THE SOLUTION OF ONE PROBLEM BY METHOD OF I. VEKUA FOR APPROXIMATION $N = 2$

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Abstract

In the paper the problem of bending for isotropic plate with constant thickness 2h is considered. Problems of bending for infinite plates with circular hole, when there is put elastic body in case of approximation N=2 of I. Vekua's theory is solved. We consider the case when the body is soldered. Obtained results compared to the corresponding results obtained by plane classical bending theory.

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In the present paper the problem of bending for isotropic plate with constant thickness $2h$ is considered. Problems of bending for infinite plates with circular hole, when there is put elastic body in case of approximation $N = 2$ of I. Vekua's theory is solved. We consider the case when the body is soldered. Obtained results compared to the corresponding results obtained by plane classical bending theory.

As well-known system of bending equations in the components of the displacement vector will be written as following

$$\begin{cases} (\lambda+2\mu)\Delta \overset{(1)}{\theta} + \frac{3\lambda}{h}\Delta \overset{(2)}{u}_3 - \frac{3\mu}{h}\left(\Delta \overset{(0)}{u}_3 + \frac{1}{h}\overset{(1)}{\theta}\right) = 0, \\ \Delta \overset{(0)}{u}_3 + \frac{1}{h}\overset{(1)}{\theta} = 0, \\ \mu\Delta \overset{(2)}{u}_3 - \frac{5\lambda}{h}\overset{(1)}{\theta} - \frac{15(\lambda+2\mu)}{h^2}\overset{(2)}{u}_3 = 0. \end{cases} \tag{1}$$

The general solutions of system (1) will be represented by means of two analytic functions $\varphi(z)$, $\psi(z)$ of complex variable and two solutions of Helmholtz's equations $\chi_1(z,\bar{z}) - \eta^2\chi_1(z,\bar{z}) = 0$, $\chi_2(z,\bar{z}) - \gamma^2\chi_2(z,\bar{z}) = 0$

$$\overset{(0)}{u}_3 = \frac{\nu}{10}\chi_1(z,\bar{z}) - \frac{1-\nu}{2Eh}(z\overline{\varphi(z)} + \bar{z}\varphi(z)) + \psi(z) + \overline{\psi(z)},$$
$$\overset{(2)}{u}_3 = \chi_1(z,\bar{z}) - \frac{2\nu h}{3E}(\varphi'(z) + \overline{\varphi'(z)}),$$

$$
\begin{aligned}
&\overset{(1)}{u}{}_{+} = i\partial_{\bar z}\chi_2(z,\bar z) - \tfrac{\nu h}{5}\partial_{\bar z}\chi_1(z,\bar z) + \tfrac{8h^2}{3E}\overline{\varphi''(z)}\\
&+\tfrac{1-\nu}{E}(z\overline{\varphi'(z)} + \varphi(z)) - 2h\overline{\psi(z)},\\
&\overset{(1)}{\theta} = -\tfrac{3\nu}{h(1-\nu)}\chi_1(z,\bar z) + \tfrac{2(1-\nu)}{E}\left(\varphi'(z) + \overline{\varphi'(z)}\right),
\end{aligned}
$$

for the complex combination of stresses we get following expression

$$
\begin{aligned}
&\overset{(1)}{\sigma}{}_{11} + \overset{(1)}{\sigma}{}_{22} = 2\left(\varphi'(z) + \overline{\varphi'(z)}\right) + \tfrac{3E\nu}{h(1-\nu^2)}\chi_1(z,\bar z),\\
&\overset{(1)}{\sigma}{}_{11} - \overset{(1)}{\sigma}{}_{22} + 2i\overset{(1)}{\sigma}{}_{12} = 4\mu\left(i\partial^2_{\bar z\bar z}\chi_2(z,\bar z) - \tfrac{\nu h}{5}\partial^2_{\bar z\bar z}\chi_1(z,\bar z) +\right.\\
&\left.+\tfrac{1-\nu}{E}z\overline{\varphi''(z)} - 2h\overline{\psi''(z)} + \tfrac{8h^2}{3E}\overline{\varphi'''(z)}\right),\\
&\overset{(0)}{\sigma}{}_{13} + i\overset{(0)}{\sigma}{}_{23} = \mu\left(\tfrac{i}{h}\partial_{\bar z}\chi_2(z,\bar z) + \tfrac{8h}{3E}\overline{\varphi''(z)}\right),\\
&\overset{(2)}{\sigma}{}_{13} + i\overset{(2)}{\sigma}{}_{23} = 2\mu\left(\partial_{\bar z}\chi_1(z,\bar z) - \tfrac{2\nu h}{3E}\overline{\varphi''(z)}\right).
\end{aligned}
$$

Let us consider the infinite plate with circular hole with radius R, when there put the same radius the elastic washer. The washer is soldered to the contour of the plate. The stresses are limited at the infinity $\overset{(0)}{\sigma}{}^{\infty}_{11} = M_1$, $\overset{(0)}{\sigma}{}^{\infty}_{22} = M_2$, $\overset{(2)}{\sigma}{}^{\infty}_{ij} = 0$. Denote by index ,,0" the elastic constants of the washer, also all the elements are involving the washer.

