INVESTIGATION OF THE INTERIOR NEUMANN TYPE BOUNDARY VALUE PROBLEM OF THERMOELASTOSTATICS FOR ANISOTROPIC BODIES

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Abstract

We consider the interior Neumann type boundary value problem (BVP) of thermoelastostatics for anisotropic bodies. We construct the general solution to the corresponding homogeneous BVP explicitly and by the potential method we investigate the existence results. The problem is reduced to the equivalent system of singular integral equations. The Fredholm properties of the corresponding matrix singular integral operator and its adjoint one are established and their null spaces are constructed efficiently. Finally, on the basis of the results obtained we derive the necessary and sufficient conditions in explicit form for the interior Neumann type nonhomogeneous boundary value problem to be solvable.

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1 Introduction

Boundary-value problems of thermoelasticity, including mixed and crack type boundary value problems, for the so called *pseudo-oscillation* and *steady state oscillation* equations are well studied in the scientific literature for isotropic and anisotropic solids (for details see [6], [3], [4], [5], [1] and the references therein).

The purpose of the present paper is a detailed analysis of the interior Neumann type boundary value problem (BVP) of *thermoelastostatics* for anisotropic bodies. We apply the potential method to this BVP and reduce it to equivalent system of singular integral equations. We prove that the corresponding matrix singular integral operator and its adjoint one are normally solvable and show that their indices equal to zero. We construct explicitly the general solution of the BVP under consideration as well as the seven dimensional null spaces of the mutually adjoint singular integral operators associated with the BVP under consideration. On the basis of these results we formulate the necessary and sufficient conditions explicitly for the nonhomogeneous interior Neumann type BVP to be solvable.

2 Problem setting and Green's formulas

Let $\Omega^+ \in \mathbb{R}^3$ be a bounded domain with smooth boundary $S = \partial\Omega^+$. For simplicity, we assume that S is a $C^{1,\alpha}$-smooth surface with $0 < \alpha \leqslant 1$. We set $\overline{\Omega^+} = \Omega^+ \cup S$ and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$. By $n(x) = (n_1(x), n_2(x), n_3(x))$ we denote the outward unit normal vector to S at the point $x \in S$. We assume that the domains $\Omega^\pm$ are occupied by anisotropic elastic media.

The basic governing homogeneous equations of the linear thermoelastostatics read as (see, e.g., [6], [4], [10], [11]):

$$c_{kjpq}\partial_j\partial_q u_p(x) - \beta_{kj}\partial_j\vartheta(x) = 0, \quad k = 1, 2, 3, \tag{2.1}$$

$$\lambda_{pq}\partial_p\partial_q\vartheta(x) = 0, \tag{2.2}$$

where $x \in \Omega^\pm$, $u = (u_1, u_2, u_3)^\top$ is the displacement vector, ϑ is the temperature distribution function, $c_{kjpq} = c_{pqkj} = c_{jkpq}$ are elastic constants, $\lambda_{pq} = \lambda_{qp}$ are heat conduction coefficients, $\beta_{pq} = \beta_{qp}$ are the material constant describing the coupling of mechanical and thermal fields, $\partial_j = \partial/\partial x_j$, $\partial = \nabla = (\partial_1, \partial_2, \partial_3)$. Throughout the paper summation over repeated indices is meant from 1 to 3 if not otherwise stated. The symbol $(\cdot)^\top$ denotes transposition.

The matrix $[\lambda_{pq}]_{3\times3}$ is assumed to be positive definite, while the quadratic form $c_{kjpq}\,\eta_{kj}\,\eta_{pq}$ is assumed to be positive definite in symmetric variables $\eta_{kj} = \eta_{jk} \in \mathbb{R}$, i.e., there are positive constants δ_1 and δ_2 such that the following inequalities hold true (cf., [4], [6], [10])

$$\begin{aligned} &\lambda_{pq}\,\xi_p\,\xi_q \geqslant \delta_1\,\xi_p\,\xi_p && \text{for all} \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3, \\ &c_{kjpq}\,\eta_{kj}\,\eta_{pq} \geqslant \delta_2\,\eta_{kj}\,\eta_{kj} && \text{for all} \quad \eta_{kj} = \eta_{jk} \in \mathbb{R}. \end{aligned} \tag{2.3}$$

The system (2.1)-(2.2) can be written in matrix form

$$A(\partial)U(x) = 0, \qquad x \in \Omega^\pm, \tag{2.4}$$

where $U = (u_1, u_2, u_3, \vartheta)^\top = (u, \vartheta)^\top$,

$$A(\partial) := \big[A_{kj}(\partial)\big]_{4\times4} = \begin{bmatrix} C(\partial) & [-\beta_{kj}\partial_j]_{3\times1} \\ [0]_{1\times3} & \Lambda(\partial) \end{bmatrix}_{4\times4}, \tag{2.5}$$

$$C(\partial)=\big[C_{kp}(\partial)\big]_{3\times3},\qquad C_{kp}(\partial)=c_{kjpq}\partial_j\partial_q,\quad \Lambda(\partial)=\lambda_{pq}\partial_p\partial_q. \tag{2.6}$$

Further, let us introduce the following boundary operators related to the thermo-mechanical stress vector and the heat flux vector

$$P(\partial,n):=\big[\,P_{kp}(\partial,n)\,\big]_{3\times4}=\big[\,[T(\partial,n)]_{3\times3},\,[-\beta_{kj}n_j]_{3\times1}\,\big]_{3\times4}, \tag{2.7}$$

$$B(\partial,n):=\big[\,B_{kp}(\partial,n)\,\big]_{4\times4}=\begin{bmatrix} T(\partial,n) & [-\beta_{kj}n_j]_{3\times1}\\ [0]_{1\times3} & \lambda(\partial,n)\end{bmatrix}_{4\times4}, \tag{2.8}$$

$$T(\partial,n):=\big[\,T_{kp}(\partial,n)\,\big]_{3\times3}=[c_{kjpq}n_j\partial_q]_{3\times3},\quad \lambda(\partial,n)=\lambda_{pq}n_p\,\partial_q\,. \tag{2.9}$$

For a given vector-function $U=(u,\vartheta)^\top$ the four dimensional vector

$$B(\partial,n)U=\big(P(\partial,n)U,\,\lambda(\partial,n)\,\vartheta\big)$$

has the following physical sense: the first three components correspond to the thermo-mechanical stress vector $\big[P(\partial,n)U\big]_k=\big[T(\partial,n)\,u\big]_k-\beta_{kj}\,n_j\,\vartheta$, $k=1,2,3$, while the fourth component corresponds to the normal component of the heat flux vector $\lambda(\partial,n)\,\vartheta$. Note that $T(\partial,n)\,u$ is the usual mechanical stress vector when the thermal effects are not taken into consideration.

Denote by $A^*(\partial)$ the operator adjoint to $A(\partial)$:

$$A^*(\partial):=\big[A^\top(-\partial)\big]=\begin{bmatrix} C(\partial) & [0]_{3\times1}\\ \big[\beta_{kj}\partial_j\big]_{1\times3} & \Lambda(\partial)\end{bmatrix}_{4\times4}, \tag{2.10}$$

and introduce a boundary operator $Q(\partial,n)$ associated with the operator $A^*(\partial)$,

$$Q(\partial,n):=\big[\,Q_{kp}(\partial,n)\,\big]_{4\times4}=\begin{bmatrix} T(\partial,n) & [\,0\,]_{3\times1}\\ [0]_{1\times3} & \lambda(\partial,n)\end{bmatrix}_{4\times4}. \tag{2.11}$$

For vector-functions $U=(u,\vartheta)^\top\in[C^2(\overline{\Omega^+})]^4$ and $U^*=(u^*,\vartheta^*)^\top\in[C^2(\overline{\Omega^+})]^4$ we have the following Green's formulas (cf., [3], [4])

$$\int\limits_{\Omega^+}\Lambda(\partial)\vartheta\ \vartheta^*\,dx=-\int\limits_{\Omega^+}\lambda_{pq}\partial_q\vartheta\,\partial_p\vartheta^*\,dx+\int\limits_{\partial\Omega^+}\{\lambda(\partial,n)\vartheta\}^+\{\vartheta^*\}^+dS, \tag{2.12}$$

$$\int\limits_{\Omega^+}C(\partial)u\cdot u^*\,dx=-\int\limits_{\Omega^+}\widetilde{E}(u,u^*)\,dx+\int\limits_{\partial\Omega^+}\{T(\partial,n)u\}^+\cdot\{u^*\}^+\,dS, \tag{2.13}$$

$$\int\limits_{\Omega^+}A(\partial)U\cdot U^*\,dx=-\int\limits_{\Omega^+}E(U,U^*)\,dx+\int\limits_{\partial\Omega^+}\{B(\partial,n)U\}^+\cdot\{U^*\}^+dS, \tag{2.14}$$

$$\int\limits_{\Omega^+} \big\{A(\partial)U \cdot U^* - U \cdot A^*(\partial)U^*\big\}\, dx = \int\limits_{\partial\Omega^+} \big\{\{B(\partial,n)U\}^+ \cdot \{U^*\}^+ - \{U\}^+ \cdot \{Q(\partial,n)U^*\}^+\big\} dS, \tag{2.15}$$

where

$$E(U,U^*) = c_{kjpq}\,\partial_p u_q\,\partial_k u_j^* - \beta_{kj}\,\vartheta\,\partial_j u_k^* + \lambda_{pq}\,\partial_q\vartheta\,\partial_p\vartheta^*,$$

$$\widetilde{E}(u,u^*) = c_{kjpq}\,\partial_p u_q\,\partial_k u_j^*\,.$$

Here and in what follows the symbols $\{\cdot\}^+$ and $\{\cdot\}^-$ denote one sided limits on S from Ω^+ and Ω^- respectively, while the central dot denotes the scalar product in $\mathbb{R}^3$.

