# ON THE SUFFICIENT CONDITIONS OF ABSOLUTE CONVERGENCE OF DOUBLE TRIGONOMETRIC SERIES

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*Abstract*

It is well known that if the function of a single variable has a continuous derivative, its trigonometric Fourier series is absolutely convergent. However, if the function of two variables has continuous partial derivatives, its double trigonometric series is not necessarily absolutely convergent (see [1], [2]). In the present paper, in particular, the sufficient conditions are found for the absolute convergence of double trigonometric series of functions of two variables with continuous partial derivatives.

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Let $L_p(I^2)$, $I = [0,1]$, $1 \le p < \infty$, be a space of periodic functions with period 1 with respect to each variable and integrable on $I^2$ with power $p$ and let $C(I^2)$ be a space of periodic functions with period 1 with respect to each variable and continuous on $I^2$. As usual, for $f \in L_p(I^2)$

$$\omega\left(\frac{1}{m},\frac{1}{n},f\right)_{L_p} = \left(\sup_{\substack{|t|\le m^{-1}\\|s|\le n^{-1}}}\int_{I^2}|\Delta_{t,s}f(x,y)|^p dx\,dy\right)^{\frac{1}{p}},$$

where

$$\Delta_{t,s}f(x,y) = f(x+t,y+s) - f(x,y+s) - f(x+t,s) + f(x,y),$$

$$\omega_1\left(\frac{1}{m},f\right)_{L_p} = \left(\sup_{|t|\le m^{-1}}\int_{I^2}|f(x+t,y)-f(x,y)|^p\,dx\,dy\right)^{\frac{1}{p}},$$

$$\omega_2\left(\frac{1}{n},f\right)_{L_p} = \left(\sup_{|s|\le n^{-1}}\int_{I^2}|f(x,y+s)-f(x,y)|^p\,dx\,dy\right)^{\frac{1}{p}}.$$

The corresponding moduli of continuity when $f \in C(I^2)$ will be denoted, respectively, by

$$\omega\left(\frac{1}{m},\frac{1}{n},f\right),\ \ \omega_1\left(\frac{1}{m},f\right),\ \ \omega_2\left(\frac{1}{n},f\right).$$

The partial variations of a function $f(x,y)$ are defined similarly as the variations of a function of single variable. If $\Pi_n$ is a decomposition of the segment $[0,1]$ with the points $0 \le x_0 < x_1 < \cdots < x_n \le 1$ and

$$\sup_{y\in[0,1]}\ \sup_{\Pi_n}\sum_{k=0}^{n-1}|f(x_k,y)-f(x_{k+1},y)| = V_1(f) < +\infty,$$

then we will write $f \in V_1(I^2)$. Analogously, if

$$\sup_{x\in[0,1]}\ \sup_{\Pi_n}\sum_{k=0}^{n-1}|f(x,y_k)-f(x,y_{k+1})| = V_2(f) < +\infty,$$

we will write $f \in V_2(I^2)$.

Consider the decomposition $\Pi_{m,n}$ of the square $[0,1]^2$ by the points $(x_k;y_k)$, $0 \le x_0 < x_1 < \cdots < x_n \le 1$, $0 \le y_0 < y_1 < \cdots < y_m \le 1$. If

$$\sup_{\Pi_{m,n}}\sum_{i=0}^{m}\sum_{k=0}^{n}|f(x_i,y_k)-f(x_{i+1},y_k)-f(x_i,y_{k+1})+f(x_{i+1},y_{k+1})| = V(f) < +\infty,$$

we will write $f \in V(I^2)$.

Let $f \in L(I^2)$ and let

$$\sum_{m=-\infty}^{\infty}\ \sum_{n=-\infty}^{\infty} c_{mn}(f)e^{2i\pi(mx+ny)}$$

be its double Fourier series. Let

$$a_{mn}(f) = (|c_{mn}(f)| + |c_{-mn}(f)| + |c_{m\,-n}(f)| + |c_{-m\,-n}(f)|)\,,\quad m \ge 1,\quad n \ge 1,$$
$$a_{m0}(f) = (|c_{m0}(f)| + |c_{-m0}(f)|)\,,\quad m \ge 1,$$
$$a_{0n}(f) = (|c_{0n}(f)| + |c_{0\,-n}(f)|)\,,\quad n \ge 1.$$

