ON THE APPLICATION OF MUSKHELISHVILI AND VEKUA-BITSADZE METHODS FOR THE NONLINEAR AND NON-SHALLOW SHELLS

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Abstract

I. Vekua has constructed several versions of the refined linear theory of thin and shallow shells, containing the regular process by means of the method of reduction of three-dimensional problems of elasticity to two-dimensional ones.

In the present paper by means of the I. Vekua method the system of differential equations for the nonlinear theory of non-shallow shells is obtained. Using the method of a small parameter, by means of Muskhelishvili and Vekua-Bitsadze methods, for any approximations of order N the complex representations of the general solutions are obtained.

We also consider the well-known Kirsch problems for plates on the basis of Reissner-Mindlin's type and of I. Vekua's refined theories.

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1 Shallow and Non-shallow Shells

A complete system of equilibrium equation and the stress-strain relations of the 3-D nonlinear theory of elasticity can be written as:

$$\hat{\nabla}_i \boldsymbol{\sigma}^i + \boldsymbol{\Phi} = 0, \qquad \boldsymbol{\sigma}^i = E^{ijpq} e_{pq} (\mathbf{R}_j + \partial_j \mathbf{u}), \tag{1}$$

$$(i, j, p, q, = 1, 2, 3)$$

where $\hat{\nabla}_i$ are covariant derivatives with respect to the space curvilinear coordinates x^i, $\boldsymbol{\sigma}^i$ and $\boldsymbol{\Phi}$ are the contravariant "constituents" of the stress vector and an external force, e_{ij} are covariant components of the strain tensor, $\mathbf{u}$ is the displacement vector:

$$2e_{ij} = \mathbf{R}_i \partial_j \mathbf{u} + \mathbf{R}_j \partial_i \mathbf{u} + \partial_i \mathbf{u} \partial_j \mathbf{u}, \tag{2}$$

$$E^{ijpq}=\lambda g^{ij}g^{pq}+\mu(g^{ip}g^{jq}+g^{iq}g^{jp}),\quad (g^{ij}=\mathbf{R}^i\mathbf{R}^j)$$

λ and μ are Lame's constants, $\mathbf{R}_i$ and $\mathbf{R}^i$ are covariant and contravariant basis vectors of surface $\hat{S}$ ($x^3 = const$) of the 3-D domain Ω, which are connected with the basis vectors $\mathbf{r}_i$ and $\mathbf{r}^i$ of the midsurface S ($x^3 = 0$) by the following relations:

$$\begin{aligned}&\mathbf{R}_i = A_{i.}^{\cdot j}\mathbf{r}_j,\quad \mathbf{R}^i = A_{\cdot j}^{i\cdot}\mathbf{r}^j,\quad \mathbf{R}^3=\mathbf{R}_3=\mathbf{r}^3=\mathbf{r}_3=\mathbf{n}\\ &A_{\alpha.}^{\cdot\beta}=a_\alpha^\beta - x_3 b_\alpha^\beta,\quad A_{i.}^{\cdot 3}=A_{\cdot 3}^{i\cdot}=\delta_{i3},\\ &A_{\cdot\beta}^{\alpha\cdot}=\vartheta^{-1}[a_\beta^\alpha + x_3(b_\beta^\alpha - 2Ha_\beta^\alpha)],\quad \vartheta=1-2Hx_3+Kx_3^2,\end{aligned}\tag{3}$$

$$(\alpha,\beta=1,2;\quad -h\le x^3=x_3\le h)$$

where a_β^α ($a_{\alpha\beta}, a^{\alpha\beta}$) and b_β^α ($b_{\alpha\beta}, b^{\alpha\beta}$) are mixed (covariant, contravariant) components of the metric tensor and tensor of curvature of the midsurface S ($x_3 = 0$), x_3 is the thickness coordinate and h is the semi-thickness of the shell Ω, H and K are middle and Gaussian curvatures of S, and $\mathbf{n}$ is unit vector of the normal to S at the point $(x^1, x^2) \in S$.

The main quadratic forms of the midsurfaces S and $\hat{S}$ have the forms:

$$\mathrm{I}=ds^2=a_{\alpha\beta}dx^\alpha dx^\beta,\qquad \mathrm{II}=k_s ds^2=b_{\alpha\beta}dx^\alpha dx^\beta\quad S(x_3=0),\tag{4}$$

$$\mathrm{I}=d\hat{s}^2=g_{\alpha\beta}dx^\alpha dx^\beta,\qquad \mathrm{II}=\hat{k}_{\hat{s}}d\hat{s}^2=\hat{b}_{\alpha\beta}dx^\alpha dx^\beta\quad \hat{S}(x_3=const),$$

where k_s and $\hat{k}_{\hat{s}}$ are the normal curvatures of the surfaces S and $\hat{S}$:

$$a_{\alpha\beta}=\boldsymbol{r}_\alpha\boldsymbol{r}_\beta,\ \ b_{\alpha\beta}=-\boldsymbol{r}_\alpha\boldsymbol{n}_\beta,\ \ k_s=b_{\alpha\beta}s^\alpha s^\beta, s^\alpha=\frac{dx^\alpha}{ds},\quad S(x_3=0),$$

$$\begin{aligned}&g_{\alpha\beta}=\boldsymbol{R}_\alpha\boldsymbol{R}_\beta=a_{\alpha\beta}-2x_3b_{\alpha\beta}+x_3^2(2Hb_{\alpha\beta}-Ka_{\alpha\beta}),\\ &\hat{b}_{\alpha\beta}=(1-2Hx_3)b_{\alpha\beta}+x_3Ka_{\alpha\beta},\quad \hat{S}(x_3=const),\\ &(b_\alpha^\gamma b_{\gamma\beta}=2Hb_{\alpha\beta}-Ka_{\alpha\beta}).\end{aligned}\tag{5}$$

The unit vectors of the tangent $\hat{\boldsymbol{s}}$ and tangential normal $\hat{\boldsymbol{l}}$ are expressed by the following formulas:

$$\hat{\boldsymbol{s}}=[(1-x_3k_s)\boldsymbol{s}+x_3\tau_l\boldsymbol{s}]\frac{ds}{d\hat{s}},\quad \hat{\boldsymbol{l}}=[(1-x_3k_s)\boldsymbol{l}-x_3\tau_s\boldsymbol{s}]\frac{ds}{d\hat{s}}\tag{6}$$

$$d\hat{s}=\sqrt{1-2x_3k_s+x_3^2(k_s^2+\tau_s^2)}ds,$$

where $\boldsymbol{s}$ and $\boldsymbol{l}$ are the unit vectors of the tangent and tangential normal of the midsurface S, $d\hat{s}$ and ds are the linear elements of the surfaces $\hat{S}$ and S, respectively, $\tau_s=-b_{\alpha\beta}l^\alpha s^\beta$ is the geodesic torsion of the surface S, $\left(l^1=\frac{1}{\sqrt{a}}s_2, l^2=\frac{1}{\sqrt{a}}s_1\right)$.

