ON VARIATIONAL FORMULATION OF BITSADZE-SAMARSKII PROBLEM FOR SECOND ORDER TWO-DIMENSIONAL ELLIPTIC EQUATIONS

T. Jangveladze, G. Lobjanidze

Ilia Chavchavadze State University
Chavchavadze Av. 32, 0179, Tbilisi, Georgia
Ilia Vekua Institute of Applied Mathematics
Ivane Javakhishvili Tbilisi State University
0186 University Street 2, Tbilisi, Georgia
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Abstract

Bitsadze-Samarskii nonlocal boundary value problem for the second order two-dimensional elliptic equation is considered. The variational formulation of this problem is stated. The necessary and sufficient condition, indicating when the function minimizing the specially constructing parametrical functional is a solution of the considered problem, is given.

Key words and phrases: Second order two-dimensional elliptic equation, nonlocal boundary value problem, variational formulation, necessary and sufficient condition.

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1 Introduction

In the known work of A. Bitsadze and A. Samarskii [1], new mathematical problems with nonlocal boundary conditions are stated and studied. Numerous scientific papers deal with the investigation and numerical solution of problems considered in [1] and its modifications and generalizations (see, e.g. [2]-[10] and references therein).

In this article variational formulation of the Bitsadze-Samarskii nonlocal boundary value problem for the second order two-dimensional elliptic equation in the rectangle is considered. The problem has the following form [1]: find the function $u(x,y)\in C^{(2)}(G)\cap C(\overline{G})$ satisfying the conditions:

$$
\begin{gathered}
Au \equiv -\frac{\partial}{\partial x}\left[k(x)\frac{\partial u(x,y)}{\partial y}\right]-\frac{\partial}{\partial y}\left[p(y)\frac{\partial u(x,y)}{\partial y}\right]+ \\
+q(y)u(x,y)=f(x,y),\quad (x,y)\in G,
\end{gathered}
\tag{1}
$$

$$
u(x,y)\,|_{\Gamma}=0,\quad u(x,y)\,|_{\Gamma_{-\xi}}=u(x,y)\,|_{\Gamma_0}\,,
$$

where $G = \{(x, y) \mid -a < x < 0,\ 0 < y < b\}$ is the rectangle, a and b are the given positive constants, Γ_t is the intersection of the line $x = t$ with the set $\overline{G} = G \cup \partial G$ (∂G is a boundary of G), $\Gamma = \partial G \backslash \Gamma_0$. We assume that:

$$\begin{gathered} f(x,y) \in C(\overline{G}), \quad k(x) \in C^{(1)}\left[-a,\, 0\right], \quad k'(0) = 0, \\ 0 < k_0 \leq k(x) \leq K_0, \quad p(y) \in C^{(1)}\left[0,\, b\right], \quad 0 < p_0 \leq p(y), \\ q(y) \in C\left[0,\, b\right], \quad q(y) \geq 0. \end{gathered}$$

Let us denote by $D(\overline{G})$ the lineal of all real functions $v(x, y)$ satisfying the following conditions:

1. $v(x, y)$ is defined almost everywhere on $\overline{G}\backslash\Gamma_0$ and the boundary value $v(0, y)$ is defined almost everywhere on Γ_0;

2. $v(x, y) \in L_2(G),\ \ v(0, y) \in L_2(0, b)$.

We note that the definition of the function $v(x, y) \in D(\overline{G})$ means the definition of the pair $(v(x, y),\, v(0, y)),\ (x, y) \in \overline{G}\backslash\Gamma_0,\ y \in [0, b]$. Two functions $v_1(x, y)$ and $v_2(x, y)$ are assumed as the same element of $D(\overline{G})$ if $v_1(x, y) = v_2(x, y)$ almost everywhere on $\overline{G}\backslash\Gamma_0$ and $v_1(0, y) = v_2(0, y)$ almost everywhere on $[0, b]$.