**Lemma 1.** *Let $f \in L_p(I^2)$, $p \in (1,2]$, $r \in (0,q]$, $q = \frac{p}{p-1}$. Then for any*

$M \geq 1$, $N \geq 1$

$$\sum_{m=M}^{2M-1}\sum_{n=N}^{2N-1} a_{mn}^r(f) \leq c_{p,r} \sum_{m=M}^{2M-1}\sum_{n=N}^{2N-1} \omega^r\left(\frac{1}{m}, \frac{1}{n}, f\right)_{L_p} (m\cdot n)^{-\frac{r}{q}}, \quad (1)$$

$$\sum_{m=M}^{2M-1} a_{m0}^r(f) \leq c_{p,r} \sum_{m=M}^{2M-1} \omega_1^r\left(\frac{1}{m}, f\right)_{L_p} m^{-\frac{r}{q}}, \quad (2)$$

$$\sum_{n=N}^{2N-1} a_{0n}^r(f) \leq c_{p,r} \sum_{n=N}^{2N-1} \omega_2^r\left(\frac{1}{n}, f\right)_{L_p} n^{-\frac{r}{q}}. \quad (3)$$ [0)]

*Proof.* It is easy to see that the series

$$-\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} 4c_{mn}(f)e^{i2\pi(mx+ny)} \sin\frac{\pi m}{4M} \sin\frac{\pi n}{4N}$$

is the Fourier series of the function

$$\delta_{M,N}f(x,y) = f\left(x+\frac{1}{8M}, y+\frac{1}{8N}\right) - f\left(x-\frac{1}{8M}, y+\frac{1}{8N}\right) - f\left(x+\frac{1}{8M}, y-\frac{1}{8N}\right) + f\left(x-\frac{1}{8M}, y-\frac{1}{8N}\right).$$

Hence, using the Hausdorff–Young theorem, we will have

$$\sum_{m=M}^{2M-1}\sum_{n=N}^{2N-1} a_{mn}^q(f)\left|\sin\frac{\pi m}{4M}\sin\frac{\pi n}{4N}\right|^q \leq c_p\left(\int_{I^2} |\delta_{M,N}f(x,y)|^p\,dx\,dy\right)^{\frac{1}{p-1}} \leq c_p\,\omega^{\frac{p}{p-1}}\left(\frac{1}{8M}, \frac{1}{8N}, f\right)_{L_p}.$$

Therefore, on account of

$$\left|\sin\frac{\pi m}{4M}\sin\frac{\pi n}{4N}\right| \geq \frac{1}{2}, \quad M \leq m \leq 2M-1, \quad N \leq n \leq 2N-1,$$

we get

$$\sum_{m=M}^{2M-1}\sum_{n=N}^{2N-1} a_{mn}^q(f) \leq c_p\omega^{\frac{p}{p-1}}\left(\frac{1}{4M}, \frac{1}{4N}, f\right)_{L_p}.$$

[0)] In what follows we will denote by $c$, $c_{\alpha,\beta}$, respectively, the absolute constants and constants depending on their indices, which are different in different inequalities.

Using the Hölder inequality, we have

$$\sum_{m=M}^{2M-1}\sum_{n=N}^{2N-1} a_{mn}^r(f) \le (MN)^{1-\frac{r}{q}}\Big(\sum_{m=M}^{2M-1}\sum_{n=N}^{2N-1} a_{mn}^q(f)\Big)^{\frac{r}{q}}.$$

From the last two inequalities we have

$$\sum_{m=M}^{2M-1}\sum_{n=N}^{2N-1} a_{mn}^r(f) \le c_p(MN)^{1-\frac{r}{q}}\omega^r\left(\frac{1}{4M},\frac{1}{4N},f\right)_{L_p}$$
$$\le c_{p,r}\sum_{m=M}^{2M-1}\sum_{n=N}^{2N-1}\omega^r\left(\frac{1}{m},\frac{1}{n},f\right)_{L_p}(mn)^{-\frac{r}{q}}.$$

Now it is easy to see that

$$\sum_{m=0}^{\infty} 2ic_{m0}(f)e^{i2\pi x}\sin\frac{m\pi}{4M}$$

is the Fourier series of the function

$$\delta_M f_1(x) = f_1\left(x+\frac{1}{8M}\right) - f_1\left(x-\frac{1}{8M}\right),$$

where $f_1(x) = \int_I f(x,y)\,dy$.