Under shallow shells we mean 3-D shell-type elastic bodies satisfying the following conditions

$$a_\alpha^\beta - x_3 b_\alpha^\beta \cong a_\alpha^\beta \quad \Rightarrow \quad \mathbf{R}_\alpha \cong \mathbf{r}_\alpha, \quad \mathbf{R}^\alpha \cong \mathbf{r}^\alpha, \quad g_{\alpha\beta} \cong a_{\alpha\beta}, \quad \hat{b}_{\alpha\beta} \cong b_{\alpha\beta}, \tag{7}$$

i.e. in the case the interior geometry of the shell does not vary in thickness and therefore such kind of shells are usually called the shells with non-varying geometry.

For the Koiter-Naghdi refined theory of shells these relations have the form:

$$\mathbf{R}^\alpha \cong (a_\beta^\alpha + x_3 b_\beta^\alpha)\mathbf{r}^\beta, \quad \mathbf{R}_\alpha \cong (a_\alpha^\beta - x_3 b_\alpha^\beta)\mathbf{r}_\beta, \tag{8}$$
$$g_{\alpha\beta} \cong a_{\alpha\beta} - 2x_3 b_{\alpha\beta}, \quad g^{\alpha\beta} \cong a^{\alpha\beta} + 2x_3 b^{\alpha\beta},$$

i.e. in this case only linear part with respect to x_3 is retained.

In the sequel, by non-shallow shells we mean 3-D shell-type elastic bodies satisfying the relations (3), (4), (5), (6).

To reduce the 3-D problems of the theory of elasticity to 2-D ones, it is necessary to rewrite the relation (1), (2) in forms of the bases of the midsurface S $(x_3 = 0)$.

The relation (1) can be written as:

$$\nabla_\alpha(\vartheta \boldsymbol{\sigma}^\alpha) + \partial_3(\vartheta \boldsymbol{\sigma}^3) + \vartheta \Phi = 0, \tag{9}$$

$$\boldsymbol{\sigma}^i = A_{i_1}^i A_{p_1}^p M^{i_1 j_1 p_1 q_1}[(\boldsymbol{r}_{q_1}\partial_p \boldsymbol{U}) + \frac{1}{2}A_{q_1}^q(\partial_p \boldsymbol{U}\partial_q \boldsymbol{U})](\boldsymbol{r}_{j_1} + A_{j_1}^j \partial_j \boldsymbol{U}), \tag{10}$$

where ∇_α are covariant derivatives on the midsurface S $(x_3 = 0)$,

$$M^{i_1 j_1 p_1 q_1} = \lambda a^{i_1 j_1} a^{p_1 q_1} + \mu(a^{i_1 p_1} a^{j_1 q_1} + a^{i_1 q_1} a^{j_1 p_1}) \ \ (a^{i_1 j_1} = \boldsymbol{r}^{i_1} \boldsymbol{r}^{j_1}) \tag{11}$$

2 I. Vekua's reduction method

In the present paper we use I. Vekua's reduction method for the nonlinear theory of non-shallow shells (I. Vekua used the method for linear theory of shallow shells) the essence of which consists, without going into detals, in the following: since the system of Legendre polynomials $P_m(\frac{x_3}{h})$ is complete in the interval $[-h, h]$, for equation (9) the equivalent infinite system of 2-D equations is obtained

$$\nabla_\alpha \overset{(m)}{\boldsymbol{\sigma}}{}^\alpha - \frac{2m+1}{h}\left(\overset{(m-1)}{\boldsymbol{\sigma}}{}^3 + \overset{(m-3)}{\boldsymbol{\sigma}}{}^3 + ...\right) + \overset{(m)}{\boldsymbol{F}} = 0, \tag{12}$$

where

$$\left(\overset{(m)}{\boldsymbol{\sigma}}{}^i, \overset{(m)}{\boldsymbol{\Phi}}\right) = \frac{2m+1}{2h}\int_{-h}^{h}\left(\vartheta \boldsymbol{\sigma}^i, \vartheta \boldsymbol{\Phi}\right) P_m\left(\tfrac{x_3}{h}\right) dx_3,$$

$$\overset{(m)}{\boldsymbol{F}} = \overset{(m)}{\boldsymbol{\Phi}} + \frac{2m+1}{2h}\left(\overset{(+)}{\vartheta}\,\overset{(+)}{\boldsymbol{\sigma}}{}_3 - (-1)^m\,\overset{(-)}{\vartheta}\,\overset{(-)}{\boldsymbol{\sigma}}{}_3\right),$$

$$\left(\overset{(\pm)}{\vartheta} = 1 \mp 2hH + Kh^2\right).$$

Thus we have obtained the infinite system of 2-D equations (12), for which the boundary conditions of the face surfaces ($x_3 = \pm h$) are satisfied, i.e. $\overset{(\pm)}{\boldsymbol{\sigma}}{}^3 = \boldsymbol{\sigma}^3(x^1, x^2, \pm h)$ is the preassigned vector field and is entered in the equilibrium equations.