Let us define on $D(\overline{G})$ the operator of symmetrical extension τ as follows

$$\tau v(x,y) = \begin{cases} v(x,y), & (x,y) \in \overline{G}, \\ -v(-x,y) + 2v(0,y), & (x,y) \in \overline{Q}, \end{cases}$$

where $Q = \{\, (x, y) \mid 0 < x < \xi,\ 0 < y < b\}$. Let us note that the operator τ associates to every function $v(x, y)$ of the lineal $D(\overline{G})$ the function $\widetilde{v}(x, y) = \tau v(x, y)$. This function is defined almost everywhere on $\overline{G} \cup \overline{Q}$ in such a way that the function $\widetilde{v}(x, y) - v(0, y)$ is the odd function with respect to the variable x almost everywhere on $[-\xi,\, \xi]$ for the almost all $y \in [0, b]$.

Let us also introduce on $D(\overline{G})$ the operator of even extension Λ as follows

$$\Lambda v(x,y = \begin{cases} v(x,y), & (x,y) \in \overline{G}, \\ v(-x,y), & (x,y) \in \overline{Q}. \end{cases}$$

It is clear that the function $\bar{v}(x, y) = \Lambda v(x, y)$ is the even function with respect to the variable x almost everywhere on $[-\xi,\, \xi]$ for the almost all $y \in [0, b]$.

For two arbitrary functions $v(x, y)$ and $w(x, y)$ from the lineal $D(\overline{G})$ we define the scalar product

$$[v, w] = \int_0^b \int_{-\xi}^{\xi} \frac{1}{\bar{k}(x)} \int_{-a}^{x} \widetilde{v}(s,y)\, \widetilde{w}(s,y) ds dx dy. \tag{2}$$

After the introduction of the scalar product (2) the lineal $D(\overline{G})$ becomes the pre-Hilbert space, which we denote by $H(\overline{G})$. The norm originated from the scalar product (2) in $H(\overline{G})$ we denote by $\|\cdot\|_H$:

$$\|v\|_H^2 = \int_0^b \int_{-\xi}^{\xi} \frac{1}{\bar{k}(x)} \int_{-a}^{x} \widetilde{v}^2(s,y)\, dsdxdy.$$

Theorem 1. *The norm $\|\cdot\|$ defined on the lineal $H(\overline{G})$ by the formula*

$$\|v\|^2 = \|v(x,y)\|^2_{L_2(G)} + \|v(0,y)\|^2_{L_2(0,b)}$$

is equivalent to the norm $\|\cdot\|_H$.

Proof. It is sufficient to note that

$$\frac{1}{K_0} \int_0^b \int_{-\xi}^{\xi} \int_{-a}^{x} \widetilde{v}^2(s,y)\, dsdxdy \le \|v\|_H^2 \le \frac{1}{k_0} \int_0^b \int_{-\xi}^{\xi} \int_{-a}^{x} \widetilde{v}^2(s,y) dsdxdy$$

and use the Theorem 1.1 from [8].

Consequence. *$H(\overline{G})$ is the Hilbert space.*

Let the area of definition of the operator A is the lineal $D_A(\overline{G})$ of the functions from the space $H(\overline{G})$, for the elements $v(x,y)$ of which the following conditions are fulfilled:

1. $v(x,y) \in C^{(2)}(\overline{G})$, $\quad \frac{\partial^2 v}{\partial x^2}(0,y) = 0, \quad \forall y \in [0,b]$;
2. $v(x,y)\big|_{\Gamma} = 0, \quad v(x,y)\big|_{\Gamma_{-\xi}} = v(x,y)\big|_{\Gamma_0}$.

Theorem 2. *The lineal $D_A(\overline{G})$ is dense in the space $H(\overline{G})$.*

The proof of the Theorem 2 is given in [8].

Hence, the operator A acts from the lineal $D_A(\overline{G})$ dense in the space $H(\overline{G})$ to the space $H(\overline{G})$.