According to the Hausdorff–Young theorem we get

$$\sum_{m=M}^{2M_1} a_{m0}^q(f)\Big|\sin\frac{m\pi}{4M}\Big|^q \le c_p\Big(\int_I |\delta_M f_1(x)|^p\Big)^{\frac{1}{p-1}}dx$$
$$\le c_p\omega\left(\frac{1}{4M},f_1\right) \le c_p\omega_1^{\frac{p}{p-1}}\left(\frac{1}{8M},f\right)_{L_p}.$$

Consequently, since

$$\sin\frac{m\pi}{4M} \ge 2^{-\frac{1}{2}},\quad M\le m\le 2M-1,$$

we have

$$\sum_{m=M}^{2M-1} a_{m0}^q(f) \le c_p\omega_1^{\frac{p}{p-1}}\left(\frac{1}{8M},f\right)_{L_p}.$$

In view of the Hölder inequality

$$\sum_{m=M}^{2M-1} a_{m0}^r(f) \le M^{1-\frac{r}{q}}\Big(\sum_{m=M}^{2M-1} a_{m0}^q(f)\Big)^{\frac{r}{q}}.$$

From the last two inequalities we get

$$\sum_{m=M}^{2M-1} a_{m0}^r(f) \le c_p M^{1-\frac{r}{q}} \omega_1^r \left(\frac{1}{4M}, f\right)_{L_p} \le c_{p,r} \sum_{m=M}^{2M-1} \omega_1^r \left(\frac{1}{m}, f\right)_{L_p} m^{-\frac{r}{q}}.$$

The validity of inequality (3) can be proved in a similar way. □

**Corollary 1.** a) *If $f'_x \in L_p(I^2)$, $p \in (1,2]$, $r \in (0,q]$, $q = \frac{p}{p-1}$, then for $M \ge 1$, $N \ge 1$*

$$\sum_{m=M}^{2M-1} \sum_{n=N}^{2N-1} a_{mn}^r(f) \le c_{p,r} \sum_{m=M}^{2M-1} \sum_{n=N}^{2N-1} \omega^r \left(\frac{1}{m}, \frac{1}{n}, f'_x\right)_{L_p} m^{-r\,\frac{q+1}{q}} n^{-\frac{r}{q}}, \quad (4)$$

$$\sum_{m=M}^{2M-1} a_{m0}^r(f) \le c_{p,r} \sum_{m=M}^{2M-1} \omega_1^r \left(\frac{1}{m}, f'_x\right)_{L_p} m^{-r\,\frac{q+1}{q}}. \quad (5)$$

b) *If $f'_y \in L_p(I^2)$, $p \in (1,2]$, $r \in (0,q]$, $q = \frac{p}{p-1}$, then for $M \ge 1$, $N \ge 1$*

$$\sum_{m=M}^{2M-1} \sum_{n=N}^{2N-1} a_{mn}^r(f) \le c_{p,r} \sum_{m=M}^{2M-1} \sum_{n=N}^{2N-1} \omega^r \left(\frac{1}{m}, \frac{1}{n}, f'_y\right)_{L_p} m^{-\frac{r}{q}} n^{-r\,\frac{q+1}{q}}, \quad (6)$$

$$\sum_{n=N}^{2N-1} a_{0n}^r(f) \le c_{p,r} \sum_{m=N}^{2N-1} \omega_2^r \left(\frac{1}{n}, f'_y\right)_{L_p} n^{-r\,\frac{q+1}{q}}. \quad (7)$$

*Proof.* Since

$$c_{mn}(f) = \frac{1}{m}\, c_{mn}(f'_x),$$

when $m \ne 0$, for $M \ge 1$, $N \ge 1$ we have

$$\sum_{m=M}^{2M-1} \sum_{n=N}^{2N-1} a_{mn}^r(f) = \sum_{m=M}^{2M-1} \sum_{n=N}^{2N-1} m^{-r} a_{mn}^r(f'_x) \le M^{-r} \sum_{m=M}^{2M-1} \sum_{n=N}^{2N-1} a_{mn}^r(f'_x).$$

