grad}\Psi_2(x), \;\; x\in\Omega^-, \tag{2.12}$$

where $\Psi_j(x), \;\; j=1,2$, is an arbitrary harmonic function.

Since $\{n(y)\cdot u'(y)\}^-=0$, $\{n(y)\cdot u''(y)\}^-=0$, the harmonic functions $\Psi_j(x), \;\; j=1,2$, satisfy, on the boundary $\partial\Omega$, the Neumann condition

$$\left\{\frac{\partial\Psi_j(y)}{\partial n(y)}\right\}^-=0, \quad y\in\partial\Omega.$$

As is known, the homogeneous Neumann problem has the solution $\Psi_j(x)=C_j=const, \quad j=1,2, \quad x\in\Omega^-$. Substituting this value of $\Psi_j(x)$ into (2.12), we obtain $u'(x)=0, \;\; u''(x)=0, \;\; x\in\Omega^-$.

Thus the homogeneous problem $(\mathbf{N})_\mathbf{0}$ has only a trivial solution. Hance it follows that problem $(\mathbf{N})$ admits no more than regular solution. □

3 Solution of the $(\mathrm{N})^-$ problem

If in the boundary conditions (2.4) we assume that $n(y)=(0,0,1)^\top$, then these boundary conditions can be rewritten as follows:

$$\{u_3'(y)\}^-=f_3'(y), \quad \{u_3''(y)\}^-=f_3''(y),$$
$$\left\{\frac{\partial u_j'(y)}{\partial x_3}\right\}^-=\frac{\partial f_3'(y)}{\partial y_j}-f_j'(y), \quad \left\{\frac{\partial u_j''(y)}{\partial x_3}\right\}^-=\frac{\partial f_3''(y)}{\partial y_j}-f_j''(y), \tag{3.1}$$
$$j=1,2.$$

In formulas (3.1) assume the following

$$\left\{\frac{\partial v(y)}{\partial x_3}\right\}^-=\lim_{\Omega^-\ni x\to y\in\partial\Omega}\frac{\partial v(x)}{\partial x_3}.$$

From equation (2.1)-(2.2), we have

$$\Delta\operatorname{rot} u'(x)=0, \quad \Delta\operatorname{rot} u''(x)=0, \quad x\in\Omega^-. \tag{3.2}$$

From the boundary conditions (3.1), we obtained

$$\begin{aligned} \big\{\operatorname{rot} u'(y)\big\}_j^- &= \delta_{1j} f_2'(y) - \delta_{2j} f_1'(y), \\ \big\{\operatorname{rot} u''(y)\big\}_j^- &= \delta_{1j} f_2''(y) - \delta_{2j} f_1''(y), \quad j = 1,2, \end{aligned} \tag{3.3}$$

$$\begin{aligned} \left\{\frac{\partial}{\partial x_3} \operatorname{rot} u'(y)\right\}_3^- &= \frac{\partial f_1'(y)}{\partial y_2} - \frac{\partial f_2'(y)}{\partial y_1}, \\ \left\{\frac{\partial}{\partial x_3} \operatorname{rot} u''(y)\right\}_3^- &= \frac{\partial f_1''(y)}{\partial y_2} - \frac{\partial f_2''(y)}{\partial y_1}. \end{aligned} \tag{3.4}$$

The Dirichlet and Neumann problems (3.2) – (3.4) has a following solution [19],[24]

$$\big[\operatorname{rot} u'(x)\big]_j = -\frac{1}{2\pi} \iint\limits_{-\infty}^{+\infty} \frac{\partial}{\partial x_3} \frac{1}{r} (\delta_{1j} f_2'(y) - \delta_{2j} f_1'(y)) dy, \quad j = 1,2,$$

$$\big[\operatorname{rot} u''(x)\big]_j = -\frac{1}{2\pi} \iint\limits_{-\infty}^{+\infty} \frac{\partial}{\partial x_3} \frac{1}{r} (\delta_{1j} f_2''(y) - \delta_{2j} f_1''(y)) dy, \quad j = 1,2,$$