Now let us formulate the interior Neumann type boundary value problem $(N)^+$:

Find a regular vector-function $U \in \big[C^2(\Omega^+)\big]^4 \cap \big[C^1(\overline{\Omega^+})\big]^4$ in the domain Ω^+ satisfying the differential equation

$$A(\partial)U(x) = 0, \qquad x \in \Omega^+, \tag{2.16}$$

and the Neumann type boundary condition

$$\big\{B(\partial,n)U(x)\big\}^+ = F(x), \qquad x \in S, \tag{2.17}$$

where $F = \big(F_1,F_2,F_3,F_4\big)^\top \in [C(S)]^4$ is a given vector-function.
If $F = 0$ we have the homogeneous Neumann type BVP.

3 Uniqueness theorem

Here we prove the following uniqueness theorem.

Theorem 3.1 *The general solution U_0 to the homogeneous Neumann type boundary value problem $(N)^+$ is representable as*

$$U_0(x) = \big(\chi(x) + \vartheta_0 v^{(0)}(x), \vartheta_0\big)^\top = \big(\chi(x), 0\big)^\top + \vartheta_0\big(v^{(0)}(x), 1\big)^\top,$$

where ϑ_0 is an arbitrary constant, $\chi(x) = a \times x + b$ is a rigid displacement vector with $a = (a_1,a_2,a_3)^\top$ and $b = (b_1,b_2,b_3)^\top$ being arbitrary constant vectors, while

$$v_k^{(0)}(x) = \alpha_{kl}\,x_l, \qquad k = 1,2,3,$$

and the constants $\alpha_{kl} = \alpha_{lk}$ are defined by the uniquely solvable system of linear algebraic equations

$$c_{kjpq}\alpha_{pq} = \beta_{kj}, \qquad k,j = 1,2,3\,.$$

Proof. It is easy to see that the homogenous BVP under consideration is decomposed into two boundary value problems:

$$\big[C(\partial)u(x)\big]_k = \beta_{kj}\partial_j u_4(x), \qquad x \in \Omega^+, \tag{3.1}$$

$$\big\{P(\partial,n)U(x)\big\}_k^+ = 0, \quad x \in S, \quad k = 1,2,3, \tag{3.2}$$

and

$$\lambda_{pq}\partial_p\partial_q u_4(x) = 0, \quad x \in \Omega^+,$$
$$\big\{\lambda(\partial,n)u_4(x)\big\}^+ = 0, \quad x \in S,$$

where $U = (u_1, u_2, u_3, u_4)^\top = (u, u_4)^\top$, with $u = (u_1, u_2, u_3)^\top$ and $u_4 := \vartheta$.

By Green's formula (2.12), with the help of positive definiteness of the matrix $[\lambda_{pq}]_{3\times3}$ and the first inequality in (2.3), we easily deduce that $u_4(x) = \vartheta_0 = const$ in Ω^+, where ϑ_0 is an arbitrary constant. Therefore (3.1)-(3.2) along with (2.7) lead to the following non-homogenous Neumann type boundary value problem of elastostatics

$$\big[C(\partial)u(x)\big]_k = 0, \qquad x \in \Omega^+, \qquad k = 1,2,3, \tag{3.3}$$

$$\big\{T(\partial,n)u(x)\big\}_k^+ = \beta_{kj}n_j(x)\,\vartheta_0, \qquad x \in S, \qquad k = 1,2,3. \tag{3.4}$$

It is well known that a non-homogenous Neumann type boundary value problem of elastostatics is not unconditionally solvable, the total stress vector and total momentum of the prescribed boundary stress vector should vanish (see, e.g., [6]). The necessary and sufficient conditions for the non-homogeneous problem (3.3)-(3.4) to be solvable read as

$$\int_S \beta_{pj}n_j\vartheta_0 dS = 0, \qquad \int_S \big(\beta_{pj}n_j\vartheta_0 x_q - \beta_{qj}n_j\vartheta_0 x_p\big)dS = 0, \quad p,q = 1,2,3,$$

which are automatically satisfied for arbitrary constant ϑ_0 due to the Gauss divergence theorem and the symmetry condition $\beta_{pq} = \beta_{qp}$. Therefore the nonhomogeneous problem (3.3)-(3.4) is solvable and the general solution can be represented as

$$u(x) = u^{(0)}(x) + \chi(x),$$

where $u^{(0)} = (u_1^{(0)}, u_2^{(0)}, u_3^{(0)})^\top$ is some particular solution of the problem (3.3)-(3.4) and $\chi(x) = a \times x + b$ is an arbitrary rigid displacement vector with $a = (a_1, a_2, a_3)^\top$ and $b = (b_1, b_2, b_3)^\top$ being arbitrary constant vectors.

Now we construct explicitly the particular solution $u^{(0)}(x)$. It is evident that if a vector $v^{(0)} = (v_1^{(0)}, v_2^{(0)}, v_3^{(0)})^\top$ solves the problem

$$\big[C(\partial)v^{(0)}(x)\big]_k = 0, \qquad x \in \Omega^+, \qquad k = 1,2,3, \tag{3.5}$$

$$\big[T(\partial,n)v^{(0)}(x)\big]_k^+ = \beta_{kj}n_j(x), \qquad x\in S, \qquad k=1,2,3, \tag{3.6}$$

then $u^{(0)}(x)=\vartheta_0 v^{(0)}(x)$ is a sought for particular solution. Let us look for the vector $v^{(0)}=(v_1^{(0)},v_2^{(0)},v_3^{(0)})^\top$ in the form

$$v_k^{(0)}(x)=\alpha_{kl}\,x_l, \qquad k=1,2,3, \tag{3.7}$$

where $\alpha_{kl}=\alpha_{lk}$ are constants to be defined. Clearly the vector-function $v^{(0)}$ automatically satisfies equation (3.5). Note that

$$\begin{aligned}\big[T(\partial,n)v^{(0)}(x)\big]_k &= T_{kp}(\partial,n)v_p^{(0)}(x)=c_{kjpq}n_j\partial_q(\alpha_{pl}x_l)\\ &= c_{kjpq}n_j\alpha_{pl}\delta_{ql}=c_{kjpq}n_j\alpha_{pq}, \qquad k=1,2,3,\end{aligned}$$

where δ_{ql} is the Kronecker delta. Therefore the boundary conditions (3.6) lead to the equations

$$c_{kjpq}n_j\,\alpha_{pq}=\beta_{kj}\,n_j, \quad k=1,2,3.$$

Whence, equating the coefficients of n_j, we arrive at the system of linear algebraic equations

$$c_{kjpq}\,\alpha_{pq}=\beta_{kj}, \qquad k,j=1,2,3. \tag{3.8}$$

Since the quadratic form $c_{kjpq}\gamma_{pq}\gamma_{kj}$ in symmetric variables $\gamma_{jk}=\gamma_{kj}$ is positive definite due to the second inequality in (2.3), we conclude that the system (3.8) is uniquely solvable and the constants α_{pq} are uniquely defined. Consequently, a particular solution $u^{(0)}(x)$ is constructed explicitly,

$$u^{(0)}(x)=\vartheta_0\,v^{(0)}(x)+\chi(x), \quad x\in\Omega^+,$$

where $v^{(0)}$ is given by (3.7) with coefficients α_{kl} defined by the system (3.8). □

Remark 3.2 *Let us introduce a space of generalized rigid displacement vectors $\mathfrak{X}$ which is defined by the basis vectors*

$$\begin{aligned}&\Psi^{(1)}=\big(1,0,0,0\big)^\top, \quad \Psi^{(2)}=\big(0,1,0,0\big)^\top, \quad \Psi^{(3)}=\big(0,0,1,0\big)^\top,\\ &\Psi^{(4)}=\big(0,-x_3,x_2,0\big)^\top, \quad \Psi^{(5)}=\big(x_3,0,-x_1,0\big)^\top,\\ &\Psi^{(6)}=\big(-x_2,x_1,0,0\big)^\top, \quad \Psi^{(7)}=\big(v_1^{(0)},v_2^{(0)},v_3^{(0)},1\big)^\top,\end{aligned} \tag{3.9}$$

where $v_k^{(0)}$, $k=1,2,3$, are the same as above and are given by (3.7).