The equations of the state (10) may be write as:

$$\begin{aligned}\overset{(m)}{\boldsymbol{\sigma}}{}^{i} = &\frac{1}{2}M^{i_1j_1p_1q_1}\sum_{m_1=0}^{\infty}\Bigg\{\left(\underset{(m_1)}{\overset{(m)}{A}}{}^{i\,p}_{i_1p_1}\boldsymbol{r}_{q_1}\cdot D_p\overset{(m_1)}{\boldsymbol{U}} + \underset{(m_1)}{\overset{(m)}{A}}{}^{i\,q}_{i_1q_1}\boldsymbol{r}_{p_1}\cdot D_q\overset{(m_1)}{\boldsymbol{U}}\right)\boldsymbol{r}_{j_1} + \\ &+\sum_{m_1=0}^{\infty}\Bigg[\underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{i\,pq}_{i_1p_1q_1}\left(D_p\overset{(m_1)}{\boldsymbol{U}}\cdot D_q\overset{(m_2)}{\boldsymbol{U}}\right)\boldsymbol{r}_{j_1} + \\ &+\left(\underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{ij\,p}_{i_1j_1p_1}\boldsymbol{r}_{q_1}\cdot D_p\overset{(m_1)}{\boldsymbol{U}} + \underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{ij\,q}_{i_1j_1q_1}\boldsymbol{r}_{p_1}\cdot D_q\overset{(m_1)}{\boldsymbol{U}}\right)D_q\overset{(m_2)}{\boldsymbol{U}} + \\ &+\sum_{m_3=0}\underset{(m_1,m_2,m_3)}{\overset{(m)}{A}}{}^{i\,j\,p\,q}_{i_1j_1p_1q_1}\left(D_p\overset{(m_1)}{\boldsymbol{U}}\,D_q\overset{(m_2)}{\boldsymbol{U}}\right)D_j\overset{(m_3)}{\boldsymbol{U}}\Bigg]\Bigg\}\end{aligned} \tag{13}$$

where

$$\overset{(m)}{\boldsymbol{U}} = \frac{2m+1}{2h}\int_{-h}^{h}\boldsymbol{U}P_m\Big(\frac{x_3}{h}\Big)dx_3,$$

$$D_i\overset{(m)}{\boldsymbol{U}} = \delta_i^{\beta}\partial_\beta\overset{(m)}{U} + \delta_i^3\overset{(m)}{U}{}';\quad \overset{(m)}{\boldsymbol{U}}{}' = \frac{2m+1}{h}\left(\overset{(m+1)}{\boldsymbol{U}} + \overset{(m+3)}{\boldsymbol{U}} + \ldots\right), \tag{14}$$

$$\underset{(m_1)}{\overset{(m)}{A}}{}^{ij}_{i_1j_1} = \frac{2m+1}{2h}\int_{-h}^{h}\vartheta A^i_{i_1}A^j_{j_1}P_{m_1}P_m dx_3$$

$$\underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{ijp}_{i_1j_1p_1} = \frac{2m+1}{2h}\int_{-h}^{h}\vartheta A^i_{i_1}A^j_{j_1}A^p_{p_1}P_{m_1}P_{m_2}P_m dx_3, \tag{15}$$

$$\underset{(m_1,m_2,m_3)}{\overset{(m)}{A}}{}^{ijpq}_{i_1j_1p_1q_1} = \frac{2m+1}{2h}\int_{-h}^{h}\vartheta A^i_{i_1}A^j_{j_1}A^p_{p_1}A^q_{q_1}P_{m_1}P_{m_2}P_{m_3}P_m dx_3.$$

The passage to finite systems can be realized by various methods one of which consists in considering of a finite series, i.e.

$$(\vartheta\boldsymbol{\sigma}^i, \boldsymbol{U}, \vartheta\boldsymbol{\Phi}) = \sum_{m=0}^{N}\left(\overset{(m)}{\boldsymbol{\sigma}}{}^i, \overset{(m)}{\boldsymbol{U}}, \overset{(m)}{\Phi}\right) P_m\left(\frac{x_3}{h}\right),$$

where N is a fixed nonnegative number. In other words, it is assumed that

$$\overset{(m)}{\boldsymbol{U}} = 0, \quad \overset{(m)}{\boldsymbol{\sigma}}{}^i = 0, \quad if \quad m > N.$$

Approximation of this type will be called approximation of order N.

The integrals of the type (15) can be calculated, for example

$$\underset{(m_1)}{\overset{(m)}{A}}{}^{\alpha\beta}_{\alpha_1\beta_1} = \frac{2m+1}{2h}\int_{-h}^{h} \vartheta^{-1} B^{\alpha}_{\alpha_1}(x_3) B^{\beta}_{\beta_1}(x_3) P_{m_1}\left(\frac{x_3}{h}\right) P_m\left(\frac{x_3}{h}\right) dx_3 =$$

$$\frac{2m+1}{2\sqrt{E}h}\left[B^{\alpha}_{\alpha_1}(hy)B^{\beta}_{\beta_1}(hy)\begin{pmatrix} P_{m_1}(y)Q_m(y), \ m_1 \le m \\ Q_{m_1}(y)P_m(y), \ m_1 \le m \end{pmatrix}\right]_{y_1}^{y_2} + \frac{L^{\alpha}_{\alpha_1}L^{\beta}_{\beta_1}}{K}\sigma^m_{m_1}, \quad (16)$$

if $E \neq 0$ $K \neq 0$ and equals $a^{\alpha}_{\alpha_1}a^{\beta}_{\beta_1}\delta^m_{m_1}$, if $E = H^2 - K = 0$; where $Q_m(y)$ is the Legendre function of the second kind, E is the Euler difference, $B^{\alpha}_{\beta}(x) = a^{\alpha}_{\beta} + xL^{\alpha}_{\beta}$, $L^{\alpha}_{\beta} = b^{\alpha}_{\beta} - 2Ha^{\alpha}_{\beta}$. Under the square brackets we mean the following:

$$[f(y)]_{y_1}^{y_2} = f(y_2) - f(y_1), \quad y_{1,2} = [(H \mp \sqrt{E})h]^{-1}.$$

Note that for Koiter-Naghdi's non-shallow shells the following expression

$$\underset{(m_1)}{\overset{(m)}{A}}{}^{\alpha\beta}_{\alpha_1\beta_1} \cong a^{\alpha}_{\alpha_1}a^{\beta}_{\beta_1}\delta^m_{m_1} + h(a^{\alpha}_{\alpha_1}b^{\beta}_{\beta_1} + a^{\beta}_{\beta_1}a^{\alpha}_{\alpha_1})\times \quad (17)$$

$$\times\left(\frac{m}{2m-1}\delta^m_{m_1-1} + \frac{m+1}{2m+3}\delta^m_{m_1+1}\right)$$

is obtained.