Lemma 1. *For an arbitrary function $v(x,y)$ of the lineal $D_A(\overline{G})$ the following identities are valid:*

$$\frac{\partial}{\partial x}\left[\widetilde{k(x)\frac{\partial v(x,y)}{\partial x}}\right] = \frac{\partial}{\partial x}\left[\bar{k}(x)\frac{\partial \widetilde{v}(x,y)}{\partial x}\right], \tag{3}$$

$$\frac{\partial}{\partial y}\left[\widetilde{p(y)\frac{\partial v(x,y)}{\partial y}}\right] = \frac{\partial}{\partial y}\left[p(y)\frac{\partial \widetilde{v}(x,y)}{\partial y}\right], \tag{4}$$

$$\widetilde{q(y)v(x,y)} = q(y)\,\widetilde{v}(x,y). \tag{5}$$

Proof. For the case $(x,y) \in \overline{G}$ the identities (3)-(5) are trivial. Let us verify validity of the identities (3)-(5) for the case $(x,y) \in \overline{Q}$. We have:

$$\frac{\partial}{\partial x}\left[\widetilde{k(x)\frac{\partial v(x,y)}{\partial x}}\right] = -\frac{\partial}{\partial x}\left(k\frac{\partial v}{\partial x}\right)(-x,y) + 2\frac{\partial}{\partial x}\left(k\frac{\partial v}{\partial x}\right)(0,y) =$$

$$= -\frac{\partial}{\partial x}\left[k(-x)\frac{\partial v(-x,y)}{\partial x}\right] + 2k'(0)\frac{\partial v}{\partial x}(0,y) + 2k(0)\frac{\partial^2 v}{\partial x^2}(0,y) =$$

$$= \frac{\partial}{\partial x}\left\{k(-x)\frac{\partial[-v(-x,y)+2v(0,y)]}{\partial x}\right\} = \frac{\partial}{\partial x}\left[\bar{k}(x)\frac{\partial\,\widetilde{v}(x,y)}{\partial x}\right],$$

$$\frac{\partial}{\partial y}\left[\widetilde{p(y)\frac{\partial v(x,y)}{\partial y}}\right] = -\frac{\partial}{\partial y}\left(p\frac{\partial v}{\partial y}\right)(-x,y) + 2\frac{\partial}{\partial y}\left(p\frac{\partial v}{\partial y}\right)(0,y) =$$

$$= -\frac{\partial}{\partial y}\left[p(y)\frac{\partial v(-x,y)}{\partial y}\right] + 2\frac{\partial}{\partial y}\left(p\frac{\partial v}{\partial y}\right)(0,y) =$$

$$= -\frac{\partial}{\partial y}\left[p(y)\frac{\partial v(-x,y)}{\partial y}\right] + \frac{\partial}{\partial y}\left[2p(y)\frac{\partial v(0,y)}{\partial y}\right] =$$

$$= \frac{\partial}{\partial y}\left\{p(y)\frac{\partial}{\partial y}\left[-v(-x,y)+2v(0,y)\right]\right\} = \frac{\partial}{\partial y}\left[p(y)\frac{\partial\widetilde{v}(x,y)}{\partial y}\right],$$

$$\widetilde{q(y)v(x,y)} = -q(y)v(-x,y) + 2q(y)v(0,y) = q(y)\,\widetilde{v}(x,y).$$

Lemma 2. *For two arbitrary functions $v(x,y)$ and $w(x,y)$ of the lineal $D_A(\overline{G})$ we have*

$$\int\limits_{-\xi}^{\xi}\frac{\partial\,\widetilde{v}(x,y)}{\partial x}\widetilde{w}(x,y)dx = 0,\ \ \forall\, y\in[0,b]\,.$$

The proof of the Lemma 2 is also given in [8].

Lemma 3. *The operator A is symmetric on the lineal $D_A(\overline{G})$.*

Proof. We have:

$$[Av,w] = \int\limits_0^b\int\limits_{-\xi}^{\xi}\frac{1}{\bar{k}(x)}\int\limits_{-a}^{x}\widetilde{Av}(s,y)\widetilde{w}(s,y)dsdxdy =$$

$$= \int\limits_0^b\int\limits_{-\xi}^{\xi}\frac{1}{\bar{k}(x)}\int\limits_{-a}^{x}\left\{-\frac{\partial}{\partial s}\left[\bar{k}(s,y)\frac{\partial\widetilde{v}(s,y)}{\partial s}\right] - \frac{\partial}{\partial y}\left[p(y)\frac{\partial\widetilde{v}(s,y)}{\partial y}\right]+\right.$$