It is evident that all the vectors-functions $\Psi^{(k)}$, $k=\overline{1,7}$, represent solutions to the homogenous Neumann type BVP $(N)^+$. Moreover, due to Theorem 3.1 any solution of the homogenous BVP $(N)^+$ can be represented as a linear combination of the vector-functions (3.9).

4 Properties of potential type operators

4.1 Fundamental matrices

Denote by $\Gamma(x) = \big[\Gamma_{kj}(x)\big]_{4\times 4}$ the fundamental matrix of the operator $A(\partial_x)$,

$$A(\partial_x)\Gamma(x-y) = I_4\,\delta(x-y),$$

where $\delta(\cdot)$ is Dirac's distribution. Here and in what follows I_k stands for the $k\times k$ unit matrix.

The fundamental matrix $\Gamma(x)$ can be constructed explicitly and is written in the form

$$\Gamma(x) = F^{-1}_{\xi\to x}\big[A^{-1}(-i\xi)\big],$$

where $\mathcal{F}^{-1}{}_{\xi\to x}$ is the inverse Fourier transform[1] and $A^{-1}(-i\xi)$ is the matrix inverse to $A(-i\xi)$,

$$A^{-1}(-i\xi) = \frac{1}{\det A(-i\xi)}A^{(c)}(-i\xi),$$

where $A^{(c)}(-i\xi)$ is the matrix of co-factors of the matrix $A(-i\xi)$.

Note that $\det A(-i\xi)$ is a homogeneous polynomial of order 8 in variables $\xi=(\xi_1,\xi_2,\xi_3)$, and, moreover, $\det A(-i\xi)\neq 0$ for $\xi\in\mathbb{R}^3\setminus\{0\}$. It is also evident that the entries of the matrix $A^{(c)}(-i\xi)$ are also homogeneous polynomials and at the origin and at infinity they have the following asymptotic behaviour:

$$\begin{aligned}
&A^{(c)}_{kj}(-i\xi) = \mathcal{O}(|\xi|^6), \quad k,j=1,2,3,\\
&A^{(c)}_{j4}(-i\xi) = \mathcal{O}(|\xi|^5), \quad A^{(c)}_{4j}(-i\xi) = 0, \quad j=1,2,3,\\
&A^{(c)}_{44}(-i\xi) = \mathcal{O}(|\xi|^6).
\end{aligned}$$

The functions $A^{(c)}_{j4}(-i\xi)$ for $j=1,2,3$, are odd polynomials in ξ. Consequently, the entries

$$K_j(\xi) := \frac{A^{(c)}_{j4}(-i\xi)}{\det A(-i\xi)}, \qquad j=1,2,3,$$

are odd functions in ξ: $K_j(-\xi) = -K_j(\xi)$, $j=\overline{1,3}$. Therefore for the homogenous functions $K_j(\xi)$ of order -3 the cancelation Tricomi conditions

$$\int\limits_{|\xi|=1} K_j(\xi)dS = 0, \qquad j=1,2,3, \tag{4.1}$$

[1]For absolutely integrable functions f and g the direct and inverse Fourier transforms are defined as follows $\mathcal{F}_{x\to\xi}[f(x)] = \int_{\mathbb{R}^3} f(x)\,e^{i\,x\cdot\xi}\,dx$ and $\mathcal{F}^{-1}{}_{\xi\to x}[g(\xi)] = (2\pi)^{-3}\int_{\mathbb{R}^3} g(\xi)\,e^{-i\,x\cdot\xi}\,d\xi$ respectively.

hold. Whence it follows that the generalized inverse Fourier transforms of these functions $\widehat{K}_j(x) \equiv \mathcal{F}^{-1}_{\xi\to x}[K_j(\xi)]$, understood in the Cauchy principal value sense, are homogenous functions of order zero and satisfy the same type cancelation Tricomi conditions (see [8, Ch. 2, Proposition 2.16])

$$\int\limits_{|x|=1} \widehat{K}_j(x)dS = 0, \qquad j = \overline{1,3}. \tag{4.2}$$

Since the generalized Fourier transform and inverse Fourier transform of a homogeneous function of order $-r < 0$ is a homogenous function of order $-3 + r$ provided $0 < r < 3$ (see, e.g., [8, Ch. 2, Proposition 2.13]), the entries of the fundamental matrix $\Gamma(x)$ are homogeneous functions in x and in a neighbourhood of the origin and infinity have the following asymptotic behaviour

$$\Gamma(x) = \begin{bmatrix} [\mathcal{O}(|x|^{-1})]_{3\times3} & [\mathcal{O}(1)]_{3\times1} \\ [0]_{1\times3} & \mathcal{O}(|x|^{-1}) \end{bmatrix}_{4\times4},$$

i.e., the entries $\Gamma_{kj}(x)$ for $k,j = \overline{1,3}$, or $k = j = 4$, are homogeneous function of order -1, while the entries $\Gamma_{k4}(x)$ for $1 \leqslant k \leqslant 3$ are homogeneous functions of zero order. Moreover, form (4.1) and (4.2) it follows that

$$\int\limits_{|x|=1} \Gamma_{k4}(x)dS = 0, \qquad k = 1,2,3. \tag{4.3}$$

It is evident that the matrix $\Gamma^*(x-y) = \big[\Gamma(y-x)\big]^\top_{4\times4}$ is the fundamental matrix of the adjoint operator $A^*(\partial)$ and

$$\Gamma^*(x) = \begin{bmatrix} [\mathcal{O}(|x|^{-1})]_{3\times3} & [0]_{3\times1} \\ [\mathcal{O}(1)]_{1\times3} & \mathcal{O}(|x|^{-1}) \end{bmatrix}_{4\times4}. \tag{4.4}$$

4.2 Classes $Z(\Omega^-)$ and $Z^*(\Omega^-)$

Here we introduce special classes $Z(\Omega^-)$ and $Z^*(\Omega^-)$ of vector-functions needed in our analysis below.

Definition 4.1 *A vector-function $U = (u_1, u_2, u_3, \vartheta)^\top$ is said to belong to the class $Z(\Omega^-)$ if it is continuous in a neighbourhood of infinity and satisfies the following asymptotic conditions*

$$u_k(x) = \mathcal{O}(1), \quad k = 1,2,3, \qquad \vartheta(x) = \mathcal{O}(|x|^{-1}) \quad as \quad |x| \to \infty,$$
$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int_{\Sigma(0,R)} u_k(x)\, d\Sigma(0,R) = 0, \quad k = 1,2,3,$$

where $\Sigma(0,R)$ is a sphere centered at the origin and radius R.

Definition 4.2 *A vector-function* $U^* = (u_1^*, u_2^*, u_3^*, \vartheta^*)^\top$ *is said to belong to the class* $Z^*(\Omega^-)$ *if it is continuous in a neighbourhood of infinity and satisfies the following asymptotic conditions*

$$u_k^*(x) = \mathcal{O}(|x|^{-1}), \quad k = 1,2,3, \qquad \vartheta^*(x) = \mathcal{O}(1) \quad as \quad |x| \to \infty, \tag{4.5}$$

$$\lim_{R\to\infty} \frac{1}{4\pi R^2} \int_{\Sigma(0,R)} \vartheta^*(x)\, d\Sigma(0,R) = 0, \quad k = 1,2,3. \tag{4.6}$$

These classes play a crucial role in the study of exterior problems (see [2]).