$$\big[\operatorname{rot} u'(x)\big]_3 = -\frac{1}{2\pi} \iint\limits_{-\infty}^{+\infty} \frac{1}{r} \left(\frac{\partial f_1'(y)}{\partial y_2} - \frac{\partial f_2'(y)}{\partial y_1}\right) dy$$

$$\big[\operatorname{rot} u''(x)\big]_3 = -\frac{1}{2\pi} \iint\limits_{-\infty}^{+\infty} \frac{1}{r} \left(\frac{\partial f_1''(y)}{\partial y_2} - \frac{\partial f_2''(y)}{\partial y_1}\right) dy$$

$$r = |x - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2}.$$

from this we obtain that

$$
\begin{aligned}
\big[x\times \operatorname{rot} u'(x)\big]_j &= -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{x_2\delta_{1j}-x_1\delta_{2j}}{r}\left(\frac{\partial f_1'(y)}{\partial y_2}-\frac{\partial f_2'(y)}{\partial y_1}\right)dy\\
&\quad -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} x_3\frac{\partial}{\partial x_3}\frac{1}{r}f_j'(y)dy, \quad j=1,2,\\
\big[x\times \operatorname{rot} u''(x)\big]_j &= \frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{x_1\delta_{2j}-x_2\delta_{1j}}{r}\left(\frac{\partial f_1''(y)}{\partial y_2}-\frac{\partial f_2''(y)}{\partial y_1}\right)dy\\
&\quad -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} x_3\frac{\partial}{\partial x_3}\frac{1}{r}f_j''(y)dy, \quad j=1,2,\\
\big[x\times \operatorname{rot} u'(x)\big]_3 &= \frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{\partial}{\partial x_3}\frac{1}{r}(x_1f_1'(y)+x_2f_2'(y))dy,\\
\big[x\times \operatorname{rot} u''(x)\big]_3 &= \frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{\partial}{\partial x_3}\frac{1}{r}(x_1f_1''(y)+x_2f_2''(y))dy.
\end{aligned}
\tag{3.5}
$$

If the following equality $\operatorname{grad}\operatorname{div} u = \Delta u + \operatorname{rot}\operatorname{rot} u$ is consider in the equations (2.1)-(2.2), we get

$$
\begin{gathered}
(a_1+b_1)\Delta u'(x) + b_1 \operatorname{rot}\operatorname{rot} u'(x) + (c+d)\Delta u''(x)\\
+d\operatorname{rot}\operatorname{rot} u''(x) = 0,\\
(c+d)\Delta u'(x) + d\operatorname{rot}\operatorname{rot} u'(x) + (a_2+b_2)\Delta u''(x)\\
+b_2\operatorname{rot}\operatorname{rot} u''(x) = 0,\\
x\in\Omega^-.
\end{gathered}
\tag{3.6}
$$

On the other hand

$$
\Delta[x\times \operatorname{rot} u'(x)] = 2\operatorname{rot}\operatorname{rot} u'(x), \quad \Delta[x\times \operatorname{rot} u''(x)] = 2\operatorname{rot}\operatorname{rot} u''(x),
$$

there fore the equations (3.6) will be rewrote so

$$
\Delta v'(x) = 0, \quad \Delta v''(x) = 0, \quad x\in\Omega^-, \tag{3.7}
$$

where

$$
\begin{aligned}
v'(x) &= 2(a_1+b_1)u'(x) + 2(c+d)u''(x) + b_1[x \times \operatorname{rot} u'(x)] \\
&\quad + d[x \times \operatorname{rot} u''(x)], \\
v''(x) &= 2(c+d)u'(x) + 2(a_2+b_2)u''(x) + d[x \times \operatorname{rot} u'(x)] \\
&\quad + b_2[x \times \operatorname{rot} u''(x)].
\end{aligned} \tag{3.8}
$$