For the integrals containing the product of three Legendre polynomials we have

$$\underset{(m_1,m_2)}{\overset{(m)}{A}}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} = \frac{2m+1}{2h}\int_{-h}^{h}\frac{B^{\alpha_1}_{\beta_1}B^{\alpha}_{\beta_2}B^{\alpha}_{\beta_3}}{1-2Hx_3+Kx_3}P_{m_1}P_{m_2}P_m dx_3 = \frac{2m+1}{K^2h^4}\times$$

$$\times\sum_{r=0}^{min(m_1,m_2)}\gamma_{m_1m_2r}\sum_{n=0}^{3}\overset{n}{\mathbb{C}}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3}h^n\frac{\partial^2}{\partial y_1\partial y_2}\left[\frac{y^n}{y_1-y_2}\begin{pmatrix} P_s(y)Q_m(y), s \le m \\ Q_s(y)P_m(y), s \ge m \end{pmatrix}\right]_{y_1}^{y_2},$$

where $s = m_1 + m_2 - 2r$,

$$\gamma_{pqr} = \frac{A_{p-r}A_rA_{q-r}}{A_{p+q-r}} \frac{2(p+q)-4r+1}{2(p+q)-2r+1}, \quad A_p = \frac{1.3\cdots 2p-1}{p!},$$

$$B^{\alpha_1}_{\beta_1}(x)B^{\alpha_2}_{\beta_2}(x)B^{\alpha_3}_{\beta_3}(x) = \sum_{n=0}^{3} \overset{n}{\mathbb{C}}{}^{\alpha_1\alpha_2\alpha_3}_{\beta_1\beta_2\beta_3} x^n.$$

For the integrals containing the product of four Legendre polynomials the corresponding presentations can be written similarly.

3 Introduce a small parameter

To introduce a small parameter $\varepsilon = \dfrac{h}{R}$, where R is a certain radius of curvature of the midsurface S, from (12) will be obtained the following finite system of 2-D equations (approximation of order N):

$$\begin{aligned} &h\nabla_\alpha \overset{(m)}{\sigma}{}^{\alpha\beta} - \varepsilon b^\beta_\alpha \overset{(m)}{\sigma}{}^{\alpha 3}R - (2m+1)\Big(\overset{(m-1)}{\sigma}{}^{3\beta} + \overset{(m-3)}{\sigma}{}^{3\beta} + \cdots\Big) + \overset{(m)}{F}{}^\beta = 0, \\ &h\nabla_\alpha \overset{(m)}{\sigma}{}^{\alpha 3} + \varepsilon b_{\alpha\beta} \overset{(m)}{\sigma}{}^{\alpha\beta}R - (2m+1)\Big(\overset{(m-1)}{\sigma}{}^{3}_{3} + \overset{(m-3)}{\sigma}{}^{3}_{3} + \cdots\Big) + \overset{(m)}{F}{}^3 = 0, \end{aligned} \tag{18}$$

$$(\overset{(m)}{\sigma}{}^{ij} = \overset{(m)}{\boldsymbol{\sigma}}{}^i \mathbf{r}^j, \quad m = 0, 1, ...)$$

To find components of the displacement vector $\overset{(m)}{\mathbf{u}}$ and stress tensor $\overset{(m)}{\sigma}{}^{ij}$ we take of following series expansions with respect to the small parameter ε:

$$\Big(\overset{(m)}{\mathbf{u}}, \overset{(m)}{\boldsymbol{\sigma}}{}^{ij}, \overset{(m)}{F}\Big) = \sum_{n=1}^{\infty} \Big(\overset{(m,n)}{\mathbf{u}}, \overset{(m,n)}{\boldsymbol{\sigma}}{}^{ij}, \overset{(m,n)}{F}\Big)\varepsilon^n.$$

Substituting the above expansions into the (13) and (18) than equalizing the coefficients of expansions for ε^n we obtain the following 2-D finite system of equilibrium equations with respect to components of displacement vector in the isometric coordinates $a_{11} = a_{22} = \Lambda(z, \bar{z})$, which has the following complex form:

$$4\mu\partial_{\overline{z}}\Big(\Lambda^{-1}\partial_z \overset{(m,n)}{u_+}\Big)+2(\lambda+\mu)\partial_{\overline{z}}\overset{(m,n)}{\theta}+2\lambda\partial_{\overline{z}}\overset{(m,n)}{u'_3}-(2m+1)\mu$$

$$\Big[2\partial_{\overline{z}}\Big(\overset{(m-1,n)}{u_3}+\overset{(m-3,n)}{u_3}+\cdots\Big)+\overset{(m-1,n)}{u'_+}+\overset{(m-3,n)}{u'_+}+\cdots\Big]+\overset{(m,n)}{F_+}=0, \quad (19)$$

$$\mu\Big(\nabla^2\overset{(m,n)}{u_3}+\overset{(m,n)}{\theta'}\Big)-(2m+1)\Big[\lambda\Big(\overset{(m-1,n)}{\theta}+\overset{(m-3,n)}{\theta}+\cdots\Big)+$$

$$(\lambda+2\mu)\Big(\overset{(m-1,n)}{u'_3}+\overset{(m-3,n)}{u'_3}+\cdots\Big)\Big]+\overset{(m,n)}{F_3}=0,$$

where $u_+=u_1+iu_2$, $\theta=\Lambda^{-1}\Big(\partial_z u_+ +\partial_{\overline{z}}\overline{u}_+\Big)$, $z=x^1+ix^2$,$2\partial_z=\partial_1-i\partial_2$, $\nabla^2=\dfrac{4}{\Lambda}\dfrac{\partial^2}{\partial z\partial\overline{z}}$.