4.3 Layer potentials

The single layer potentials V, V^*, and double layer potentials W, W^*, related to the fundamental solutions $\Gamma(x-y)$ and $\Gamma^*(x-y)$ read as follows

$$V(h)(x) = V_S(h)(x) := \int_S \Gamma(x-y)\, h(y)\, dS_y, \qquad x \in \mathbb{R}^3 \setminus S, \tag{4.7}$$

$$W(h)(x) = W_S(h)(x) := \int_S \big[Q\big(\partial_y, n(y)\big)\Gamma^\top(x-y)\big]^\top h(y)\, dS_y, \qquad x \in \mathbb{R}^3 \setminus S, \tag{4.8}$$

$$V^*(\varphi^*)(x) = V_S^*(\varphi^*)(x) := \int_S \Gamma^*(x-y)\, \varphi^*(y)\, dS_y, \quad x \in \mathbb{R}^3 \setminus S, \tag{4.9}$$

$$W^*(\varphi^*)(x) = W_S^*(\varphi^*)(x) := \int_S \big[B\big(\partial_y, n(y)\big)[\Gamma^*(x-y)]^\top\big]^\top \varphi^*(y)\, dS_y, \qquad x \in \mathbb{R}^3 \setminus S, \tag{4.10}$$

where $h = (h_1, h_2, h_3, h_4)^\top$ and $\varphi^* = (\varphi_1^*, \varphi_2^*, \varphi_3^*, \varphi_4^*)^\top$ are the corresponding densities defined on S, while the operators $B(\partial, n)$ and $Q(\partial, n)$ are defined in (2.8) and (2.11) respectively.

The properties of these potentials are described by the following assertions.

Theorem 4.3 *The single and double layer potentials,* V *and* W*, solve the homogeneous equation* $A(\partial)U(x) = 0$ *in* $\mathbb{R}^3 \setminus S$ *and belong to the class* $Z(\Omega^-)$*.*

Proof. It can be found in [2]. □

Theorem 4.4 *The single and double layer potentials,* V^* *and* W^**, solve the homogeneous equation* $A^*(\partial)U^*(x) = 0$ *in* $\mathbb{R}^3 \setminus S$ *and belong to the class* $Z^*(\Omega^-)$*.*

Proof. The first part of the theorem follows from the relations

$$\begin{aligned}\big[A^*(\partial_x)V^*(\varphi^*)(x)\big]_k &= A^*_{kj}(\partial_x)\big[V^*(\varphi^*)(x)\big]_j \\ &= A^*_{kj}(\partial_x)\int\limits_S \Gamma^*_{jp}(x-y)\varphi^*_p(y)dS_y \\ &= \int\limits_S A^*_{kj}(\partial_x)\Gamma^*_{jp}(x-y)\varphi^*_p(y)dS_y = 0, \qquad k=\overline{1,4},\end{aligned}$$

since

$$A^*_{kj}(\partial_x)\Gamma^*_{jp}(x-y) = \big[A^*(\partial_x)\Gamma^*(x-y)\big]_{kp} = 0, \qquad k,p=\overline{1,4}, \qquad x\neq y.$$

Similarly for the double layer potential we have,

$$\begin{aligned}\big[A^*(\partial_x)W^*(\varphi^*)(x)\big]_k &= A^*_{kj}(\partial_x)\Big[\int\limits_S \big[B\big(\partial_y,n(y)\big)[\Gamma^*(x-y)]^\top\big]^\top\varphi^*(y)dS_y\Big]_j \\ &= A^*_{kj}(\partial_x)\int\limits_S \big[B(\partial_y,n(y))[\Gamma^*(x-y)]^\top\big]_{pj}\varphi^*_p(y)dS_y \\ &= A^*_{kj}(\partial_x)\int\limits_S B_{pm}\big(\partial_y,n(y)\big)\Gamma^*_{jm}(x-y)\varphi^*_p(y)dS_y \\ &= \int\limits_S B_{pm}\big(\partial_y,n(y)\big)A^*_{kj}(\partial_x)\Gamma^*_{jm}(x-y)\varphi^*_p(y)dS_y = 0, \\ &\qquad k=\overline{1,4}.\end{aligned}$$

To prove the second part of the theorem, let us use the asymptotic property of the fundamental matrix $\Gamma^*(x-y)$ at infinity (see (4.4)). It can be shown that if y belongs to a compact set, say S, and $|x|$ is sufficiently large, then the following relation holds

$$\big[Q(\partial_y,n(y))[\Gamma^*(x-y)]^\top\big]^\top = \begin{bmatrix} [\mathcal{O}(|x|^{-2})]_{3\times3} & [\mathcal{O}(|x|^{-1})]_{3\times1} \\ [0]_{1\times3} & \mathcal{O}(|x|^{-2}) \end{bmatrix}_{4\times4}.$$

Therefore

$$\big[W^*(\varphi^*)(x)\big]_k = \begin{cases} \mathcal{O}(|x|^{-2}), & k=\overline{1,3}, \\ \mathcal{O}(|x|^{-1}), & k=4, \end{cases} \qquad \text{as } |x|\to\infty.$$

Whence the inclusion $W^*(\varphi^*)\in Z^*(\Omega^-)$ follows immediately.

To prove the same type inclusion for the single layer potential we proceed as follows. First, let us note that $\Gamma^*(x-y)=\Gamma^*(x)+\mathcal{O}(|x|^{-1})$ for

$y \in S$ and $|x|$ sufficiently large. Therefore, in accordance with (4.4), we have

$$[V^*(\varphi^*)(x)]_k = \mathcal{O}(|x|^{-1}), \qquad k = 1, 2, 3,$$

$$\begin{aligned}
[V^*(\varphi^*)(x)]_4 &= \sum_{p=1}^{4} \int\limits_S \Gamma^*_{4p}(x-y)\,\varphi^*_p(y)\,dS_y \\
&= \sum_{p=1}^{4} \int\limits_S \Big[\Gamma^*_{4p}(x) + \mathcal{O}(|x|^{-1})\Big]\varphi^*_p(y)\,dS_y \\
&= \sum_{p=1}^{3} \Gamma^*_{4p}(x) \int\limits_S \varphi^*_p(y)\,dS_y + \Gamma^*_{44}(x) \int\limits_S \varphi^*_4(y)\,dS_y + \mathcal{O}(|x|^{-1}) \\
&= \Gamma^*_{4p}(x) \int\limits_S \varphi^*_p(y)dS + \mathcal{O}(|x|^{-1}).
\end{aligned}$$

Whence, due to (4.3) we get

$$\begin{aligned}
&\lim_{R\to\infty} \frac{1}{4\pi R^2} \int\limits_{\Sigma(0,R)} [V^*(\varphi^*)(x)]_4\, d\Sigma(0,R) = \\
&\quad = \lim_{R\to\infty} \frac{1}{4\pi R^2} \int\limits_{\Sigma(0,R)} \Big\{\Gamma^*_{4p}(x) \int\limits_S \varphi^*_p(y)dS + \mathcal{O}(|R|^{-1})\Big\} d\Sigma(0,R) \\
&\quad = \lim_{R\to\infty} \frac{1}{4\pi R^2} \Big\{ \int\limits_{\Sigma(0,R)} \Gamma^*_{4p}(x) d\Sigma(0,R) \int\limits_S \varphi^*_4(y)dS + \int\limits_{\Sigma(0,R)} \mathcal{O}(|R|^{-1}) d\Sigma(0,R)\Big\} \\
&\quad = \lim_{R\to\infty} \frac{1}{4\pi R^2} \int\limits_{\Sigma(0,R)} \mathcal{O}(|R|^{-1}) d\Sigma(0,R) = 0,
\end{aligned}$$

which completes the proof. □

By standard arguments it can be shown that if regular vector-functions U and U^* solve the homogeneous equations $A(\partial)U(x) = 0$ and $A^*(\partial)U^*(x) = 0$ in Ω^+, respectively, then the following integral representation formulas, called also *Green's third formulas*, hold

$$W\big(\{U\}^+\big)(x) - V\big(\{BU\}^+\big)(x) = \begin{cases} U(x) & \text{for} \quad x \in \Omega^+, \\ 0 & \text{for} \quad x \in \Omega^-. \end{cases} \tag{4.11}$$

$$W^*\big(\{U^*\}^+\big)(x) - V^*\big(\{QU^*\}^+\big)(x) = \begin{cases} U^*(x) & \text{for} \quad x \in \Omega^+, \\ 0 & \text{for} \quad x \in \Omega^-. \end{cases} \tag{4.12}$$

The counterparts of these formulas hold also true for the unbounded domain Ω^- assuming that vector-functions possess appropriate asymptotic properties at infinity.