$$\left.+q(y)\widetilde{v}(s,y)\right\}\widetilde{w}(s,y)dsdxdy = \int\limits_0^b\int\limits_{-\xi}^{\xi}\frac{1}{\bar{k}(x)}\left[-\bar{k}(s,y)\frac{\partial\widetilde{v}(s,y)}{\partial s}\widetilde{w}(s,y)\,\Big|_{-a}^{x} +\right.$$

$$\left.+\int\limits_{-a}^{x}\bar{k}(s)\frac{\partial\widetilde{v}(s,y)}{\partial s}\frac{\partial\widetilde{w}(s,y)}{\partial s}ds\right]dxdy-$$

$$-\int_{-\xi}^{\xi}\frac{1}{\bar{k}(x)}\int_{-a}^{x}\int_{0}^{b}\frac{\partial}{\partial y}\left[p(y)\frac{\partial\widetilde{v}(s,y)}{\partial y}\right]\widetilde{w}(s,y)dydsdx+$$

$$+\int_{0}^{b}\int_{-\xi}^{\xi}\frac{1}{\bar{k}(x)}\int_{0}^{x}q(y)\widetilde{v}(s,y)\,\widetilde{w}(s,y)dsdxdy=$$

$$=\int_{0}^{b}\int_{-\xi}^{\xi}\frac{1}{\bar{k}(x)}\int_{0}^{x}\bar{k}(s)\frac{\partial\,\widetilde{v}(s,y)}{\partial s}\frac{\partial\widetilde{w}(s,y)}{\partial s}dsdxdy+$$

$$+\int_{0}^{b}\int_{-\xi}^{\xi}\frac{1}{\bar{k}(x)}\int_{0}^{x}p(y)\frac{\partial\widetilde{v}(s,y)}{\partial y}\frac{\partial\widetilde{w}(s,y)}{\partial y}dsdxdy+$$

$$+\int_{0}^{b}\int_{-\xi}^{\xi}\frac{1}{\bar{k}(x)}\int_{0}^{x}q(y)\widetilde{v}(s,y)\widetilde{w}(s,y)dsdxdy=$$

$$=\int_{0}^{b}\int_{-\xi}^{\xi}\frac{1}{\bar{k}(x)}\int_{0}^{x}\left[\bar{k}(s)\frac{\partial\,\widetilde{v}(s,y)}{\partial s}\frac{\partial\widetilde{w}(s,y)}{\partial s}+p(y)\frac{\partial\widetilde{v}(s,y)}{\partial y}\frac{\partial\widetilde{w}(s,y)}{\partial y}+\right.$$

$$\left.+q(y)\widetilde{v}(s,y)\widetilde{w}(s,y)\right]dsdxdy=[Aw,v]=[v,Aw]\,.$$

Theorem 3. *The operator A is positively defined on the lineal $D_A(\overline{G})$.*
Proof. We have

$$[Av,v]\geq\frac{k_0}{K_0}\int_{0}^{b}\int_{-\xi}^{\xi}\int_{0}^{x}\left(\frac{\partial\,\widetilde{v}(s,y)}{\partial s}\right)^2dsdxdy+$$
$$+\frac{p_0}{K_0}\int_{0}^{b}\int_{-\xi}^{\xi}\int_{0}^{x}\left(\frac{\partial\widetilde{v}(s,y)}{\partial y}\right)^2dsdxdy. \tag{6}$$

The following Poincare-Friedrichs type inequalities take place [8]:

$$\int_{0}^{b}\int_{-\xi}^{\xi}\int_{-a}^{x}\widetilde{v}^2(s,y)dsdxdy\leq\frac{(\xi+a)^2}{2}\int_{0}^{b}\int_{-\xi}^{\xi}\int_{-a}^{x}\left(\frac{\partial\widetilde{v}(s,y)}{\partial s}\right)^2dsdxdy\,, \tag{7}$$

$$\int_{0}^{b}\int_{-\xi}^{\xi}\int_{-a}^{x}\widetilde{v}^2(s,y)dsdxdy\leq\frac{b^2}{2}\int_{0}^{b}\int_{-\xi}^{\xi}\int_{-a}^{x}\left(\frac{\partial\widetilde{v}(s,y)}{\partial y}\right)^2dsdxdy. \tag{8}$$

Taking into account (7) and (8) from (6) we receive

$$[Av, v] \geq \left(\frac{k_0}{K_0} \cdot \frac{2}{(\xi + a)^2} + \frac{p_0}{K_0} \cdot \frac{2}{b^2} \right) [v, v] \, .$$

So, the proof of theorem 3 is over.