Now using Lemma 1 for the function $f'_x(x,y)$ we get

$$\sum_{m=M}^{2M-1} \sum_{n=N}^{2N-1} a_{mn}^r(f'_x) \le c_{p,r} \sum_{m=M}^{2M-1} \sum_{n=N}^{2N-1} \omega^r \left(\frac{1}{m}, \frac{1}{n}, f'_x\right)_{L_p} (mn)^{-\frac{r}{q}}$$

$$\le c_{p,r} M^r \sum_{m=M}^{2M-1} \sum_{n=N}^{2N-1} \omega^r \left(\frac{1}{m}, \frac{1}{n}, f'_x\right)_{L_p} m^{-r\,\frac{q+1}{q}} n^{-\frac{r}{q}}.$$

From the last two inequalities it follows that inequality (4) is true. The validity of inequalities (5)–(7) is proved in a similar way. □

**Theorem 1.** *Let* $f \in L_p(I^2)$, $p \in (0,2]$, $r \in (0,q]$. *Then for any* $\mu = 0,1,\dots$, $\nu = 0,1,\dots$, $\mu_1 \geq \mu$, $\nu_1 \geq \nu$ *we have*

$$\sum_{m=2^\mu}^{2^{\mu_1}-1} \sum_{n=2^\nu}^{2^{\nu_1}-1} a_{mn}^r(f) \leq c_{p,r} \sum_{m=2^\mu}^{2^{\mu_1}-1} \sum_{n=2^\nu}^{2^{\nu_1}-1} \omega^r \left(\frac{1}{m}, \frac{1}{n}, f\right)_{L_p} (mn)^{-\frac{r}{q}}, \tag{8}$$

$$\sum_{m=2^\mu}^{2^{\mu_1}-1} a_{m0}^r(f) \leq c_{p,r} \sum_{m=2^\mu}^{2^{\mu_1}-1} \omega_1^r \left(\frac{1}{m}, f\right)_{L_p} m^{-\frac{r}{q}}, \tag{9}$$

$$\sum_{n=2^\nu}^{2^{\nu_1}-1} a_{0n}^r(f) \leq c_{p,r} \sum_{n=2^\nu}^{2^{\nu_1}-1} \omega_2^r \left(\frac{1}{n}, f\right)_{L_p} n^{-\frac{r}{q}}. \tag{10}$$

*Proof.* Assuming in (1) that $M = 2^i$, $N = 2^k$, from (1) we get

$$\sum_{m=2^i}^{2^{i+1}-1} \sum_{n=2^k}^{2^{k+1}-1} a_{mn}^r(f) \leq c_{p,r} \sum_{m=2^i}^{2^{i+1}-1} \sum_{n=2^k}^{2^{k+1}-1} \omega^r \left(\frac{1}{m}, \frac{1}{n}, f\right)_{L_p} (mn)^{-\frac{r}{q}}.$$

Summing up this inequality when $i$ changes from $\mu$ to $\mu_1$, and $k$ changes from $\nu$ to $\nu_1$, we get the validity of (8). Inequalities (9) and (10) are obtained, respectively, from (2) and (3) in a similar way. □

Denote by $A_r$ the set of those functions $f$ for which

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn}^r(f) < \infty. \tag{11}$$

Theorem 1 yields the following

**Corollary 2.** *Let the conditions of Theorem* 1 *be satisfied. Then if*

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega^r \left(\frac{1}{m}, \frac{1}{n}, f\right)_{L_p} (mn)^{-\frac{r}{q}} < \infty,$$

$$\sum_{m=1}^{\infty} \omega_1^r \left(\frac{1}{m}, f\right)_{L_p} m^{-\frac{r}{q}} < \infty,$$

$$\sum_{n=1}^{\infty} \omega_2^r \left(\frac{1}{n}, f\right)_{L_p} n^{-\frac{r}{q}} < \infty,$$

*we have* $f \in A_r$.

*Proof.* Assuming in inequalities (8)–(10) $\mu = 0$, $\nu = 0$ and passing to the limit when $\mu_1 \to \infty$ and $\nu_1 \to \infty$ we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^r(f) \le c_{p,r} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \omega^r \left(\frac{1}{m}, \frac{1}{n}, f\right)_{L_p} (mn)^{-\frac{r}{q}},$$
$$\sum_{m=1}^{\infty} a_{m0}^r(f) \le c_{p,r} \sum_{m=1}^{\infty} \omega_1^r \left(\frac{1}{m}, f\right)_{L_p} m^{-\frac{r}{q}},$$
$$\sum_{n=1}^{\infty} a_{0n}^r(f) \le c_{p,r} \sum_{n=1}^{\infty} \omega_2^r \left(\frac{1}{n}, f\right)_{L_p} n^{-\frac{r}{q}}.$$

This and the conditions of the corollary yield the validity of inequality (11). □

Corollary 2 when $p = 2$ and $r = 1$ was obtained in [3].