If consider (3.1),(3.3)-(3.4) boundary conditions, we will get

$$
\left\{v_3'(y)\right\}^- = f_3^{(1)}(y), \quad \left\{v_3''(y)\right\}^- = f_3^{(2)}(y), \tag{3.9}
$$

$$
\begin{gathered}
\left\{\frac{\partial}{\partial x_3} v_j'(y)\right\}^- = f_j^{(1)}(y), \quad \left\{\frac{\partial}{\partial x_3} v_j''(y)\right\}^- = f_j^{(2)}(y), \quad j=1,2, \\
y \in \partial\Omega,
\end{gathered} \tag{3.10}
$$

where

$$
\begin{aligned}
f_3^{(1)}(y) &= 2(a_1+b_1)f_3'(y) + 2(c+d)f_3''(y) - b_1\left(y_1 f_1'(y) + y_2 f_2'(y)\right) \\
&\quad - d\left(y_1 f_1''(y) + y_2 f_2''(y)\right), \\
f_3^{(2)}(y) &= 2(c+d)f_3'(y) + 2(a_2+b_2)f_3''(y) - c\left(y_1 f_1'(y) + y_2 f_2'(y)\right) \\
&\quad - b_2\left(y_1 f_1''(y) + y_2 f_2''(y)\right), \\
f_j^{(1)}(y) &= 2(a_1+b_1)\left(\frac{\partial f_3'(y)}{\partial y_j} - f_j'(y)\right) + 2(c+d)\left(\frac{\partial f_3''(y)}{\partial y_j} - f_j''(y)\right) \\
&\quad + (y_2\delta_{1j} - y_1\delta_{2j})\left[b_1\left(\frac{\partial f_1'(y)}{\partial y_2} - \frac{\partial f_2'(y)}{\partial y_1}\right) + d\left(\frac{\partial f_1''(y)}{\partial y_2} - \frac{\partial f_2''(y)}{\partial y_1}\right)\right], \\
f_j^{(2)}(y) &= 2(c+d)\left(\frac{\partial f_3'(y)}{\partial y_j} - f_j'(y)\right) + 2(a_2+b_2)\left(\frac{\partial f_3''(y)}{\partial y_j} - f_j''(y)\right) \\
&\quad + (y_2\delta_{1j} - y_1\delta_{2j})\left[d\left(\frac{\partial f_1'(y)}{\partial y_2} - \frac{\partial f_2'(y)}{\partial y_1}\right) + b_2\left(\frac{\partial f_1''(y)}{\partial y_2} - \frac{\partial f_2''(y)}{\partial y_1}\right)\right],
\end{aligned}
$$

$$
j = 1, 2.
$$

The Dirichlet problems (3.7),(3.9) and Neumann problems(3.7),(3.10) has a following solution

$$v_3'(x) = -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{\partial}{\partial x_3}\frac{1}{r} f_3^{(1)}(y)dy,$$
$$v_3''(x) = -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{\partial}{\partial x_3}\frac{1}{r} f_3^{(2)}(y)dy;$$
$$v_j'(x) = -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{1}{r} f_j^{(1)}(y)dy, \quad j=1,2,$$
$$v_j''(x) = -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{1}{r} f_j^{(2)}(y)dy, \quad j=1,2. \tag{3.11}$$

From (3.8), we have

$$\begin{aligned} u'(x) &= \zeta_2 v'(x) - \zeta_3 v''(x) + \zeta_4\left[x\times \operatorname{rot} u'(x)\right] \\ &\quad +\zeta_5\left[x\times \operatorname{rot} u''(x)\right], \\ u''(x) &= \zeta_1 v''(x) - \zeta_3 v'(x) + \zeta_6\left[x\times \operatorname{rot} u'(x)\right] \\ &\quad +\zeta_7\left[x\times \operatorname{rot} u''(x)\right], \end{aligned} \tag{3.12}$$

where

$\zeta_1 = (a_1+b_1)/2d_2, \quad \zeta_2 = (a_2+b_2)/2d_2, \quad \zeta_3 = (c+d)/2d_2,$
$\zeta_4 = \left(d(c+d) - b_1(a_2+b_2)\right)/2d_2, \quad \zeta_5 = \left(b_2(c+d) - d(a_2+b_2)\right)/2d_2,$
$\zeta_6 = \left(b_1(c+d) - d(a_1+b_1)\right)/2d_2, \quad \zeta_7 = \left(d(c+d) - b_2(a_1+b_1)\right)/2d_2.$