We have thereby obtained a standard situation: A is a positively defined operator on the lineal $D_A(\overline{G})$, which is dense in the Hilbert space $H(\overline{G})$. We follow the well-known scheme [11]. Let us consider the problem of minimization of following quadratic parametrical functional

$$\begin{gathered} J(v) = \int\limits_0^b \int\limits_{-\xi}^{\xi} \frac{1}{\bar{k}(x)} \int\limits_{-a}^{x} \left[\bar{k}(s) \left(\frac{\partial \widetilde{v}(s,y)}{\partial s} \right)^2 + p(y) \left(\frac{\partial \, \widetilde{v}(s,y)}{\partial y} \right)^2 + \right. \\ \left. + q(y) \widetilde{v}^{\,2}(s,y) - 2\widetilde{\Phi}(s,y) \, \widetilde{v}(s,y) \right] dsdxdy \, , \end{gathered} \tag{9}$$

where the function $\Phi(x,y) \in H(\overline{G})$ is defined by the following form

$$\Phi(x,y) = \begin{cases} f(x,y), & (x,y) \in \overline{G} \backslash \Gamma_0, \\ f_0(y), & (x,y) \in \Gamma_0. \end{cases}$$

For every function $f_0(y) \in L_2(0,b)$ there exists a unique function in the energetic space $H_A(\overline{G})$, which minimizes the quadratical functional $J(v)$. The space $H_A(\overline{G})$ consists of all elements of the Sobolev space $W_2^1(G)$ which satisfies boundary conditions of the problem (1).

After some transformations, functional (9) may be rewritten in the following form

$$\begin{gathered} J(v) = 2 \int\limits_{-\xi}^{0} \frac{dx}{k(x)} \int\limits_0^b \int\limits_{-a}^{0} \left[k(x) \left(\frac{\partial v(x,y)}{\partial x} \right)^2 + p(y) \left(\frac{\partial v(x,y)}{\partial y} \right)^2 + \right. \\ \left. + \, q(y) v^2(x,y) - 2 f(x,y) v(x,y) \right] dxdy - \\ -4 \int\limits_{-\xi}^{0} \frac{xdx}{k(x)} \int\limits_0^b \left[p(y) \left(\frac{\partial v(0,y)}{\partial y} \right)^2 + q(y) v^2(0,y) - 2 f_0(y) v(0,y) \right] dy - \\ -4 \int\limits_0^b \left[p(y) \frac{\partial v(0,y)}{\partial y} \int\limits_{-\xi}^{0} \frac{1}{k(x)} \int\limits_x^0 \frac{\partial v(s,y)}{\partial y} dsdx + \right. \\ + q(y) v(0,y) \int\limits_{-\xi}^{0} \frac{1}{k(x)} \int\limits_x^0 v(s,y) dsdx - \\ \left. - v(0,y) \int\limits_{-\xi}^{0} \frac{1}{k(x)} \int\limits_x^0 f(s,y) dsdx - f_0(y) \int\limits_{-\xi}^{0} \frac{1}{k(x)} \int\limits_x^0 v(s,y) dsdx \right] dy. \end{gathered} \tag{10}$$

Suppose that $u(x,y)$ is a solution of problem (1). Let us introduce the notation

$$\varphi_0(y) = -\frac{\partial}{\partial y}\left[p(y)\frac{\partial u(0,y)}{\partial y}\right] + q(y)u(0,y)\,.$$

Theorem 4. *The function v(x,y) which minimizes the functional (10), is a solution of problem (1) if and only if the following condition is fulfilled*

$$-\frac{\partial}{\partial y}\left[p(y)\frac{\partial v(0,y)}{\partial y}\right] + q(y)v(0,y) = f_0(y)\,. \tag{11}$$