Obviously, in passing from the n-th step of approximation to the $(n+1)$-th step only the right-hand side of equations are changed.

Below the upper index n will be omitted.

The general solution of the homogeneous system (19) we can find the form

$$\overset{(m)}{u_+}=\partial_{\overline{z}}\overset{(m)}{V_+}+\Big(\frac{1}{\pi}\iint\limits_S\frac{\overline{\varphi'_0(\zeta)}-æ_1\varphi'_0(\zeta)}{\overline{\zeta}-\overline{z}}dS_\zeta-\overline{\psi'_0(z)}\Big)\delta_{0m}-$$

$$\Big(\frac{1}{\pi}\iint\limits_S\frac{\varphi'_1(\zeta)+\overline{\varphi'_1(\zeta)}}{\overline{\zeta}-\overline{z}}dS_\zeta+\eta_1\overline{\varphi''_1(z)}-2\overline{\psi'_1(z)}\Big)\delta_{1m}+ \quad (20)$$

$$æ_2\overline{\varphi''_0(z)}\delta_{2m}+\eta_2\overline{\varphi''_1(z)}\delta_{3m},$$

$$\overset{(m)}{u_3}=\overset{(m)}{V_3}-\Big(\frac{1}{\pi}\iint\limits_S(\varphi'_1(\zeta)+\overline{\varphi'_1(\zeta)})ln|\zeta-z|dS_\zeta-\psi_1(z)-\overline{\psi_1(z)}\Big)\delta_{0m}$$

$$-\frac{3}{2}æ_2\Big[(\varphi'_0(z)+\overline{\varphi'_0(z)})\delta_{1m}-(\varphi'_1(z)+\overline{\varphi'_1(z)})\delta_{2m}\Big],\qquad (m=0,1,...,N)$$

$$\overset{(0)}{V_1}=\overset{(0)}{V_2}=0,\quad \overset{(0)}{u_3}=\psi_1(z)+\overline{\psi_1(z)},\quad if\ N=0,$$

$$\Bigg(dS_\zeta=\Lambda(\zeta,\overline{\zeta})d\xi d\eta,\ \zeta=\xi+i\eta,\ \overset{(m)}{V_+}=\overset{(m)}{V_1}+i\overset{(m)}{V_2}\Bigg).$$

where $\varphi'_0(z)$,$\varphi'_1(z)$,$\psi'_0(z)$,$\psi'_1(z)$ are holomorphic functions of z and express the biharmonic solution of the system (19). Then $æ_1$, $æ_2$, η_1, η_2 are known constants.

Note that for a plate (i.e. $\Lambda=1$) the expression of $\overset{(0)}{u_+}$ coincides with the well-known representation of Kolosov-Muskhelishvili [1]

$$\overset{(0)}{u}_+ = \frac{\lambda+3\mu}{\lambda+\mu}\varphi_0(z) - z\overline{\varphi_0'(z)} - \overline{\psi_0(z)}.$$

Substituting expressions (20) into (19) the matrix equations for $\overset{(m)}{V_i}$ are obtained

$$\nabla^2 V + AV = X, \quad \nabla^2\Omega + B\Omega = Y, \tag{21}$$

where V and Ω are column-matrices of the form

$$V = \big(\overset{(0)}{V_1}, \overset{(1)}{V_1}, ..., \overset{(N)}{V_1}, \overset{(0)}{V_3}, \overset{(1)}{V_3}, ..., \overset{(N)}{V_3}\big)^T, \quad \Omega = \big(\overset{(0)}{V_2}, \overset{(1)}{V_2}, ..., \overset{(N)}{V_2}\big)^T,$$

and A and B are block-matrices $2N+2\times 2N+2$ and $N+1\times N+1$ respectively.

Colomn-matrices V and Ω are expressed by $3N-1$ metaharmonic functions W_i and χ_i:

$$W = \big(W_1, W_2, ..., W_{2N-1}\big)^T, \quad Q = \big(\chi_1, \chi_2, ..., \chi_N\big)^T,$$

which satisfy the following matrix equations

$$\nabla^2 W - \mathbb{C}W = 0, \quad \nabla^2 Q - \mathbb{D}Q = 0, \tag{22}$$

where $\mathbb{C} = \{c_{ij}\}_{2N-1,2N-1}$ and $\mathbb{D} = \{d_{ij}\}_{N,N}$.

Using now the formulae Vekua-Bitsacze for the homogenous matrix equations (21) we obtain the following complex representation of the general solutions

$$W = 2Re\{f(z) + \frac{\mathbb{C}}{4}\int_{z_0}^{z}\int_{\overline{z}_0}^{\overline{z}} \Lambda(t,\overline{t})R(z,\overline{z},t,\overline{t})f(t)dtd\overline{t}\},$$

$$Q = 2Re\{g(z) + \frac{\mathbb{D}}{4}\int_{z_0}^{z}\int_{\overline{z}_0}^{\overline{z}} \Lambda(t,\overline{t})r(z,\overline{z},t,\overline{t})g(t)dtd\overline{t}\},$$

where R and r are the Riemann's matrix functions of the equations (22), $f(z)$ and $g(z)$ are holomorphic column-matrices:

$$f(z) = (f_1(z), \cdots, f_N(z), f_{N+1}(z), \cdots f_{2N-1}(z))^T, \quad g(z) = (g_1(z), \cdots, g_N(z))^T.$$