For example, if a regular vector-function U solves the homogenous equation $A(\partial)U(x) = 0$ in Ω^- and belongs to the space $Z(\Omega^-)$ then (see [2])

$$-W\big(\{U\}^-\big) + V\big(\{BU\}^-\big) = \begin{cases} U(x) & \text{for} \quad x \in \Omega^-, \\ 0 & \text{for} \quad x \in \Omega^+. \end{cases}$$

Now we derive the similar integral representation formula of a regular solution to the homogeneous equation $A^*(\partial)U^*(x) = 0$ in Ω^- provided that the vector-function U^* belongs to the class of vector-functions satisfying the asymptotic properties (4.5)-(4.6), i.e. $U^* \in Z^*(\Omega^-)$. To this purpose, let us write the integral representation formula (4.12) for the bounded domain $\Omega_R^- := \Omega^- \cap B(0,R)$, where R is a sufficiently large positive number and $B(0,R) := \{x \in \mathbb{R}^3 : |x| < R\}$ is a ball centered at the origin and radius R, such that $\overline{\Omega^+} \subset B(0,R)$,

$$W^*_{\partial\Omega_R^-}\big(\{U^*\}^+\big)(x) - V^*_{\partial\Omega_R^-}\big(\{QU^*\}^+\big)(x) = \begin{cases} U^*(x) & \text{for} \quad x \in \Omega_R^-, \\ 0 & \text{for} \quad x \in \mathbb{R}^3 \setminus \overline{\Omega_R^-}, \end{cases}$$

where $\partial\Omega_R^- = S \cup \Sigma_R$ with $\Sigma_R = \partial\, B(0,R)$. From this relation we have

$$U^*(x) = -W_S^*(\{U^*\}_S^-) + V_S^*(\{QU^*\}_S^-) + \Phi_R(x), \qquad x \in \Omega_R^-, \tag{4.13}$$

$$0 = -W_S^*(\{U^*\}_S^-) + V_S^*(\{QU^*\}_S^-) + \Phi_R(x), \quad x \in \Omega^+ \cup \big[\mathbb{R}^3 \setminus \overline{B(0,R)}\,\big], \tag{4.14}$$

where V_S^* and W_S^* with $S = \partial\Omega^+$ are the single and double layer potentials defined by formulas (4.9) and (4.10) respectively, while

$$\Phi_R(x) := W^*_{\Sigma_R}\big(\{U^*\}^+_{\Sigma_R}\big)(x) - V^*_{\Sigma_R}\big(\{QU^*\}^+_{\Sigma_R}\big)(x) \tag{4.15}$$

with $V^*_{\Sigma_R}$ and $W^*_{\Sigma_R}$ being again the single and double layer potentials with the integration surface Σ_R. From equality (4.15) it follows that

$$A^*(\partial)\Phi_R(x) = 0, \qquad x \notin \Sigma_R. \tag{4.16}$$

Moreover, from (4.13) and (4.14) we have

$$\begin{aligned} \Phi_R(x) &= U^*(x) + W_S^*\big(\{U^*\}_S^-\big) - V_S^*\big(\{QU^*\}_S^-\big), \qquad x \in \Omega_R^-, \\ \Phi_R(x) &= W_S^*\big(\{U^*\}_S^-\big) - V_S^*\big(\{QU^*\}_S^-\big), \qquad x \in \Omega^+ \cup \big[\mathbb{R}^3 \setminus \overline{B(0,R)}\,\big]. \end{aligned} \tag{4.17}$$

These relations imply that for sufficiently large numbers $R_1 < R_2$,

$$\Phi_{R_1}(x) = \Phi_{R_2}(x) \quad \text{for} \quad |x| < R_1 < R_2. \tag{4.18}$$

Therefore, for arbitrary $x \in \mathbb{R}^3$ the following limit exists

$$\Phi(x) := \lim_{R\to\infty} \Phi_R(x) = \begin{cases} U^*(x) + W_S^*\big(\{U^*\}_S^-\big) - V_S^*\big(\{QU^*\}_S^-\big), & x \in \Omega^-, \\ W_S^*\big(\{U^*\}_S^-\big) - V_S^*\big(\{QU^*\}_S^-\big), & x \in \Omega^+. \end{cases}$$

Consequently,

$$A^*(\partial)\Phi(x) = 0, \qquad x \in \Omega^+ \cup \Omega^-.$$

On the other hand, from (4.18) we get

$$\Phi(x) = \lim_{R\to\infty} \Phi_R(x) = \Phi_{R_1}(x)$$

for arbitrary $x \in \mathbb{R}^3$ with $R_1 > |x|$ and $\overline{\Omega^+} \subset B(0, R_1)$. From (4.15) and (4.16) then we conclude

$$A^*(\partial)\Phi(x) = 0, \qquad x \in \mathbb{R}^3. \tag{4.19}$$

At the same time from (4.17) we see that

$$\Phi \in Z^*(\mathbb{R}^3), \tag{4.20}$$

since $U \in Z^*(\Omega^-)$ and $W_S^*, V_S^* \in Z^*(\Omega^-)$ due to Theorem 4.4. Further we show that

$$\Phi(x) = 0, \qquad \forall x \in \mathbb{R}^3.$$

Indeed, from the relations (4.19) by the Fourier transform we have

$$A^*(-i\xi)\widehat{\Phi}(\xi) = 0, \qquad \xi \in \mathbb{R}^3,$$

where the Fourier transform $\widehat{\Phi}(\xi)$ is a vector-function from the Schwartz space of tempered distributions in view of (4.20). Since the determinant $\det A(-i\xi)$ is nonsingular for $\xi \in \mathbb{R}^3 \setminus \{0\}$ (see Subsection 4.1), it follows that the support of the distribution $\widehat{\Phi}$ is the origin $\xi = 0$. Consequently, $\widehat{\Phi}$ is a linear combination of the Dirac distribution and its derivatives,

$$\widehat{\Phi}(\xi) = \sum_{|\alpha|\leqslant M} C_\alpha \delta^{(\alpha)}(\xi),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index, $C_\alpha = (C_{\alpha,1}, C_{\alpha,2}, C_{\alpha,3}, C_{\alpha,4})^\top$ is a constant vector, M is some nonnegative integer, and $\delta^{(\alpha)}$ stands for the α-th order derivative of Dirac's distribution δ. Therefore the vector-function $\Phi(x)$ is a polynomial in x,

$$\Phi(x) = \sum_{|\alpha|\leqslant M} C_\alpha x^\alpha, \qquad x \in \mathbb{R}^3.$$

Since $\Phi \in Z^*(\mathbb{R}^3)$, the conditions (4.5) and (4.6) imply that

$$\Phi(x) = 0, \qquad x \in \mathbb{R}^3.$$

Now, passing to the limit in (4.13) as $R \to \infty$, we arrive at the general integral representation formula of a regular solution satisfying the property $Z^*(\Omega^-)$,

$$-W^*\big(\{U^*\}^-\big) + V^*\big(\{QU^*\}^-\big) = \begin{cases} U^*(x) & \text{for} \quad x \in \Omega^-, \\ 0 & \text{for} \quad x \in \Omega^+. \end{cases} \tag{4.21}$$

The mapping properties of the above introduced layer potentials are described by the following assertions.

Theorem 4.5 *Let $S \in C^{k+1,\alpha}$ with $k \geqslant 1$ and $0 < \beta < \alpha \leqslant 1$. Then the following operators are continuous*

$$\begin{aligned} V &: [C^{k,\beta}(S)]^4 \to [C^{k+1,\beta}(\overline{\Omega^\pm})]^4, \quad W : [C^{k,\beta}(S)]^4 \to [C^{k,\beta}(\overline{\Omega^\pm})]^4, \\ V^* &: [C^{k,\beta}(S)]^4 \to [C^{k+1,\beta}(\overline{\Omega^\pm})]^4, \quad W^* : [C^{k,\beta}(S)]^4 \to [C^{k,\beta}(\overline{\Omega^\pm})]^4. \end{aligned}$$

Proof. It is word for word of the proof of the corresponding theorems in [6], [3], [4]. □