If we consider the equality (3.5), (3.11) in (2.11), we get

$$U(x) = -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \mathbf{K}(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{y})\mathbf{dy}, \quad \mathbf{x}\in\mathbf{\Omega}^-, \tag{3.13}$$

where

$$\mathbf{K}(\mathbf{x},\mathbf{y}) = \begin{bmatrix} \mathbf{K^{(1)}}(\mathbf{x},\mathbf{y}) & \mathbf{K^{(2)}}(\mathbf{x},\mathbf{y}) \\ \mathbf{K^{(3)}}(\mathbf{x},\mathbf{y}) & \mathbf{K^{(4)}}(\mathbf{x},\mathbf{y}) \end{bmatrix}_{\mathbf{6\times 6}}, \quad \mathbf{K^{(p)}}(\mathbf{x},\mathbf{y}) = \left[\mathbf{K^{(p)}_{lj}}(\mathbf{x},\mathbf{y})\right]_{\mathbf{3\times 3}},$$

$p=1,2,3,4, \quad f=(f',f'')^\top, \quad f'=(f_1',f_2',f_3')^\top, \quad f''=(f_1'',f_2'',f_3'')^\top,$

$$\begin{aligned} \mathbf{K^{(1)}_{lj}}(\mathbf{x},\mathbf{y}) &= (1-\delta_{l3})(1-\delta_{3j})\left(\delta_{lj}(\zeta_4-1)\frac{1}{r} - \zeta_4\frac{\partial^2 r}{\partial x_l\partial x_j}\right) \\ &\quad + (1-\delta_{l3})\delta_{3j}\frac{\partial}{\partial x_l}\frac{1}{r} - \zeta_4(1-\delta_{3j})\delta_{3l}x_3\frac{\partial}{\partial x_j}\frac{1}{r} + \delta_{l3}\delta_{3j}\frac{\partial}{\partial x_3}\frac{1}{r} \end{aligned}$$

$$\mathbf{K_{lj}^{(2)}}(\mathbf{x},\mathbf{y}) = \zeta_5(1-\delta_{l3})(1-\delta_{3j})\left(\delta_{lj}\frac{1}{r} - \frac{\partial^2 r}{\partial x_l \partial x_j}\right) - \zeta_5(1-\delta_{3j})\delta_{l3}x_3\frac{\partial}{\partial x_j}\frac{1}{r},$$

$$\mathbf{K_{lj}^{(3)}}(\mathbf{x},\mathbf{y}) = \zeta_6(1-\delta_{l3})(1-\delta_{3j})\left(\delta_{lj}\frac{1}{r} - \frac{\partial^2 r}{\partial x_l \partial x_j}\right) - \zeta_6(1-\delta_{3j})\delta_{l3}x_3\frac{\partial}{\partial x_j}\frac{1}{r},$$

$$\mathbf{K_{lj}^{(4)}}(\mathbf{x},\mathbf{y}) = (1-\delta_{l3})(1-\delta_{3j})\left(\delta_{lj}(\zeta_7-1)\frac{1}{r} - \zeta_7\frac{\partial^2 r}{\partial x_l \partial x_j}\right)$$
$$+ (1-\delta_{l3})\delta_{3j}\frac{\partial}{\partial x_l}\frac{1}{r} - \zeta_7(1-\delta_{3j})\delta_{3l}x_3\frac{\partial}{\partial x_j}\frac{1}{r} + \delta_{l3}\delta_{3j}\frac{\partial}{\partial x_3}\frac{1}{r}.$$