Proof. At first let us proof sufficiency of the equality (11). Let $f_0(y)$ be such that minimization function $v(x,y)$ of the functional (10) satisfies condition (11). Let us show that function $v(x,y)$ is the solution of problem (1). The variation of a functional $J(v)$ for all functions $h(x,y) \in H_A(\overline{G})$ has the form:

$$\delta J(v) = 4\int\limits_{-\xi}^{0}\frac{dx}{k(x)}\int\limits_{0}^{b}\int\limits_{-a}^{0}\left[k(x)\frac{\partial v(x,y)}{\partial x}\frac{\partial h(x,y)}{\partial x} + p(y)\frac{\partial v(x,y)}{\partial y}\frac{\partial h(x,y)}{\partial y} + \right.$$

$$\left. + q(y)v(x,y)h(x,y) - f(x,y)h(x,y)\right]dxdy - 4\int\limits_{-\xi}^{0}\frac{xdx}{k(x)}\times$$

$$\times 2\int\limits_{0}^{b}\left[p(y)\frac{\partial v(0,y)}{\partial y}\frac{\partial h(0,y)}{\partial y} + q(y)v(0,y)h(0,y) - f_0(y)h(0,y)\right]dy -$$

$$\begin{aligned} -4\int\limits_{0}^{b}\Bigg[& p(y)\frac{\partial h(0,y)}{\partial y}\int\limits_{-\xi}^{0}\frac{1}{k(x)}\int\limits_{x}^{0}\frac{\partial v(s,y)}{\partial y}dsdx + \\ & + p(y)\frac{\partial v(0,y)}{\partial y}\int\limits_{-\xi}^{0}\frac{1}{k(x)}\int\limits_{x}^{0}\frac{\partial h(s,y)}{\partial y}dsdx + \\ & + q(y)h(0,y)\int\limits_{-\xi}^{0}\frac{1}{k(x)}\int\limits_{x}^{0}v(s,y)dsdx + \\ & + q(y)v(0,y)\int\limits_{-\xi}^{0}\frac{1}{k(x)}\int\limits_{x}^{0}h(s,y)dsdx - h(0,y)\int\limits_{-\xi}^{0}\frac{1}{k(x)}\int\limits_{x}^{0}f(s,y)dsdx - \\ & - f_0(y)\int\limits_{-\xi}^{0}\frac{1}{k(x)}\int\limits_{x}^{0}h(s,y)dsdx\Bigg]dy = 0\,. \end{aligned} \tag{12}$$

Let us take in (12) any function $h(x,y) \in H_A(\overline{G})$, which satisfies condition: $h(x,y) \equiv 0$ for $x \in [-\xi, 0]$. We get

$$\int\limits_0^b \int\limits_{-a}^{-\xi} \left[k(x) \frac{\partial v(x,y)}{\partial x} \frac{\partial h(x,y)}{\partial x} + p(y) \frac{\partial v(x,y)}{\partial y} \frac{\partial h(x,y)}{\partial y} + \right. \\ \left. + q(y) v(x,y) h(x,y) - f(x,y) h(x,y) \right] dxdy = 0\,. \tag{13}$$

From (13) it is clear that $v(x,y)$ is a weak solution [11] of the equation $Au = f(x,y)$ on the rectangle $]-a,-\xi] \times]0,b]$.

Analogously, if we take $h(x,y) \equiv 0$ for $x \in [-a,-\xi]$ and keeping in mind that $h(-\xi,y) = h(0,y) = 0$, , then we get that $v(x,y)$ is a solution of the equation $Au = f(x,y)$ on the rectangle $]-\xi,0] \times]0,b]$, too.

Now, let us show that if the condition (11) is fulfilled, then $v(x,y)$ is a solution of problem (1) (the equation is also fulfilled in points $(-\xi,y)$, $y \in]0,b]$). The restrictions of the function on rectangles $[-a,-\xi] \times [0,b]$ and $[-\xi,0] \times [0,b]$ we denote by $v_1(x,y)$ and $v_2(x,y)$, respectively. Keeping in mind that the functions $v_1(x,y)$ and $v_2(x,y)$ are solutions of the equation $Au = f(x,y)$ on the suitable rectangles, the variation of a functional $J(v)$ for all functions gives