Theorem 4.6 *Let $S \in C^{2,\alpha}$ with $0 < \beta < \alpha \leqslant 1$, $h \in [C^{0,\beta}(S)]^4$, and $g \in [C^{1,\beta}(S)]^4$. Then the following relations hold for all $x \in S$:*

$$\{V(h)(x)\}^\pm = \mathcal{H}(h)(x), \tag{4.22}$$

$$\{B(\partial_x, n(x))V(h)(x)\}^\pm = [\mp 2^{-1} I_4 + \mathcal{K}]h(x), \tag{4.23}$$

$$\{W(h)(x)\}^\pm = [\pm 2^{-1} I_4 + \mathcal{N}]h(x), \tag{4.24}$$

$$\{B(\partial_x, n(x))W(g)(x)\}^+ = \{B(\partial_x, n(x))W(g)(x)\}^- =: \mathcal{L}g(x), \tag{4.25}$$

$$\{V^*(\varphi^*)(x)\}^\pm = \mathcal{H}^*(\varphi^*)(x), \tag{4.26}$$

$$\{Q(\partial_x, n(x))V^*(\varphi^*)(x)\}^\pm = [\mp 2^{-1} I_4 + \mathcal{N}^*]\varphi^*(x), \tag{4.27}$$

$$\{W^*(\varphi^*)(x)\}^\pm = [\pm 2^{-1} I_4 + \mathcal{K}^*]\varphi^*(x), \tag{4.28}$$

$$\{Q(\partial_x, n(x))W^*(\psi^*)(x)\}^+ = \{Q(\partial_x, n(x))W^*(\psi^*)(x)\}^- =: \mathcal{L}^*\psi^*(x), \tag{4.29}$$

where $\mathcal{H}$ and $\mathcal{H}^*$ are weakly singular integral operators, $\mathcal{K}$, $\mathcal{N}$, $\mathcal{K}^*$, and $\mathcal{N}^*$ are singular integral operators, while $\mathcal{L}$ and $\mathcal{L}^*$ are singular integro-differential operators,

$$\mathcal{H}h(x) := \int\limits_S \Gamma(x - y)h(y)dS_y, \tag{4.30}$$

$$\mathcal{K}h(x) := \int_S [B(\partial_x, n(x))\Gamma(x-y)]h(y)dS_y, \tag{4.31}$$

$$\mathcal{N}h(x) := \int_S [Q(\partial_y, n(y))\Gamma^\top(x-y)]^\top h(y)dS_y, \tag{4.32}$$

$$\mathcal{L}g(x) := \lim_{\Omega^\pm \ni z \to x \in S} B(\partial_z, n(x) \int_S [Q(\partial_y, n(y))\Gamma^\top(x-y)]^\top g(y)dS_y, \tag{4.33}$$

$$\mathcal{H}^*\varphi^*(x) := \int_S \Gamma^*(x-y)\varphi^*(y)dS_y, \tag{4.34}$$

$$\mathcal{K}^*\varphi^*(x) := \int_S \Big[B(\partial_y, n(y))[\Gamma^*(x-y)]^\top\Big]^\top \varphi^*(y)dS_y, \tag{4.35}$$

$$\mathcal{N}^*\varphi^*(x) := \int_S [Q(\partial_x, n(x))\Gamma^*(x-y)]\varphi^*(y)dS_y, \tag{4.36}$$

$$\mathcal{L}^*\psi^*(x) := \lim_{\Omega^\pm \ni z \to x \in S} Q(\partial_x, n(x)$$
$$\times \int_S \Big[B(\partial_y, n(y))[\Gamma^*(x-y)]^\top\Big]^\top \varphi^*(y)dS_y. \tag{4.37}$$

Proof. The jump relations (4.22), (4.23), (4.24), (4.26), (4.27),and (4.28) can be proved by standard arguments (see [6], [3], [4]). The so called Liapunov-Tauber type theorem (4.25) for the double layer potential W is proved in [2]. Here we present a very simple proof of the Liapunov-Tauber type theorem (4.29) for the adjoint double layer potential W^*.

Let $U^* := W^*(h)$ with $h \in [C^{1,\alpha}(S)]^4$. Evidently $U^* \in [C^{1,\alpha}(\overline{\Omega^\pm})]^4 \cap Z^*(\Omega^-)$ and it satisfies the homogeneous equation (2.4). Therefore we can write a general integral representation formulas (4.12) and (4.21) for the vector-function U^* in $\Omega^\pm$. By adding these formulas termwise we get

$$U^*(x) = W^*([U^*]_S)(x) - V^*([QU^*]_S)(x), \qquad x \in \Omega^+ \cup \Omega^-, \tag{4.38}$$

where $[\Psi]_S$ denotes the jump of a function Ψ across the surface S, $[\Psi]_S := \{\Psi\}^+ - \{\Psi\}^-$. Note that due to the jump relations for the double layer potential we have

$$[U^*]_S = [W^*(h)]_S = \{W^*(h)\}^+ - \{W^*(h)\}^- = h.$$

Therefore from (4.38) it follows that

$$W^*(h)(x) = W^*(h)(x) - V^*([QW^*(h)]_S)(x), \qquad x \in \Omega^+ \cup \Omega^-,$$

implying $V^*([QW^*(h)]_S)(x) = 0$ for $x \in \Omega^+ \cup \Omega^-$. Let us set $\Phi := [QW^*(h)]_S$. Then evidently $V^*(\Phi)(x) = 0$ for all $x \in \Omega^+ \cup \Omega^-$ and in view of (4.27) we deduce

$$0 = \{QV^*(\Phi)\}^- - \{QV^*(\Phi)\}^+ = \Phi$$

$$\equiv [QW^*(h)]_S = \{QW^*(h)\}^+ - \{QW^*(h)\}^-,$$

and, consequently, $\{QW^*(h)(x)\}^+ = \{QW^*(h)(x)\}^-$ for $x \in S$ which completes the proof. □

Remark 4.7 *Note that if $S \in C^{2,\alpha}$ with $0 < \beta < \alpha \leqslant 1$, then the operators*

$$\mathcal{H},\ \ \mathcal{H}^*\ :\ [C^{k,\beta}(S)]^4 \to [C^{k+1,\beta}(S)]^4,$$

are invertible (see [4, Remark 12.2]) and the operators

$$\mathcal{H}^{-1},\ \ [\mathcal{H}^*]^{-1}\ :\ [C^{k+1,\beta}(S)]^4 \to [C^{k,\beta}(S)]^4$$

are continuous.

4.4 Uniqueness theorems for the Dirichlet problems

Here we recall the following uniqueness results for the Dirichlet interior and exterior boundary value problems.

Theorem 4.8 *The homogenous interior and exterior Dirichlet type boundary value problems*

$$A(\partial)U(x) = 0 \quad in \quad \Omega^{\pm}, \qquad\qquad \{U(x)\}^{\pm} = 0 \quad on \quad S,$$

where $A(\partial)$ is defined in (2.5)-(2.6), possess only the trivial solution in the class of regular vector-functions $\big[C^2(\Omega^{\pm})\big]^4 \cap \big[C^1(\overline{\Omega^{\pm}})\big]^4 \cap Z(\Omega^-)$.

proof. It can be found in [2]. □

Theorem 4.9 *The homogenous interior and exterior Dirichlet type "adjoint" boundary value problems*

$$A^*(\partial)U^*(x) = 0 \quad in \quad \Omega^{\pm}, \qquad\qquad \{U^*(x)\}^{\pm} = 0 \quad on \quad S,$$

where $A^*(\partial)$ is defined in (2.10), possess only the trivial solution in the class of regular vector-functions $\big[C^2(\Omega^{\pm})\big]^4 \cap \big[C^1(\overline{\Omega^{\pm}})\big]^4 \cap Z^*(\Omega^-)$.

Proof. We start with the interior problem. Let $U = (u, \vartheta)^\top \in \big[C^2(\Omega^+)\big]^4 \cap \big[C^1(\overline{\Omega^+})\big]^4$ be a regular solution of the homogenous interior Dirichlet type boundary value problem, i.e.,

$$A^*(\partial)U^*(x) = 0 \quad \text{for} \quad x \in \Omega^+, \qquad \{U^*(x)\}^+ = 0 \quad \text{for} \quad x \in S.$$

From (2.10) it then follows that the original problem is decomposed into two boundary value problems:

$$\begin{aligned} &\big[C(\partial)u^*(x)\big]_k = 0, \qquad x \in \Omega^+, \\ &\big\{u_k^*(x)\big\}^+ = 0, \quad x \in S, \quad k = 1, 2, 3, \end{aligned} \tag{4.39}$$

and

$$\lambda_{pq}\partial_p\partial_q u_4^*(x) = \beta_{kj}\partial_j u_k^*(x), \quad x \in \Omega^+, \tag{4.40}$$

$$\big\{u_4^*(x)\big\}^+ = 0, \quad x \in S. \tag{4.41}$$

By Green's formula (2.13) in view of the boundary condition (4.39) we easily deduce $u^*(x) = 0$ for $x \in \Omega^+$. Therefore equation (4.40) takes the form $\lambda_{pq}\partial_p\partial_q u_4^*(x) = 0$ for $x \in \Omega^+$ and by Green's formula (2.12) along with the boundary condition (4.41) we conclude $u_4^*(x) = 0$ in Ω^+.

