ON ONE PROBLEM OF STEFAN

B. Dochviri, E. Nadaraya, O. Purtukhia,G. Sokhadze

Ivane Javakhishvili Tbilisi State University
0186 University Street 2, Tbilisi, Georgia
A. Razmadze Mathematical Institute Georgian
National Academy of Science

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Abstract

The problem of a disruption of a Wiener process is considered. This problem is reduce to the so-called Stefan problem. The explicit form of sufficient statistics also is obtained.

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1 Statement of the problem

We consider one problem of disorder (disruption) for a standard Wiener process. At first we bring the one formulation of this problem and some later we will consider our formulation of problem and basic result.

We shall assume that on a probability space $(\Omega, \Im, P)$ a random variable $\theta = \theta(\omega)$ with values $[0, \infty)$ and a standard Wiener process $w = w_t, t \geq 0$, mutually independent are given such that

$$P(\theta = 0) = \pi, \quad P(\theta \geq t | \theta > 0) = e^{-\lambda t}, \tag{1}$$

where λ is the known constant, $0 < \lambda < \infty$, $0 \leq \pi \leq 1$, and

$$w_0 \equiv 0, \quad E(w_t) = 0, \quad E[(w_t - w_s)^2] = t - s, \ \ 0 \leq s \leq t. \tag{2}$$

We also assume that we observe the random process $\xi_t, t \geq 0$, with the following stochastic differential

$$d\xi_t = r\chi(t - \theta)dt + \sigma dw_t, \tag{3}$$

where $r \neq 0, \sigma^2 > 0$, are known constants and

$$\chi(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases}$$

In other words, the structure of the process ξ is such that

$$\xi_t = \begin{cases} \sigma w_t, & t < 0, \\ r(t-\theta) + \sigma w_t, & t \geq 0. \end{cases} \tag{4}$$

We shall consider the problem of the earliest detection of θ (the disruption problem) for the Bayes formulation. Let the payoff is defined by

$$\rho = \rho^{\pi} = inf\{P(\tau < \theta) + cE[max(\tau - \theta, 0)]\}, \quad c > 0, \tag{5}$$

where infimum is taken over the class of all stopping times $\tau \in \mathcal{M}^{\xi}$ (with respect to the filtration $\Im_t^{\xi} = \sigma\{\xi_s : s \leq t\}$). The value $P(\tau < \theta)$ is naturally interpreted as probability of false alarm of disruption and the value $E[max(\tau - \theta, \theta)]$ - as the average delay of detecting the occurrence of disruption correctly, i.e. when $\tau \geq \theta$. We say that the stopping time $\tau_{\pi}^* \in \mathcal{M}^{\xi}$ is π-Bayes (Optimal stopping time) if its risk function

$$\rho^{\pi}(\tau_{\pi}^*) = P(\tau_{\pi}^* < \theta) + cE[max(\tau_{\pi}^* - \theta, 0)] \tag{6}$$

coincides with ρ^{π}.

Proposition ([1], Theorem 4.10). The π-Bayes stopping time is

$$\tau_{\pi}^* = inf\{t \geq 0 : \pi_t \geq A^*\},$$

where $\pi_t = P(\theta \leq t | \Im_t^{\xi})$ and the threshold A^* is a some unknown threshold. The payoff ϱ^{π} is the unique solution of the following Stefan problem

$$\lambda(1-\pi)f^{'}(\pi) + \frac{r^2}{2\sigma^2}[\pi(1-\pi)]^2 f^{''}(\pi) = -c\pi, \quad 0 \leq \pi \leq A^*,$$

$$f(\pi) = 1 - \pi, \quad A^* \leq \pi \leq 1,$$

$$f^{'}(A^*) = -1, \quad f^{''}(0) = 0,$$

where $f(\pi)$ is unknown function from the class of nonnegative convex upward, twice continuously differentiable functions.

2 Main results

Now we assume that we have also the random variable $\eta = \eta(\omega)$ is given such that θ, w, η are mutually independent and

$$P(\eta \leq x) = 1 - e^{-\nu x}, \tag{7}$$

where $\nu > 0$ is the known constant constant.

We shall consider the earliest detection problem of moment θ for the Wiener process, when we have the following risk function

$$\rho^{\pi}(\tau) = cP(\tau < \theta) + \nu \int_0^{\infty} P(\theta \le \tau \le \theta + \eta) e^{-\nu\eta} d\eta, \;\; c > 0. \tag{8}$$

Let the payoff is defined by

$$\rho = \sup_{\tau \in \mathcal{M}^{\xi}} \rho(\tau). \tag{9}$$

Lemma. Let the variables $\rho(\tau)$ and ρ are defined respectively by (8)and (9). Then we have:

1) the payoff

$$\rho = \rho(\pi, \psi) = \sup_{\tau \in \mathcal{M}^{\pi,\psi}} E[g(\pi_\tau, \psi_\tau)], \tag{10}$$

where

$$g(\pi_t, \psi_t) = (1 - \pi_t)(c + \lambda\psi_t) \tag{11}$$

and supremum is taken over the class of stopping times $\mathcal{M}^{\pi,\psi}$ with respect to the filtration $\Im_t^{\pi,\psi} = \sigma\{(\pi_s, \psi_s) : s \le t\}$;