**Theorem 2.** *Let $f'_x, f'_y \in L_p(I^2)$, $p \in (1, 2]$, $r \in (0, q]$. Then for any $\mu = 0, 1, \dots$, $\nu = 0, 1, \dots$, $\mu_1 \ge \mu$, $\nu_1 \ge \nu$ we have*

$$\sum_{m=2^\mu}^{2^{\mu_1}-1}\sum_{n=2^\nu}^{2^{\nu_1}-1} a_{mn}^r(f) \le c_{p,r} \sum_{m=2^\mu}^{2^{\mu_1}-1}\sum_{n=2^\nu}^{2^{\nu_1}-1} \omega^r \left(\frac{1}{m}, \frac{1}{n}, f'_x\right)_{L_p} m^{-r\frac{q+1}{q}} n^{-\frac{r}{q}}, \quad (12)$$
$$\sum_{m=2^\mu}^{2^{\mu_1}-1}\sum_{n=2^\nu}^{2^{\nu_1}-1} a_{mn}^r(f) \le c_{p,r} \sum_{m=2^\mu}^{2^{\mu_1}-1}\sum_{n=2^\nu}^{2^{\nu_1}-1} \omega^r \left(\frac{1}{m}, \frac{1}{n}, f'_y\right)_{L_p} m^{-\frac{r}{q}} n^{-r\frac{q+1}{q}}, \quad (13)$$
$$\sum_{m=2^\mu}^{2^{\mu_1}-1} a_{m0}^r(f) \le c_{p,r} \sum_{m=2^\mu}^{2^{\mu_1}-1} \omega_1^r \left(\frac{1}{m}, f'_x\right)_{L_p} m^{-r\frac{q+1}{q}}. \quad (14)$$
$$\sum_{n=2^\nu}^{2^{\nu_1}-1} a_{0n}^r(f) \le c_{p,r} \sum_{n=2^\nu}^{2^{\nu_1}-1} \omega_2^r \left(\frac{1}{n}, f'_y\right)_{L_p} n^{-r\frac{q+1}{q}}. \quad (15)$$

This theorem is obtained from Corollary 1 in the same way as Theorem 1 is obtained from Lemma 1.

**Corollary 3.** *Let the conditions of Theorem 2 be fulfilled. Then if*

$$\sum_{m=1}^{\infty} \omega_1^r \left(\frac{1}{m}, f'_x\right)_{L_p} m^{-r\frac{q+1}{q}} < \infty,$$
$$\sum_{n=1}^{\infty} \omega_2^r \left(\frac{1}{n}, f'_y\right)_{L_p} n^{-r\frac{q+1}{q}} < \infty,$$

*and if one of the following conditions*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\omega^r\left(\frac{1}{m},\frac{1}{n},f'_x\right)_{L_p} m^{-r\frac{q+1}{q}}n^{-\frac{r}{q}}<\infty,$$
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\omega^r\left(\frac{1}{m},\frac{1}{n},f'_y\right)_{L_p} m^{-\frac{r}{q}}n^{-r\frac{q+1}{q}}<\infty$$

*is satisfied, then* $f\in A_r$.

The validity of this corollary is obtained from Theorem 2 in the same way as Corollary 2 was obtained from Theorem 1.

**Corollary 4.** *Let* $f'_x, f'_y\in C(I^2)$. *Then if*

$$\sum_{m=1}^{\infty}\omega_1^r\left(\frac{1}{m},f'_x\right)m^{-\frac{3}{2}r}<\infty,$$
$$\sum_{n=1}^{\infty}\omega_2^r\left(\frac{1}{n},f'_y\right)n^{-\frac{3}{2}r}<\infty,$$

*and if one of the following conditions*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\omega^r\left(\frac{1}{m},\frac{1}{n},f'_x\right)m^{-\frac{3}{2}r}n^{-\frac{r}{2}}<\infty,$$
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\omega^r\left(\frac{1}{m},\frac{1}{n},f'_y\right)m^{-\frac{r}{2}}n^{-\frac{3}{2}r}<\infty$$

*is satisfied, then* $f\in A_r$.