In the case of the exterior problem, we have to apply similar arguments and take into account that $U^* \in \big[C^2(\Omega^-)\big]^4 \cap \big[C^1(\overline{\Omega^-})\big]^4 \cap Z^*(\Omega^-)$. Evidently, the counterpart of Green's formula (2.13) for exterior domain Ω^- holds true for vector-function u^* satisfying the decay condition (4.5) and we again deduce that $u^*(x) = 0$ for $x \in \Omega^-$. Consequently, for the unknown function u_4^* we arrive at the following exterior boundary value problem

$$\lambda_{pq}\partial_p\partial_q u_4^*(x) = 0 \quad \text{for} \quad x \in \Omega^-, \qquad \big\{u_4^*(x)\big\}^- = 0 \quad \text{for} \quad x \in S,$$

where u_4^* is a regular bounded function at infinity satisfying the condition (4.6). Therefore, due to Lemma A.1 in [9], we conclude that $u_4^*(x) = C + \mathcal{O}(|x|^{-1})$ with C being a constant summand, which implies $u_4^*(x) = 0$ in Ω^-. □

5 Existence results for the interior Neumann type problem

Here we investigate the existence of solutions to the nonhomogeneous Neumann type BVP by the potential method and derive the necessary and sufficient conditions for the problem to be solvable.

5.1 Reduction to the system of integral equations

We look for a solution to the nonhomogeneous Neumann type BVP (2.16)–(2.17) in the form of single layer potential

$$U(x) = V(\varphi)(x) \equiv \int_S \Gamma(x-y)\,\varphi(y)\,dS_y, \qquad x \in \Omega^+, \tag{5.1}$$

where $\varphi = \big(\varphi_1, \varphi_2, \varphi_3, \varphi_4\big)^\top$ is a sought for density, $\Gamma(x-y)$ is the fundamental matrix of the operator $A(\partial)$.

Due to the jump relations of the single layer potential (see Theorem 4.6) and in view of the boundary condition (2.17) we arrive at the integral equation for the unknown density vector φ,

$$-2^{-1}\,\varphi(x) + \mathcal{K}\varphi(x) = F(x), \qquad x \in S, \tag{5.2}$$

where $\mathcal{K}$ is the singular integral operator defined by (4.31).

We start the investigation of the nonhomogeneous equation (5.2) with the study of the homogeneous integral equation

$$-2^{-1}\,\varphi(x) + \mathcal{K}\,\varphi(x) = 0, \qquad x \in S, \tag{5.3}$$

and the corresponding adjoint equation

$$-2^{-1}\,\psi(x) + \mathcal{K}^*\psi(x) = 0, \qquad x \in S, \tag{5.4}$$

where $\mathcal{K}^*$ is the operator adjoint to $\mathcal{K}$ in the sense of $L_2(S)$ inner product, i.e.,

$$\big(\mathcal{K}\,\varphi, \psi\big)_{L_2(S)} = \big(\varphi, \mathcal{K}^*\psi\big)_{L_2(S)}. \tag{5.5}$$

From (5.5) we easy deduce that

$$\mathcal{K}^*\psi(x) := \int_S \Big[B(\partial_y, n(y))[\Gamma^*(x-y)]^\top\Big]^\top \psi(y)dS_y,$$

where $\Gamma^*(x-y)$ is the fundamental matrix of the operator $A^*(\partial)$.

We recall here that $[-2^{-1}\,I_4 + \mathcal{K}]$ and $[-2^{-1}\,I_4 + \mathcal{K}^*]$ with $I_4 = [\delta_{kj}]_{4\times 4}$ being the unit matrix are the singular integral operators of normal type with index equal to zero (see [4]).

5.2 Null spaces of the integral operators

First we prove the following assertion.

Theorem 5.1 *Let $S \in C^{2,\alpha}$ with $0 < \alpha \leqslant 1$. Then the operator $[-2^{-1} I_4 + \mathcal{K}]$ possesses a seven dimensional null space with the basis $\{\mathcal{H}^{-1}\Psi_S^{(k)}\}_{k=1}^{7}$, where $\mathcal{H}^{-1}$ is the operator inverse to $\mathcal{H}$, while $\Psi_S^{(k)}$ is the restriction on S of the vector-function $\Psi^{(k)}$ defined in* (3.9),

$$\Psi_S^{(k)}(x) := \Psi^{(k)}(x), \qquad x \in S. \tag{5.6}$$

Proof. Let $\varphi_0 \in \ker[-2^{-1} I_4 + \mathcal{K}]$ and construct the single layer potential $V(\varphi_0)$. Since φ_0 solves the homogeneous equation (5.3), it follows that the vector $U_0 = V(\varphi_0)$ solves the homogeneous interior Neumann type BVP (2.16)–(2.17) with $F = 0$. Therefore by Theorem 3.1 and Remark 3.2 we get

$$U_0(x) = V(\varphi_0)(x) = \sum_{k=1}^{7} C_k \Psi^{(k)}(x), \qquad x \in \Omega^+,$$

where C_k are some constants. Due to the continuity property of the single layer potential across the surface S and using the equalities (5.6) we obtain

$$\{U_0(x)\}^+ = \{V(\varphi_0)(x)\}^+ \equiv \mathcal{H}(\varphi_0)(x) = \sum_{k=1}^{7} C_k \Psi_S^{(k)}(x), \qquad x \in S,$$

where $\mathcal{H}$ is the integral operator defined in (4.30). By invertibility of the operator $\mathcal{H}$ we deduce

$$\varphi_0(x) = \sum_{k=1}^{7} C_k \left(\mathcal{H}^{-1}\Psi_S^{(k)}\right)(x), \qquad x \in S. \tag{5.7}$$

Since the system $\{\Psi^{(k)}(x)\}_{k=1}^{7}$ is linearly independent in the domain Ω^+, the restriction onto S of the same system is linearly independent as well. Indeed, if there are constants $b_1, b_2, \cdots, b_7$, such that $\sum_{k=1}^{7} |b_k| \neq 0$ and

$$\sum_{k=1}^{7} b_k \Psi_S^{(k)}(x) = 0, \qquad x \in S,$$

then it follows that the vector-function

$$U(x) = \sum_{k=1}^{7} b_k \Psi^{(k)}(x), \qquad x \in \Omega^+,$$

solves the Dirichlet interior boundary value problem. Consequently, $U(x) = 0$ in Ω^+ due to the uniqueness Theorem 4.8, which contradicts to the linear independence of the system $\{\Psi^{(k)}(x)\}_{k=1}^{7}$ in Ω^+.

Next we show that the system

$$\big\{\mathcal{H}^{-1}\Psi_S^{(k)}(x)\big\}_{k=1}^{7}, \qquad x \in S, \tag{5.8}$$

is also linearly independent. In fact, let $d_1, d_2, ..., d_7$, be constants such that $\sum_{k=1}^{7} |d_k| \neq 0$ and

$$\sum_{k=1}^{7} d_k \mathcal{H}^{-1}\Psi_S^{(k)}(x) = 0, \qquad x \in S.$$

Applying the operator $\mathcal{H}$ to this equation we get

$$\sum_{k=1}^{7} d_k \Psi_S^{(k)}(x) = 0, \qquad x \in S,$$

which in turn contradicts to the linear independence of the system (5.8).

Introduce the notation

$$\varphi^{(k)}(x) := \mathcal{H}^{-1}\Psi_S^{(k)}(x), \qquad x \in S, \quad k = \overline{1,7}. \tag{5.9}$$

From the above reasonings it follows that the system $\big\{\varphi^{(k)}(x)\big\}_{k=1}^{7}$ is linearly independent on S, which implies that the homogeneous equation (5.3) possesses at least 7 linearly independent solutions, i.e., $\dim\ker[-2^{-1} I_4 + \mathcal{K}] \geqslant 7$.

On the other hand, from the representation (5.7) of an arbitrary element of $\ker[-2^{-1} I_4 + \mathcal{K}]$, it is evident that the system $\big\{\varphi^{(k)}\big\}_{k=1}^{7}$ is basis of the null-space $\ker[-2^{-1} I_4 + \mathcal{K}]$, yielding that $\dim\ker[-2^{-1} I_4 + \mathcal{K}] = 7$. This completes the proof.

□

Remark 5.2 *An arbitrary element φ_0 of the null-space* $\ker[-2^{-1} I_4 + \mathcal{K}]$ *is representable as*

$$\varphi_0(x) = \sum_{k=1}^{7} C_k\, \varphi^{(k)}(x), \quad x \in S,$$

where C_k are real constants and $\varphi^{(k)}$ are defined in (5.9).

Now we investigate the homogenous adjoint equation (5.4).