Here we used the identities

$$\iint\limits_{-\infty}^{+\infty}\frac{1}{r}\frac{\partial f_l(y)}{\partial y_j}dy = \iint\limits_{-\infty}^{+\infty}\frac{\partial}{\partial x_j}\frac{1}{r}f_l(y)dy, \quad l,j=1,2,$$

$$\iint\limits_{-\infty}^{+\infty}(x_l-y_l)\frac{1}{r}\frac{\partial f_k(y)}{\partial y_j}dy = \iint\limits_{-\infty}^{+\infty}\frac{\partial^2 r}{\partial x_l \partial x_j}f_k(y)dy, \quad l,j,k=1,2.$$

Calculate the stress vector $T(\partial,n)U(x)$ by (3.13), we get

$$T(\partial,n)U(x) = -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}\mathbf{L}(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{y})\mathbf{dy},$$

where

$$\mathbf{L}(\mathbf{x},\mathbf{y}) = \begin{bmatrix}\mathbf{L^{(1)}}(\mathbf{x},\mathbf{y}) & \mathbf{L^{(2)}}(\mathbf{x},\mathbf{y})\\ \mathbf{L^{(3)}}(\mathbf{x},\mathbf{y}) & \mathbf{L^{(4)}}(\mathbf{x},\mathbf{y})\end{bmatrix}_{\mathbf{6\times 6}}, \quad \mathbf{L^{(l)}}(\mathbf{x},\mathbf{y}) = \left[\mathbf{L_{kj}^{(l)}}(\mathbf{x},\mathbf{y})\right]_{\mathbf{3\times 3}},$$
$$l=1,2,3,4,$$

$$\mathbf{L_{kj}^{(l)}}(x,y) = (1-\delta_{k3})(1-\delta_{3j})\Bigg[(c\zeta_6 + a_1\zeta_4 - a_1)\delta_{kj}\frac{\partial}{\partial x_3}\frac{1}{r}$$
$$-2(\mu_1\zeta_4+\mu_3\zeta_6)\frac{\partial^3 r}{\partial x_k \partial x_j \partial x_3}\Bigg] + 2\mu_1(1-\delta_{k3})\delta_{3j}\frac{\partial^2}{\partial x_k \partial x_3}\frac{1}{r}$$
$$-(1-\delta_{3j})\delta_{k3}\left[\left(-2\mu_1 + a_1 + \frac{b_1}{2}\right)\frac{\partial}{\partial x_j}\frac{1}{r} + 2(\mu_1\zeta_4+\mu_3\zeta_6)x_3\frac{\partial^2}{\partial x_j \partial x_3}\frac{1}{r}\right]$$
$$+2\mu_1\delta_{k3}\delta_{3j}\frac{\partial^2}{\partial x_3^2}\frac{1}{r},$$

$$\mathbf{L^{(2)}_{kj}}(x,y)=(1-\delta_{k3})(1-\delta_{3j})\Bigg[(a_1\zeta_5+c\zeta_7-c)\delta_{kj}\frac{\partial}{\partial x_3}\frac{1}{r}$$

$$-2(\mu_1\zeta_5+\mu_3\zeta_7)\frac{\partial^3 r}{\partial x_k\partial x_j\partial x_3}\Bigg]+2\mu_3(1-\delta_{k3})\delta_{3j}\frac{\partial^2}{\partial x_k\partial x_3}\frac{1}{r}$$

$$-(1-\delta_{3j})\delta_{k3}\left[\left(-2\mu_3+c+\frac{d}{2}\right)\frac{\partial}{\partial x_j}\frac{1}{r}+2(\mu_1\zeta_5+\mu_3\zeta_7)x_3\frac{\partial^2}{\partial x_j\partial x_3}\frac{1}{r}\right]$$

$$+2\mu_3\delta_{k3}\delta_{3j}\frac{\partial^2}{\partial x_3^2}\frac{1}{r},$$

$$\mathbf{L^{(3)}_{kj}}(x,y)=(1-\delta_{k3})(1-\delta_{3j})\Bigg[(a_2\zeta_6+c\zeta_4-c)\delta_{kj}\frac{\partial}{\partial x_3}\frac{1}{r}$$