The validity of this corollary follows from Corollary 3 if we assume that $p=q=2$ and take into consideration that the modulus of continuity in the norm of the space $L_2(I^2)$ are majorized by corresponding modulus in the norm of the space $C(I^2)$.

**Theorem 3.** *Let $f'_x, f'_y \in L_p(I^2)$, $p \in (1,2]$, $r \in \left(\frac{q}{q+1}\,, q\right]$. Then we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^r(f) \\ \leq c_{p,r}\sum_{m=1}^{\infty} m^{-r\frac{q+2}{q}+1}\left[\omega^r\left(\frac{1}{m}\,,\frac{1}{m}\,,f'_x\right)_{L_p} + \omega^r\left(\frac{1}{m}\,,\frac{1}{m}\,,f'_y\right)_{L_p}\right], \tag{16}$$

$$\sum_{m=1}^{\infty} a_{m0}^r(f) \leq c_{p,r}\sum_{m=1}^{\infty}\omega_1^r\left(\frac{1}{m}\,,f'_x\right)_{L_p} m^{-r\frac{q+1}{q}}, \tag{17}$$

$$\sum_{n=1}^{\infty} a_{0n}^r(f) \leq c_{p,r}\sum_{m=1}^{\infty}\omega_1^r\left(\frac{1}{n}\,,f'_y\right)_{L_p} n^{-r\frac{q+1}{q}}. \tag{18}$$

*Proof.* We have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^r(f) \leq \left(\sum_{m=1}^{\infty}\sum_{n=1}^{m} + \sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\right)a_{mn}^r(f) \\ = \left(\sum_{n=1}^{\infty}\sum_{m=n}^{\infty} + \sum_{m=1}^{\infty}\sum_{n=m}^{\infty}\right)a_{mn}^r(f) = I_1 + I_2; \tag{19}$$

$$I_1 = \sum_{\nu=0}^{\infty}\sum_{n=2^\nu}^{2^{\nu+1}-1}\sum_{m=n}^{\infty} a_{mn}^r(f) \leq \sum_{\nu=0}^{\infty}\sum_{n=2^\nu}^{2^{\nu+1}-1}\sum_{m=2^\nu}^{\infty} a_{mn}^r(f).$$

In equality (12) setting $\mu = \nu$, $\nu_1 = \nu + 1$ and passing to the limit when $\mu_1 \to \infty$ we get

$$\sum_{m=2^\nu}^{\infty}\sum_{n=2^\nu}^{2^{\nu+1}-1} a_{mn}^r(f) < c_{p,r}\sum_{m=2^\nu}^{\infty}\sum_{n=2^\nu}^{2^{\nu+1}-1}\omega^r\left(\frac{1}{m}\,,\frac{1}{n}\,,f'_x\right)_{L_p} m^{-r\frac{q+1}{q}} n^{-\frac{r}{q}} \\ \leq c_{p,r}\sum_{n=2^\nu}^{2^{\nu+1}-1}\omega^r\left(\frac{1}{n}\,,\frac{1}{n}\,,f'_x\right)_{L_p} n^{-\frac{r}{q}}\sum_{m=2^\nu}^{\infty} m^{-r\frac{q+1}{q}}.$$

Since $r \in \left(\frac{q}{q+1}\,, q\right)$ we have $r\frac{q+1}{q} > 1$. Therefore from the last two inequalities we get

$$I_1 \le c_{p,r} \sum_{\nu=0}^{\infty} \sum_{n=2^\nu}^{2^{\nu+1}-1} \omega^r \left(\frac{1}{n}, \frac{1}{n}, f'_x\right)_{L_p} n^{-\frac{r}{q}} 2^{\nu(-r\frac{q+1}{q}+1)}$$

$$\le c_{p,r} \sum_{\nu=0}^{\infty} \sum_{n=2^\nu}^{2^{\nu+1}-1} \omega^r \left(\frac{1}{n}, \frac{1}{n}, f'_x\right)_{L_p} n^{-r\frac{q+2}{q}+1}$$

$$= c_{p,r} \sum_{n=1}^{\infty} \omega^r \left(\frac{1}{n}, \frac{1}{n}, f'_x\right)_{L_p} n^{-r\frac{q+2}{q}+1}.$$

In a similar way we can get

$$I_2 \le c_{p,r} \sum_{m=1}^{\infty} \omega^r \left(\frac{1}{m}, \frac{1}{m}, f'_y\right)_{L_p} m^{-r\frac{q+2}{q}+1}.$$

From the last two inequalities and from (19) we get the validity of inequality (16).