Then particular solutions of the matrix equations (21) have the form

$$\hat{V}(z,\overline{z}) = \frac{1}{4}\int_{z_0}^{z}\int_{\overline{z}_0}^{\overline{z}} \Lambda(t,\overline{t})R(t,\overline{t},z,\overline{z})X(t,\overline{t})dtd\overline{t},$$

$$\overset{\wedge}{\Omega}(z,\overline{z}) = \frac{1}{4}\int_{z_0}^{z}\int_{\overline{z}_0}^{\overline{z}} \Lambda(t,\overline{t})r(t,\overline{t},z,\overline{z})Y(t,\overline{t})dtd\overline{t}.$$

where

$$R(z,\overline{z},t,\overline{t}) = E + \frac{\mathbb{C}}{4}\int\limits_{t}^{z}\int\limits_{\overline{t}}^{\overline{z}}\Lambda(t_1,\overline{t}_1)dt_1 d\overline{t}_1 +$$

$$\left(\frac{\mathbb{C}}{4}\right)^2 \int\limits_{t}^{z}\int\limits_{\overline{t}}^{\overline{z}}\Lambda(t_1,\overline{t}_1)\Big(\int\limits_{t}^{t_1}\int\limits_{\overline{t}}^{\overline{t}_1}\Lambda(t_2,\overline{t}_2)dt_2 d\overline{t}_2\Big)dt_1 d\overline{t}_1 \cdot\cdot\cdot,$$

$$r(z,\overline{z},t,\overline{t}) = E + \frac{\mathbb{D}}{4}\int\limits_{t}^{z}\int\limits_{\overline{t}}^{\overline{z}}\Lambda(t_1,\overline{t})dt_1 d\overline{t}_1 +$$

$$\left(\frac{\mathbb{D}}{4}\right)^2 \int\limits_{t}^{z}\int\limits_{\overline{t}}^{\overline{z}}\Lambda(t_1,\overline{t}_1)\Big(\int\limits_{t}^{t_1}\int\limits_{\overline{t}}^{\overline{t}_1}\Lambda(t_2,\overline{t}_2)dt_2 d\overline{t}_2\Big)dt_1 d\overline{t}_1 + \cdots.$$

4 The refined theories of plates and the Kirsch's problem

Now we consider various refined theories of plates and the kizsch's problem for the concentration of stresses near the hole.

The system of Reissner-Mindlin's equations for tension-pressure coincides to the classical theory of generalized plane stress.

For bending of plates the system of Reissner-Mindlin equation can be written in complex form [3]:

$$\begin{aligned}&\partial_z(M_{11}-M_{22}+2iM_{12})+\partial_{\bar z}(M_{11}+M_{22})-Q_+=M_+,\\&\partial_z Q_+ + \partial_{\bar z}\bar Q_+ = M_3, \quad (Q_+=Q_1+iQ_2), \quad (2\partial_z=\partial_1-i\partial_2),\end{aligned} \tag{23}$$

where

$$\begin{aligned}&M_{11}-M_{22}+2iM_{12}=\frac{8\mu h^3}{3}\partial_{\bar z}V_+, \quad (V_+=V_1+iV_2),\\&M_{11}+M_{22}=\frac{4(\lambda^2+\mu)}{3}h^3\rho, \quad (\rho=2Re\partial_z V_+),\\&Q_+=\frac{5\mu h}{3}(2\partial_{\bar z}V_3+V_+), \quad (Reissner)\\&Q_+=\frac{4\mu h}{3}(2\partial_{\bar z}V_3+V_+). \quad (Mindlin)\end{aligned} \tag{24}$$

The system of equilibrium equations with respect to components of displacement vector has the complex form:

$$\begin{aligned}&\mu\Delta V_+ + 2(\lambda^*+\mu)\partial_{\bar z}\rho-\frac{5\mu}{h^2}(2\partial_{\bar z}V_3+V_+)=\frac{3}{2h^3}M_+,\\&\mu(\Delta V_3+\rho)=\frac{3}{5h}M_3.\end{aligned} \tag{25}$$

The general solution of the homogenous system (24) have the complex form

$$\begin{aligned} &V_+ = i\partial_{\bar z}\omega + \varphi(z) + z\overline{\varphi'(z)} + \frac{8(\lambda^* + 2\mu)h^2}{5\mu}\overline{\varphi''(z)} - 2\overline{\psi'(z)},\\ &V_3 = -\frac{1}{2}(\bar z\varphi(z) + \bar\varphi(\bar z)) + \psi(z) + \bar\psi(\bar z),\\ &\Delta\omega - \frac{5}{2h^2}\omega = 0, \end{aligned} \tag{26}$$

where $\varphi(z)$ and $\psi(z)$ are analytic functions of z.

The boundary conditions for Cirsch's problem on the hole's contour Γ have the form

$$M_{ll} + iM_{ls} = 0, \quad Q_{ln} = 0 \tag{27}$$

and in infinite we have

$$M_{11}^{\infty} = M_1, \quad (M_{12} = M_{22} = Q_+)^{\infty} = 0.$$

Now we consider this problem by I. Vekua's refined theory of plates.

I. Vekua's first method (so called "simplified scheme")

$$\begin{aligned} &\partial_z\left(\overset{(m)}{\sigma}_{11} - \overset{(m)}{\sigma}_{22} + 2i\overset{(m)}{\sigma}_{12}\right) + \partial_{\bar z}\left(\overset{(m)}{\sigma}_{11} + \overset{(m)}{\sigma}_{22}\right) - \overset{(m)}{\underline{\sigma}}_{+} + \overset{(m)}{F}_{+} = 0,\\ &\partial_z\overset{(m)}{\sigma}_{+} + \partial_{\bar z}\overset{(m)}{\bar\sigma}_{+} - \overset{(m)}{\underline{\sigma}}_{33} + \overset{(m)}{F}_3 = 0, \quad \left(\overset{(m)}{\sigma}_{+} = \overset{(m)}{\sigma}_{13} + i\overset{(m)}{\sigma}_{23}\right) \end{aligned} \tag{28}$$

where

$$\begin{aligned} &\overset{(m)}{\sigma}_{11} - \overset{(m)}{\sigma}_{22} + 2i\overset{(m)}{\sigma}_{12} = 4\mu\partial_{\bar z}\overset{(m)}{u}_{+}, \quad \left(\overset{(m)}{u}_{+} = \overset{(m)}{u}_{1} + i\overset{(m)}{u}_{2}\right)\\ &\overset{(m)}{\sigma}_{11} + \overset{(m)}{\sigma}_{22} = 2(\lambda+\mu)\overset{(m)}{\theta} + 2\lambda D_3\overset{(m)}{u}_{3}, \quad \left(\overset{(m)}{\theta} = 2Re\partial_z\overset{(m)}{u}_{+}\right)\\ &\overset{(m)}{\sigma}_{+} = \mu\left(2\partial_{\bar z}\overset{(m)}{u}_{3} + D_3\overset{(m)}{u}_{+}\right),\\ &\overset{(m)}{\sigma}_{33} = \lambda\overset{(m)}{\theta} + (\lambda+2\mu)D_3\overset{(m)}{u}_{3}, \end{aligned} \tag{29}$$