2) the process $\pi = (\pi_t)$ and $\psi = (\psi_t)$, $t \ge 0$, have the following stochastic differentials

$$d\pi_t = \lambda(1 - \pi_t)dt + \frac{r}{\sigma^2}\pi_t(1 - \pi_t)(d\xi_t - r\pi_t dt), \tag{12}$$

$$d\psi_t = [1 + (\lambda - \nu)\psi_t]dt + \frac{r}{\sigma^2}\psi_t d\xi_t. \tag{13}$$

Proof. 1) Let us consider the process $\pi_t = P(\theta \le t|\Im_t^{\xi})$ ($\Im_t^{\xi} = \sigma\{\xi_s, s \le t\}$) and the following variables

$$u(t) = exp\{t(\frac{r}{2\sigma^2} - \lambda) + \frac{r}{\sigma^2}\xi_t\}, \tag{14}$$

$$\widetilde{u}(t) = u(t)e^{\nu t}, \tag{15}$$

$$\psi_t = \int_0^t \widetilde{u}(s)ds. \tag{16}$$

We have

$$P(\theta \le \tau \le \theta + \eta) = E[f_\eta(\tau - \theta)], \tag{17}$$

where the function

$$f_\eta(t) = \begin{cases} 1, & 0 \le t \le \eta, \\ 0, & t \ge \eta, \end{cases}$$

and

$$P(\tau < \theta) = E[P(\tau \leq \theta | \Im_\tau^\xi)] = E(1 - \pi_\tau). \tag{18}$$

Therefore

$$\nu \int_0^\infty P(\theta \leq \tau \leq \theta + \eta) e^{-\nu\eta} d\eta = E \int_0^\tau e^{-\nu(\tau - s)} \cdot p(s|\Im_\tau^\xi) ds, \tag{19}$$

where the conditional density

$$p(s|\Im_t^\xi) = \lambda(1 - \pi_t) \cdot \frac{u(s)}{u(t)} \tag{20}$$

(see [1]).

Using the (17)-(20) we obtain (10).

2) By Theorem 4.9[1] we have (12) and by making use of notations (14)-(16) we obtain (13).

Theorem. Let the payoff ρ is defined by (9). Then the function $\rho = \rho(\pi, \psi)$ is solution of the following Stefan problem

$$\lambda(1 - \pi_t)\rho_\pi^{'} + [1 + (\lambda - \nu)\psi + \frac{r^2}{\sigma^2}\pi\psi]\rho_\pi^{'} + \frac{r^2}{\sigma^2}[\pi(1 - \pi)]^2 \rho_{\pi\pi}^{''} +$$

$$\frac{r^2}{\sigma^2}\psi^2 \rho_{\psi\psi}^{''} + \frac{r^2}{\sigma^2}\pi(1 - \pi)\psi\rho_{\pi\psi}^{''} = \lambda(c + \rho), \tag{21}$$

with boundary conditions

$$\rho(\pi, \psi)|_A = (1 - \pi)(c + \lambda\psi)|_A,$$

$$\rho_\pi^{'}|_A = -(c + \lambda\psi)|_A,$$

$$\rho_\psi^{'}|_A = \lambda(1 - \pi)|_A,$$

where A is some unknown threshold.

Proof. We shall note first, that

$$d\pi_t = \lambda(1 - \pi_t)dt + \frac{r}{\sigma}\pi_t(1 - \pi_t)d\overline{w}_t,$$

$$d\psi_t = [1 + (\lambda - \nu)\psi_t + \frac{r^2}{\sigma^2}\pi_t\psi_t]dt + \frac{r}{\sigma}\psi_t d\overline{w}_t,$$

where $\overline{w}_t$, $t \geq 0$, is the innovation Wiener process (see [1]).

The process $(\pi, \psi) = (\pi_t, \psi_t)$, $t \geq 0$, is a two dimensional standard Markov process. Next, by Theorems 3.16[1], 5.7[2] and above given Lemma we have in the domain of continuing observations $\mathcal{U}\rho(\pi, \psi) = \mathcal{D}\rho(\pi, \psi)$, where $\mathcal{U}$ is the characteristic operator of the process (π, ψ) and $\mathcal{D}$ is the differential operator with the coefficients:

$$E(d\pi) = \lambda(1-\pi)dt, \quad E[(d\pi)^2] = \frac{r^2}{\sigma^2}\pi^2(1-\pi^2),$$

$$E(d\psi) = [1-(\lambda-\nu)\psi + \frac{r^2}{\sigma^2}\pi\psi]dt, \quad E[(d\psi)^2]\frac{r^2}{\sigma^2}\psi^2 dt,$$

$$E(d\pi d\psi) = \frac{r^2}{\sigma^2}\pi(1-\pi)\psi dt.$$

Using this variables we obtain the proof of Theorem.

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