$$4 \int\limits_{-\xi}^{0} \frac{dx}{k(x)} \left[\int\limits_0^b k(-\xi) \frac{\partial v_1}{\partial x}(-\xi,y) h(-\xi,y) + k(0) \frac{\partial v_2}{\partial x}(0,y) h(0,y) - \right. \\ \left. - k(-\xi) \frac{\partial v_2}{\partial x}(-\xi,y) h(-\xi,y) \right] dy - \\ -4 \int\limits_0^b \int\limits_{-\xi}^{0} \frac{1}{k(x)} \int\limits_x^0 \left\{ -\frac{\partial}{\partial y} \left[p(y) \frac{\partial v(s,y)}{\partial y} \right] + q(y) v(x,y) - \right. \\ \left. - f(s,y) \right\} h(0,y) dsdxdy - 4 \int\limits_0^b \int\limits_{-\xi}^{0} \frac{1}{k(x)} \int\limits_x^0 \left\{ -\frac{\partial}{\partial y} \left[p(y) \frac{\partial v(0,y)}{\partial y} \right] + \right. \\ \left. + q(y) v(0,y) - f_0(y) \right\} h(s,y) dsdxdy = \\ = 4 \int\limits_{-\xi}^{0} \frac{dx}{k(x)} \left[\int\limits_0^b k(-\xi) \frac{\partial v_1}{\partial x}(-\xi,y) h(-\xi,y) + k(0) \frac{\partial v_2}{\partial x}(0,y) h(0,y) - \right. \\ \left. - k(-\xi) \frac{\partial v_2}{\partial x}(-\xi,y) h(-\xi,y) \right] dy - \\ -4 \int\limits_0^b \int\limits_{-\xi}^{0} \frac{1}{k(x)} \int\limits_x^0 \frac{\partial}{\partial s} \left[k(s) \frac{\partial v(s,y)}{\partial s} \right] h(0,y) dsdxdy =$$

$$= 4\int\limits_{-\xi}^{0}\frac{dx}{k(x)}\int\limits_{0}^{b}\left\{\left[k(-\xi)\frac{\partial v_1}{\partial x}(-\xi,y)-k(-\xi)\frac{\partial v_2}{\partial x}(-\xi,y)\right]h(-\xi,y)+\right.$$

$$\left.+k(0)\frac{\partial v_2}{\partial x}(0,y)h(0,y)-k(0)\frac{\partial v}{\partial x}(0,y)h(0,y)\right\}dy=$$

$$= 4\int\limits_{-\xi}^{0}\frac{dx}{k(x)}\int\limits_{0}^{b}k(-\xi)\left[\frac{\partial v_1}{\partial x}(-\xi,y)-\frac{\partial v_2}{\partial x}(-\xi,y)\right]h(-\xi,y)dy=0\,.$$

From this we easily get

$$\frac{\partial v_1}{\partial x}(-\xi,y)=\frac{\partial v_2}{\partial x}(-\xi,y)\,,\ \ \forall y\in[0,b]\ . \tag{14}$$

Finally, from (14) we conclude that $v(x,y)$ is a solution of problem (1).

Now, let us show necessity of the condition (11). Let $f_0(y)$ be such that minimization function $v(x,y)$ of the functional (10) is the solution of the problem (1). We shall show that the condition (11) is fulfilled. Using formula of integrating by parts we get

$$\delta J(v)=4\int\limits_{-\xi}^{0}\frac{dx}{k(x)}\int\limits_{0}^{b}\left\{k(0)\frac{\partial v}{\partial x}(0,y)h(0,y)-\right.$$

$$\left.-\int\limits_{-a}^{0}\frac{\partial}{\partial x}\left[k(x)\frac{\partial v(x,y)}{\partial x}\right]h(x,y)dx\right\}dy+$$

$$+4\int\limits_{-\xi}^{0}\frac{dx}{k(x)}\int\limits_{0}^{b}\int\limits_{-a}^{0}\left\{-\frac{\partial}{\partial y}\left[p(y)\frac{\partial v(x,y)}{\partial y}\right]+q(y)v(x,y)-\right.$$