$$\overset{(m)}{\underline{\sigma}}_{3} = \frac{2m+1}{h}\left(\overset{(m-1)}{\sigma}_{3} + \overset{(m-3)}{\sigma}_{3} + \cdots\right), \quad D_3\overset{(m)}{u}\frac{2m+1}{h}\sum_{s=m}^{N}\frac{1-(-1)^{s+m}}{2}\overset{(s)}{u},$$

$$\overset{(m)}{\boldsymbol{\sigma}}{}^{i} = \frac{2m+1}{2h}\int_{-h}^{h}\boldsymbol{\sigma}^i P_m\left(\tfrac{x_3}{h}\right)dx_3, \quad \overset{(m)}{\boldsymbol{F}} = \overset{(m)}{\boldsymbol{\Phi}} + \frac{2m+1}{2h}\left(\overset{(+)}{\sigma}_3 - (-1)^m\overset{(-)}{\sigma}_3\right).$$

II. Vekua's second method (so-called "normed moments method")

The sistem of equilibrium equations coincides with (27), then

$$\begin{aligned}
&\overset{(m)}{\sigma}_{11} - \overset{(m)}{\sigma}_{22} + 2i\overset{(m)}{\sigma}_{12} = 4\mu\partial_{\bar z}\overset{(m)}{u}_{+},\\
&\overset{(m)}{\sigma}_{11} + \overset{(m)}{\sigma}_{22} = 2(\lambda+\mu)\overset{(m)}{\theta} + 2\lambda D_3\overset{(m)}{u}_{3} -\\
&\quad 2\varepsilon_{N,m}\sum_{s=0}^{N}(1-(-1)^{s+m})\Big(\frac{\lambda^2}{\lambda+2\mu}\overset{(s)}{\theta} + \lambda D_3\overset{(s)}{u}_{3}\Big),\\
&\overset{(m)}{\sigma}_{+} = \mu\Big(2\partial_{\bar z}\overset{(m)}{u}_{3} + D_3\overset{(m)}{u}_{+} - \varepsilon_{N,m}\sum_{s=0}^{N}(1-(-1)^{s+m})\Big(2\partial_{\bar z}\overset{(s)}{u}_{3} + D_3\overset{(s)}{u}_{+})\Big),\\
&\overset{(m)}{\sigma}_{33} = \lambda\overset{(m)}{\theta} + (\lambda+2\mu)D_3\overset{(m)}{u}_{3} - \varepsilon_{N,m}\sum_{s=0}^{N}(1-(-1)^{s+m})\Big(\lambda\overset{(s)}{\theta} + (\lambda+2\mu)D_3\overset{(s)}{u}_{3}\Big),
\end{aligned} \tag{30}$$

$$\varepsilon_{N,m} = \frac{2m+1}{N(N+1)}\Big(1 - \frac{(-1)^{N+m}}{N+1}\Big).$$

It is easy to see that equations (27), (29) constitute a normal system of $6N+6$-th order if the conditions

$$1 - 2\varepsilon_{N,m} \neq 0, \quad (m = 0,1,...,N) \tag{31}$$

are satisfied. These conditions are violated when $N = 0,1,2$ and they are satisfied when $N > 2$. For $N = 3$ we have

$$\varepsilon_{3,0} = \frac{1}{12}, \quad \varepsilon_{3,1} = \frac{3}{20}, \quad \varepsilon_{3,2} = \frac{5}{12}, \quad \varepsilon_{3,3} = \frac{7}{20}.$$

The system of equilibrium equations for $N = 3$ takes the complex form

$$\begin{aligned}
&\Delta\overset{(0)}{u}_{+} + 2(\lambda^*+\mu)\partial_{\bar z}\overset{(0)}{\theta} = 0,\\
&\Delta\overset{(1)}{u}_{+} + 2(\lambda^*+\mu)\partial_{\bar z}\overset{(1)}{\theta} - \frac{\mu}{2h}\Big[2\partial_{\bar z}\Big(5\overset{(0)}{u}_{3} - \overset{(2)}{u}_{3}\Big) + \frac{5}{h}\overset{(1)}{u}_{+}\Big] = 0,\\
&\Delta\overset{(2)}{u}_{+} + 2(\lambda^*+\mu)\partial_{\bar z}\overset{(2)}{\theta} - \frac{\mu}{2h}\Big[2\partial_{\bar z}\Big(7\overset{(1)}{u}_{3} - 3\overset{(3)}{u}_{3}\Big) + \frac{5}{h}\overset{(2)}{u}_{+}\Big] = 0,\\
&\Delta\overset{(3)}{u}_{+} + 2(\lambda^*+\mu)\partial_{\bar z}\overset{(3)}{\theta} = 0,\\
&\Delta\Big(5\overset{(0)}{u}_{3} - \overset{(2)}{u}_{3}\Big) + \frac{5}{h}\overset{(0)}{\theta} = 0, \quad \Delta\Big(7\overset{(1)}{u}_{3} - 3\overset{(3)}{u}_{3}\Big) + \frac{21}{h}\overset{(2)}{\theta} = 0,\\
&\lambda\Big(7\overset{(1)}{\theta} - 3\overset{(3)}{\theta}\Big) + \frac{21(\lambda+2\mu)}{h}\overset{(2)}{u}_{3} = 0,\\
&\lambda\Big(5\overset{(0)}{\theta} - \overset{(2)}{\theta}\Big) + \frac{5(\lambda+2\mu)}{h}\overset{(1)}{u}_{3} = 0.
\end{aligned} \tag{32}$$