$$-2(\mu_2\zeta_6+\mu_3\zeta_4)\frac{\partial^3 r}{\partial x_k\partial x_j\partial x_3}\Bigg]+2\mu_3(1-\delta_{k3})\delta_{3j}\frac{\partial^2}{\partial x_k\partial x_3}\frac{1}{r}$$

$$-(1-\delta_{3j})\delta_{k3}\left[\left(c+\frac{d}{2}-2\mu_3\right)\frac{\partial}{\partial x_j}\frac{1}{r}+2(\mu_2\zeta_6+\mu_3\zeta_4)x_3\frac{\partial^2}{\partial x_j\partial x_3}\frac{1}{r}\right]$$

$$+2\mu_3\frac{\partial^2}{\partial x_3^2}\frac{1}{r},$$

$$\mathbf{L^{(4)}_{kj}}(x,y)=(1-\delta_{k3})(1-\delta_{3j})\Bigg[(c\zeta_5+a_2\zeta_7-a_2)\delta_{kj}\frac{\partial}{\partial x_3}\frac{1}{r}$$

$$-2(\mu_3\zeta_5+\mu_2\zeta_7)\frac{\partial^3 r}{\partial x_k\partial x_j\partial x_3}\Bigg]+2\mu_2(1-\delta_{k3})\delta_{3j}\frac{\partial^2}{\partial x_k\partial x_3}\frac{1}{r}$$

$$-(1-\delta_{3j})\delta_{k3}\left[\left(a_2+\frac{b_2}{2}-2\mu_2\right)\frac{\partial}{\partial x_j}\frac{1}{r}+2(\mu_3\zeta_5+\mu_2\zeta_7)x_3\frac{\partial^2}{\partial x_j\partial x_3}\frac{1}{r}\right]$$

$$+2\mu_2\delta_{k3}\delta_{3j}\frac{\partial^2}{\partial x_3^2}\frac{1}{r}.$$

Assume that the functions $f_j'(y), f_j''(y)\in C^{0,\alpha}(\partial\Omega)$, $f_3'(y)$, $f_3''(y)\in C^{1,\alpha}(\partial\Omega)$, $j=1,2$, $0<\alpha<1$, then by straight forward verification we establish that the vector $U(x)$ represented in form (3.13) is a solution of system (2.1)-(2.2) in the domain Ω^-. If in the functions $\frac{\partial u_j'(x)}{\partial x_3}$, $\frac{\partial u_j''(x)}{\partial x_3}$, $j=1,2$, $u_3'(x), u_3''(x)$ from (3.13) we pass to the limit as $x\to y\in\partial\Omega$ $(x_3\to 0)$ and taking into account [19], [24]

$$\lim_{x\to y}\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}\frac{\partial}{\partial x_3}\frac{1}{r}f(y)dy=-f(y),\quad y\in\partial\Omega,$$

we obtain that the vector $U(x)$ represented in form (3.13) satisfies the boundary conditions (3.1).

If the boundary vector-function satisfies the conditions

$$\left|f_j'(y)\right| < \frac{A}{1+|y|^2}, \quad \left|f_j''(y)\right| < \frac{A}{1+|y|^2}, \quad j=1,2,$$
$$\left|f_3'(y)\right| < \frac{A}{1+|y|}, \quad \left|f_3''(y)\right| < \frac{A}{1+|y|}, \quad y\in\partial\Omega, \quad A=const>0,$$

then the vector $U(x)$ represented by formula (3.13) is a regular solution of problem (**N**) which satisfies the following decay conditions at infinity

$$u_j'(x), u_j''(x) = O\left(|x|^{-1}\ln|x|\right), \quad j=1,2, \quad u_3'(x), u_3''(x) = O\left(|x|^{-1}\right),$$
$$\partial_k u_j'(x), \partial_k u_j''(x) = O\left(|x|^{-2}\right), \quad j=1,2,$$
$$\partial_k u_3'(x), \partial_k u_3''(x) = O\left(|x|^{-2}\ln|x|\right),$$
$$k=1,2,3.$$

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