In equalities (14) and (15) assuming, respectively, that $\mu = 0$, $\nu = 0$ and passing to the limit when $\mu_1 \to \infty$, $\nu_1 \to \infty$ we get the validity of inequalities (17) and (18). Theorem 3 is proved. □

**Corollary 5.** *Let the conditions of Theorem* 3 *be satisfied. Then if*

$$\sum_{m=1}^{\infty} \left[\omega^r \left(\frac{1}{m}, \frac{1}{m}, f'_x\right)_{L_p} + \omega^r \left(\frac{1}{m}, \frac{1}{m}, f'_y\right)_{L_p}\right] m^{-r\frac{q+2}{q}+1} < \infty, \qquad (20)$$

*then* $f \in A_r$,

*Proof.* From (16) and (20) we get

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a^r_{mn}(f) < \infty.$$

Since $r \in \left(\frac{q}{q+1}, q\right)$ we have $r\frac{q+1}{q} > 1$. Therefore from (17) and (18) we have

$$\sum_{m=1}^{\infty} a^r_{m0}(f) < \infty,$$

$$\sum_{n=1}^{\infty} a^r_{0n}(f) < \infty.$$

From the last three inequalities we get the validity of Corollary 5. □

**Corollary 6.** *Let $f'_x, f'_y \in C(I^2)$, $r > \frac{2}{3}$ and*

$$\sum_{m=1}^{\infty} \left[\omega^r\left(\frac{1}{m}, \frac{1}{m}, f'_x\right) + \omega^r\left(\frac{1}{m}, \frac{1}{m}, f'_y\right)\right] m^{-2r+1} < \infty.$$

*Then $f \in A_r$.*

This corollary follows from Corollary 5 if we take into consideration that the modulus of continuity in the norm of the space $L_p(I^2)$ are majorized by the corresponding modulus in the norm of the space $C(I^2)$ and take $p = q = 2$.

**Corollary 7.** *Let $f'_x, f'_y \in C(I^2)$ and $r > \frac{2}{3}$. If for some $i \in \{1; 2\}$*

$$\sum_{m=1}^{\infty} \omega_i^r\left(\frac{1}{m}, f'_x\right) m^{-2r+1} < \infty$$

*and for some $j \in \{1, 2\}$*

$$\sum_{m=1}^{\infty} \omega_j\left(\frac{1}{m}, f'_y\right) m^{-2r+1} < \infty,$$

*then $f \in A_r$.*

This corollary follows from Corollary 6, since for any $i, j \in \{1, 2\}$

$$\omega(\delta, \delta, f'_x) \leq 2\omega_i(\delta, f'_x), \tag{21}$$

$$\omega(\delta, \delta, f'_x) \leq 2\omega_j(\delta, f'_y). \tag{22}$$

Let

$$\Lambda_\alpha = \left\{f : \omega(\delta, \delta, f'_x) = O(\delta^\alpha),\ \ \omega(\delta, \delta, f'_y) = O(\delta^\alpha)\right\}, \quad \alpha \in (0, 2],$$

$$\lambda_\alpha = \big\{f : \omega_i(\delta, f'_x) = O(\delta^\alpha) \text{ for some } i \in \{1, 2\},$$
$$\omega_j(\delta, f'_y) = O(\delta^\alpha) \text{ for some } j \in \{1, 2\}\big\}, \quad \alpha \in (0, 1].$$

From (21) and (22) it follows that when $\alpha \in (0, 1]$, then $\lambda_\alpha \subset \Lambda_\alpha$.
The following corollaries are special cases of Corollaries 6 and 7.