Theorem 5.3 *Let $S \in C^{2,\alpha}$ with $0 < \alpha \leqslant 1$. Then the null space of the adjoint operator $-2^{-1} I_4 + \mathcal{K}^*$ is seven dimensional with the basis vector-functions*

$$\Phi^{(k)} := \Psi^{(k)}, \ \ k = \overline{1,6}, \qquad \Phi^{(7)} := \big(0,0,0,1\big)^{\top}, \tag{5.10}$$

where $\Psi^{(k)}$, $k = \overline{1,6}$, are defined in (3.9).

Proof. The equality $\dim\ker\big[-2^{-1}\,I_4+\mathcal{K}^*\big]=7$ follows from Theorem 5.1 since the index of the operator $-2^{-1}\,I_4+\mathcal{K}$ equals to zero.

To construct the basis of the null space of the operator $-2^{-1}\,I_4+\mathcal{K}^*$ explicitly we proceed as follow. Let $\psi_0\in\big[C^{1,\alpha}(S)\big]^4$ be a solution to the corresponding homogenous equation (5.4), i.e.,

$$-2^{-1}\,\psi_0(x)+\mathcal{K}^*\psi_0(x)=0,\qquad x\in S.$$

Then it follows that the vector-function U_0^* defined by the adjoint double layer potential

$$U_0^*(x)=(u^*,\vartheta^*)^\top:=W^*(\psi_0)(x),\qquad x\in\Omega^\pm,$$

solves the homogenous differential equation

$$A^*(\partial)\,U_0^*(x)=0,\qquad x\in\Omega^\pm,$$

and, in view of the relation

$$\big\{U_0\big\}^-=\big\{W^*(\psi_0)\big\}^-=-2^{-1}\,\psi_0+\mathcal{K}^*\psi_0=0\quad\text{on}\quad S,$$

satisfies the homogeneous exterior Dirichlet boundary condition. Since

$$U_0^*=W^*(\psi_0)\in\big[C^{1,\alpha}(\overline{\Omega^\pm})\big]^4\cap\big[C^2(\Omega^\pm)\big]^4\cap Z^*(\Omega^-)\,,$$

by the uniqueness Theorem 4.9 for the adjoint exterior Dirichlet boundary value problem we deduce

$$U_0^*(x)=W^*(\psi_0)(x)=0,\qquad x\in\Omega^-.\tag{5.11}$$

Applying the Liapunov-Tauber type theorem (4.29) we get

$$\big\{Q(\partial,n)W^*(\psi_0)\big\}^-=\big\{Q(\partial,n)W^*(\psi_0)\big\}^+=0,$$

implying that the vector-function U_0^* solves the following adjoint interior Neumann type boundary value problem:

$$\begin{aligned}&A^*(\partial)\,U_0^*(x)=0 &&\text{in}\quad\Omega^+,\\ &\big\{Q(\partial,n)U_0^*\big\}^+=0 &&\text{on}\quad S.\end{aligned}\tag{5.12}$$

In turn, in accordance with (2.10)–(2.11), this problem can be decomposed into two interior Neumann type boundary value problems

$$\begin{aligned}&C(\partial)\,u^*(x)=0, && x\in\Omega^+,\\ &\big\{T(\partial,n)\,u^*(x)\big\}^+=0, && x\in S,\end{aligned}$$

and

$$\begin{aligned} \Lambda(\partial)\,\vartheta^* &= 0, \qquad x \in \Omega^+, \\ \lambda(\partial, n)\,\vartheta^* &= 0, \qquad x \in S. \end{aligned}$$

The general solutions to these problems read as

$$u^* = a \times x + b = \sum_{k=1}^{6} C_k \widetilde{\Psi}^{(k)}, \qquad \vartheta^* = C_7 = const,$$

where $a = (a_1, a_2, a_3)^\top$ and $b = (b_1, b_2, b_3)^\top$ are arbitrary constant vectors, $C_j = b_j$, $j = 1, 2, 3$, $C_j = a_j$, $j = 4, 5, 6$, C_7 is an arbitrary constant, while

$$\begin{aligned} &\widetilde{\Psi}^{(1)} = \big(1, 0, 0\big)^\top, \qquad \widetilde{\Psi}^{(2)} = \big(0, 1, 0\big)^\top, \qquad \widetilde{\Psi}^{(3)} = \big(0, 0, 1\big)^\top, \\ &\widetilde{\Psi}^{(4)} = \big(0, -x_3, x_2\big)^\top, \quad \widetilde{\Psi}^{(5)} = \big(x_3, 0, -x_1\big)^\top, \quad \widetilde{\Psi}^{(6)} = \big(-x_2, x_1, 0\big)^\top. \end{aligned}$$

Therefore the linearly independent system of vectors $\big\{\Phi^{(k)}(x)\big\}_{k=1}^{7}$, where $\Phi^{(k)} := \Psi^{(k)}$, $k = \overline{1,6}$, with $\Psi^{(k)}$ defined in (3.9), and $\Phi^{(7)} := \big(0, 0, 0, 1\big)^\top$, represents a basis of the space of solutions to the homogeneous boundary value problem (5.12) in Ω^+. This implies that

$$W^*(\psi_0)(x) = \sum_{k=1}^{7} C_k \Phi^{(k)}(x) = 0, \qquad x \in \Omega^+.$$

It is easy to verify that the system $\big\{\Phi_S^{(k)}(x)\big\}_{k=1}^{7}$ with $\Phi_S^{(k)}(x) := \Phi^{(k)}(x)$ for $x \in S$ is linearly independent on S as well.

Taking into account (5.11) and using the jump formulas for the adjoint double layer potential, we get

$$\big\{W^*(\psi_0)(x)\big\}^+ - \big\{W^*(\psi_0)(x)\big\}^- = \psi_0(x) = \sum_{k=1}^{7} C_k \Phi_S^{(k)}(x), \qquad x \in S.$$

The later yields that the system $\big\{\Phi_S^{(k)}(x)\big\}_{k=1}^{7}$ defined in (5.10) represents a basis of the null space $\ker\big[-2^{-1} I_4 + \mathcal{K}^*\big]$.

□

5.3 Necessary and sufficient conditions of solvability

Here we present basic existence theorems which immediately follow from the results established in Subsection 5.2.

Theorem 5.4 *Let $S \in C^{2,\alpha}$ and $F \in [C^{0,\beta}(S)]^4$ with $0 < \beta < \alpha \leqslant 1$. The necessary and sufficient conditions for the non-homogenous singular integral equation* (5.2) *to be solvable read as*

$$\Big(F, \Phi_S^{(k)}\Big)_{L_2(S)} \equiv \int_S F(x) \cdot \Phi_S^{(k)}(x)\, dS = 0, \qquad k = \overline{1,7}, \tag{5.13}$$

where $\big\{\Phi^{(k)}(x)\big\}_{k=1}^{7}$ is the basis of the null space $\ker\big[-2^{-1} I_4 + \mathcal{K}^*\big]$ *defined in* (5.10).

If φ_0 is some particular solution to equation (5.2), *then the general solution φ is representable in the form*

$$\varphi(x) = \varphi^{(0)}(x) + \sum_{k=1}^{7} C_k \mathcal{H}^{-1} \Psi_S^{(k)}(x), \quad x \in S, \quad k = \overline{1,7},$$

where C_k are arbitrary constants, $\mathcal{H}^{-1}$ is the operator inverse to $\mathcal{H}$ and the vector-functions $\Psi_S^{(k)}$ are defined in (3.9).

Proof. The assertion follows directly from Theorems 5.1 and 5.3 due to the general theory of normally solvable singular integral equations (see, e.g., [6, Ch. IV], [7]). □

This theorem immediately implies the following basic existence results for the interior Neumann type BVP under consideration.

Theorem 5.5 *Let $S \in C^{2,\alpha}$ and $F \in C^{0,\beta}(S)$ with $0 < \beta < \alpha \leqslant 1$. The conditions* (5.13) *are necessary and sufficient for solvability of the nonhomogeneous Neumann type interior boundary value problem* (2.16)–(2.17). *A solution to the problem is representable in the form of a single layer potential* (5.1), *where the unknown density φ is defined by the singular integral equation*(5.2). *If U^0 is some particular solution to the nonhomogeneous problem* (2.16)–(2.17), *then the general regular solution to the same problem can be represented as*

$$U(x) = U^0(x) + \sum_{k=1}^{7} C_k \Psi^{(k)}(x), \quad x \in \Omega^+,$$

where C_k are arbitrary constants, while $\Psi^{(k)}$ are the generalized rigid displacement vector-functions defined in (3.9).

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