$$\left.-f(x,y)\right\}h(x,y)dxdy-4\int\limits_{-\xi}^{0}\frac{xdx}{k(x)}\cdot 2\int\limits_{0}^{b}\left\{-\frac{\partial}{\partial y}\left[p(y)\frac{\partial v(0,y)}{\partial y}\right]+\right.$$

$$\left.+q(y)v(0,y)-f_0(y)\right\}h(0,y)dy-$$

$$-4\int\limits_{0}^{b}\int\limits_{-\xi}^{0}\frac{1}{k(x)}\int\limits_{x}^{0}\left\{-\frac{\partial}{\partial y}\left[p(y)\frac{\partial v(s,y)}{\partial y}\right]+q(y)v(s,y)\ -\right.$$

$$\left.-f(s,y)\right\}h(0,y)dsdxdy-4\int\limits_{0}^{b}\int\limits_{-\xi}^{0}\frac{1}{k(x)}\int\limits_{x}^{0}\left\{-\frac{\partial}{\partial y}\left[p(y)\frac{\partial v(0,y)}{\partial y}\right]+\right.$$

$$\left.+q(y)v(0,y)-f_0(y)\right\}h(s,y)dsdxdy=$$

$$=4\int\limits_{-\xi}^{0}\frac{dx}{k(x)}\int\limits_{0}^{b}k(0)\frac{\partial v}{\partial x}(0,y)h(0,y)dy-$$

$$
\begin{gathered}
-4\int\limits_{-\xi}^{0}\frac{xdx}{k(x)}\cdot 2\int\limits_{0}^{b}\left\{-\frac{\partial}{\partial y}\left[p(y)\frac{\partial v(0,y)}{\partial y}\right]+\right. \\
\left.+q(y)v(0,y)-f_0(y)\right\}h(0,y)dy- \qquad (15)\\
-4\int\limits_{0}^{b}\int\limits_{-\xi}^{0}\frac{1}{k(x)}\int\limits_{x}^{0}\frac{\partial}{\partial s}\left[k(s)\frac{\partial v(s,y)}{\partial s}\right]h(0,y)dsdxdy- \\
-4\int\limits_{0}^{b}\int\limits_{-\xi}^{0}\frac{1}{k(x)}\int\limits_{x}^{0}\left\{-\frac{\partial}{\partial y}\left[p(y)\frac{\partial v(0,y)}{\partial y}\right]+q(y)v(0,y)-f_0(y)\right\}\times \\
\times h(s,y)dsdxdy=-8\int\limits_{-\xi}^{0}\frac{xdx}{k(x)}\int\limits_{0}^{b}\left\{-\frac{\partial}{\partial y}\left[p(y)\frac{\partial v(0,y)}{\partial y}\right]+\right. \\
\left.+q(y)v(0,y)-f_0(y)\right\}h(0,y)dy-4\int\limits_{0}^{b}\int\limits_{-\xi}^{0}\frac{1}{k(x)}\int\limits_{x}^{0}\left\{-\frac{\partial}{\partial y}\left[p(y)\frac{\partial v(0,y)}{\partial y}\right]+\right. \\
\left.+q(y)v(0,y)-f_0(y)\right\}h(s,y)dsdxdy=0.
\end{gathered}
$$

The equality (15) is true for all $h(s,y)$, thereby we conclude that (11) is fulfilled. For this reason it is sufficient to take the following function

$$
h(s,y)=s(s+\xi)\left\{-\frac{\partial}{\partial y}\left[p(y)\frac{\partial v(0,y)}{\partial y}\right]+q(y)v(0,y)-f_0(y)\right\}.
$$

Remark. It is interesting to consider the question about existence of such function $f_0(y)$ for which corresponding minimization function $v(x,y)$ of the functional (10) is the solution of problem (1). The answer is positive. Especially, taking $f_0(y)=-\frac{\partial}{\partial y}\left[p(y)\frac{\partial u(0,y)}{\partial y}\right]+q(y)u(0,y)$, where $u(x,y)$ is the solution of problem (1) and computing variation of the functional J for this $f_0(y)$, we obtain $\delta J(u)=0$. So, the function $u(x,y)$ is the minimization function. Problem (1) has an unique solution. Thereby the function $f_0(y)$ is unique too.

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