**Corollary 8.** *If $f \in \Lambda_\alpha$, $0 < \alpha \leq 1$, and $r > \frac{2}{2+\alpha}$, then $f \in A_r$.*

**Corollary 9.** *If $f \in \lambda_\alpha$, then $f \in A_r$ for $r > \frac{2}{2+\alpha}$.*

**Lemma 2.** *Let the function $g \in V(I^2) \cap C(I^2)$. Then*

$$\omega\left(\frac{1}{m}, \frac{1}{n}, g\right)_{L_2} \leq c_g \omega^{\frac{1}{2}}\left(\frac{1}{m}, \frac{1}{n}, g\right)(mn)^{-\frac{1}{2}}. \tag{23}$$

*Proof.* Let

$$\Delta_{h,\eta}g(x,y) = |g(x+h,y+\eta) - g(x,y+\eta) - g(x+h,y) + g(x,y)|\,.$$

Then

$$\begin{aligned}\omega\left(\frac{1}{m},\frac{1}{n},f\right)_{L_2} &= \sup_{\substack{h<m^{-1}\\ \eta<n^{-1}}}\left(\int_{I^2}\Delta^2_{h,\eta}g(x,y)\,dx\,dy\right)^{\frac{1}{2}}\\ &\le \sup_{\substack{h<m^{-1}\\ \eta<n^{-1}}}\sup_{(x,y)\in I^2}\Delta^{\frac{1}{2}}_{h,\eta}g(x,y)\left(\int_{I^2}\Delta_{h,\eta}g(x,y)\,dx\,dy\right)^{\frac{1}{2}}\\ &\le \omega^{\frac{1}{2}}\left(\frac{1}{m},\frac{1}{n},f\right)\sup_{\substack{h<m^{-1}\\ \eta<n^{-1}}}\left(\int_{I^2}\Delta_{h,\eta}g(x,y)\,dx\,dy\right)^{\frac{1}{2}}.\end{aligned}$$

It is easy to see that

$$\begin{aligned}&\int_{I^2}\Delta_{h,\eta}g(x,y)\,dx\,dy\\ &= m^{-1}n^{-1}\int_{I^2}\sum_{\mu=0}^{m-1}\sum_{\nu=0}^{n-1}\big|g(x+(\mu+1)h,y+(\nu+1)\eta)-g(x+\mu h,y+(\nu+1)\eta)\\ &\qquad - g(x+(\mu+1)h,y+\nu\eta)+g(x+\mu h,y+\nu\eta)\big|\,dx\,dy\\ &\qquad\qquad \le V(g)m^{-1}n^{-1}.\end{aligned}$$

From the last two inequalities we obtain the validity of inequality (23). □

**Theorem 4.** *Let the functions $f'_x$ and $f'_y$ belong to the class $V(I^2)\cap C(I^2)$. Then if*

$$\sum_{m=1}^{\infty}\left[\omega^{\frac{r}{2}}\left(\frac{1}{m},\frac{1}{m},f'_x\right)+\omega^{\frac{r}{2}}\left(\frac{1}{m},\frac{1}{m},f'_y\right)\right]m^{-3r+1}<\infty,$$

*then $f\in A_r$.*

*Proof.* Using inequality (23) for functions $f'_x$ and $f'_y$ we will have

$$\begin{aligned}&\sum_{m=1}^{\infty}\left[\omega^{r}\left(\frac{1}{m},\frac{1}{n},f'_x\right)_{L_2}+\omega^{r}\left(\frac{1}{m},\frac{1}{n},f'_y\right)_{L_2}\right]m^{-2r+1}\\ &\quad\le C_{f'_x,f'_y}\sum_{m=1}^{\infty}\left[\omega^{\frac{r}{2}}\left(\frac{1}{m},\frac{1}{m},f'_x\right)+\omega^{\frac{r}{2}}\left(\frac{1}{m},\frac{1}{m},f'_y\right)\right]m^{-3r+1}<\infty.\end{aligned}$$

From this and Corollary 5 when $p=q=2$ we get the validity of Theorem 4. □

**Corollary 10.** *Let the functions* $f'_x$ *and* $f'_y$ *belong to the class* $V(I^2) \cap C(I^2)$*. Then if* $r > \frac{2}{3}$*, then* $f \in A_r$*.*

**Corollary 11.** *Let* $f'_x \in V_i(I^2) \cap C(I^2)$ *for some* $i = 1, 2$ *and* $f'_y \in V_j(I^2) \cap C(I^2)$ *for some* $j = 1, 2$*. Then if* $r > \frac{2}{3}$*, then* $f \in A_r$*.*

This corollary follows from Corollary 10 if we take into account that $V_k(I^2) \subset V(I^2)$, $k = 1, 2$.

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