The general solution of equation (31) is expressed by the formulas

a) for the tension-pressure of plates

$$\overset{(0)}{u}_+ = \frac{5\lambda+6\mu}{3\lambda+2\mu}\varphi_0(z) - z\overline{\varphi_0'(z)} - \overline{\psi_0'(z)},$$
$$\overset{(2)}{u}_+ = \frac{ih}{21}\partial_{\bar z}\omega + \frac{\lambda+2\mu}{3\lambda+2\mu}\Big(\varphi_2(z) + z\overline{\varphi_2'(z)}\Big) + \overline{\psi_2'(z)},$$
$$\Big(\Delta\omega - \frac{21}{h^2}\omega = 0\Big),$$
$$\overset{(1)}{u}_3 = \frac{4h}{5}\frac{\lambda}{3\lambda+2\mu}Re\Big[\varphi_2'(z) - 5\varphi_0'(z)\Big],$$
$$\overset{(3)}{u}_3 = \frac{7}{h}Re\Big[f_1(z) + \frac{\lambda+2\mu}{3\lambda+2\mu}\bar z\varphi_2'(z)\Big],$$

b) for, the bending of plates

$$\overset{(1)}{u}_+ = \frac{ih}{5}\partial_{\bar z}w + \frac{\lambda+2\mu}{3\lambda+2\mu}\Big(\varphi_1(z) + z\overline{\varphi_1'(z)}\Big) + \overline{\psi_1'(z)},$$
$$\Big(\Delta w - \frac{5}{2h^2}w = 0\Big),$$
$$\overset{(3)}{u}_+ = \frac{5\lambda+6\mu}{3\lambda+2\mu}\varphi_3(z) - z\overline{\varphi_3'(z)} - \overline{\psi_3'(z)},$$
$$\overset{(0)}{u}_3 = 2Re\Big[f_2(z) - \frac{\lambda+2\mu}{2(3\lambda+2\mu)}\bar z\varphi_1'(z)\Big],$$
$$\overset{(2)}{u}_3 = -\frac{4h}{21}\frac{\lambda}{3\lambda+2\mu}Re\Big[7\varphi_1'(z) - 3\varphi_3'(z)\Big],$$

where $\varphi_k(z), \psi_k(z), (k=0,1,2,3)$ and $f_\alpha(z), (\alpha=1,2)$ are arbitrary analytic functions of $z = x_1 + ix_2$.

Th Cirsch's problem for these cases can be written as:

a) boundary conditions in infinite:

$$\overset{(0)}{\sigma}_{11}^{\infty} = P_1, \quad \overset{(0)}{\sigma}_{22}^{\infty} = P_2, \quad \Big(\overset{(0)}{\sigma}_{12} = \overset{(0)}{\sigma}_{3i}\Big)^{\infty} = 0, \quad (tension-pressure)$$

or

$$\overset{(1)}{\sigma}_{11}^{\infty} = M_1, \quad \overset{(1)}{\sigma}_{22}^{\infty} = M_2, \quad \Big(\overset{(0)}{\sigma}_{12} = \overset{(0)}{\sigma}_{3i}\Big)^{\infty} = 0, \quad (bending)$$
$$(i=1,2,3)$$

and

b) boundary conditions on the circular hole ($|z| = R$):

$$\overset{(m)}{\sigma}_{rr} + i\overset{(m)}{\sigma}_{r\vartheta} = 0, \quad \overset{(m)}{\sigma}_{r3} = 0,$$
$$(m=0,1,2,3)$$

P_1, P_2, M_1, M_2-are constants.

Conclusion

1. a) I. Vekua's approximation of order $N = 0$ (first method) gives the system of plane deformation equations. The coefficient of stress concentration K, considers with well-known meaning

$$K = \frac{max \overset{(0)}{\sigma}_{\vartheta\vartheta}}{P} = 3, \quad (P_1 = P,\ P_2 = 0).$$

b)I. Vekua's approximation of order $N = 0$ (second method) and Reissner's method describe the generalized plane stress, i.e. $K = 3$.

2. a) I. Vekua's approximation of order $N = 1$ (first method) for the tension-pressure of plates gives the following formula for K

$$K = 1 + 2\frac{2\varkappa K_0(\varkappa) + [4 + 5(1-\sigma^2)\varkappa^2]K_1(\varkappa)}{2(1-\sigma^2)\varkappa K_0(\varkappa) + [4 + (1-\sigma^2)\varkappa^2]K_1(\varkappa)},$$

where $\varkappa^2 = \dfrac{\sigma}{1-\sigma}\dfrac{R^2}{h^2}$, i.e. $K = K(h, R, \sigma)$ depends on h, R, σ (Poisson's coefficient), and when $\frac{h}{R} \to 0$ or $\frac{h}{R} \to \infty \Rightarrow K = 3$ (because $K_n(x) = \sqrt{\dfrac{\pi}{2x}}e_{-x}(1 + O(\dfrac{1}{x})))$.

b) for the plate's bending I. Vekua's approximation $N = 1$ and Reissner"s method give us:
(I. Vekua's $N = 1$)

$$K = 1 + 2\frac{K_2(\varkappa)}{K_2(\varkappa) + 2(1-\sigma)K_0(\varkappa)}, \quad \varkappa = \frac{3R^2}{h^2}$$

and when $\frac{h}{R} \to 0$ we have $K = \frac{5-2\sigma}{3-2\sigma}$

(E. Reissner)

$$K = 1 + 2\frac{(1+\sigma)K_2(\varkappa)}{(1+\sigma)K_2(\varkappa) + 2\sigma K_0(\varkappa)} \quad \varkappa = \frac{5R^2}{2h^2}$$

and when $\frac{h}{R} \to 0$, then Reissner's coefficient K coincides with classical results, $(K_{cl} = \dfrac{5+2\sigma}{3+2\sigma})$.

3. a) I. Vekua's approximation of order $N = 2$ (first and second methods) for the tension-pressure solves 3-D problems, when $P_1 = P_2 = const$.

b) for bending of plate coincides with Reissner result.

4. a) I. Vekua's approximation of order $N = 3$ solves 3-D problems for the tension-pressure when $P_1 = const$, $P_2 = 0$, (II-method).

b)I. Vekua's approximation of order $N = 3$ (II-method) for bending of plate solves 3-D problems, when $M_1 = const$, $M_2 = 0$.

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