We have following boundary conditions

$$
\begin{cases}
\overset{(1)}{\sigma}{}_{rr} = \overset{(1)}{\sigma}{}^{0}_{rr},\\
\overset{(1)}{\sigma}{}_{r\theta} = \overset{(1)}{\sigma}{}^{0}_{r\theta},\\
\overset{(1)}{u}{}_{r} = \overset{(1)}{u}{}^{0}_{r},\\
\overset{(1)}{u}{}_{\theta} = \overset{(1)}{u}{}^{0}_{\theta},\\
\overset{(0)}{u}{}_{3} = \overset{(0)}{u}{}^{0}_{3},\\
\overset{(2)}{u}{}_{3} = \overset{(2)}{u}{}^{0}_{3}.
\end{cases}
$$

The boundary conditions will be written as following

$$
\begin{aligned}
&\varphi'(z) + \overline{\varphi'(z)} + \tfrac{3E\nu}{2h(1-\nu^2)}\chi_1(z,\bar z) + \mu\left[i\partial^2_{\bar z\bar z}\chi_2(z,\bar z) - \tfrac{\nu h}{5}\partial^2_{\bar z\bar z}\chi_1(z,\bar z)\right.\\
&\left.- + \tfrac{1-\nu}{E}z\overline{\varphi''(z)} - 2h\overline{\psi''(z)} + \tfrac{8h^2}{3E}\overline{\varphi'''(z)}\right]e^{-2i\theta}\\
&+\mu\left[-i\partial^2_{zz}\chi_2(z,\bar z) - \tfrac{\nu h}{5}\partial^2_{zz}\chi_1(z,\bar z) + \tfrac{1-\nu}{E}\bar z\varphi''(z) -\right.\\
&\left.-2h\psi''(z) + \tfrac{8h^2}{3E}\varphi'''(z)\right]e^{2i\theta} = \varphi^{0\prime}(z) + \overline{\varphi^{0\prime}(z)} + \tfrac{3E^0\nu^0}{2h(1-\nu^{0 2})}\chi^0_1(z,\bar z) +\\
&+\mu^0\left[i\partial^2_{\bar z\bar z}\chi^0_2(z,\bar z) - \tfrac{\nu^0 h}{5}\partial^2_{\bar z\bar z}\chi^0_1(z,\bar z) + \tfrac{1-\nu^0}{E^0}z\overline{\varphi^{0\prime\prime}(z)} - 2h\overline{\psi^{0\prime\prime}(z)}\right.\\
&\left.+\tfrac{8h^2}{3E^0}\overline{\varphi^{0\prime\prime\prime}(z)}\right]e^{-2i\theta} + +\mu^0\left[-i\partial^2_{zz}\chi^0_2(z,\bar z) - \tfrac{\nu^0 h}{5}\partial^2_{zz}\chi^0_1(z,\bar z)\right.\\
&\left.+\tfrac{1-\nu^0}{E^0}\bar z\varphi^{0\prime\prime}(z) - 2h\psi^{0\prime\prime}(z) + \tfrac{8h^2}{3E^0}\varphi^{0\prime\prime\prime}(z)\right]e^{2i\theta}
\end{aligned}
$$

$$
\begin{aligned}
&\Big[i\partial^2_{\bar z\bar z}\chi_2(z,\bar z)-\tfrac{\nu h}{5}\partial^2_{\bar z\bar z}\chi_1(z,\bar z)+\tfrac{1-\nu}{E}z\overline{\varphi''(z)}\\
&-2h\overline{\psi''(z)}+\tfrac{8h^2}{3E}\overline{\varphi'''(z)}\Big]e^{-2i\theta}-\\
&-\Big[-i\partial^2_{zz}\chi_2(z,\bar z)-\tfrac{\nu h}{5}\partial^2_{zz}\chi_1(z,\bar z)+\tfrac{1-\nu}{E}\bar z\varphi''(z)\\
&-2h\psi''(z)+\tfrac{8h^2}{3E}\varphi'''(z)\Big]e^{2i\theta}=\\
&\Big[i\partial^2_{\bar z\bar z}\chi^0_2(z,\bar z)-\tfrac{\nu^0 h}{5}\partial^2_{\bar z\bar z}\chi^0_1(z,\bar z)+\tfrac{1-\nu^0}{E^0}z\overline{\varphi^{0\prime\prime}(z)}\\
&-2h\overline{\psi^{0\prime\prime}(z)}+\tfrac{8h^2}{3E^0}\overline{\varphi^{0\prime\prime\prime}(z)}\Big]e^{-2i\theta}-\\
&-\Big[-i\partial^2_{zz}\chi^0_2(z,\bar z)-\tfrac{\nu^0 h}{5}\partial^2_{zz}\chi^0_1(z,\bar z)+\tfrac{1-\nu^0}{E^0}\bar z\varphi^{0\prime\prime}(z)\\
&-2h\psi^{0\prime\prime}(z)+\tfrac{8h^2}{3E^0}\varphi^{0\prime\prime\prime}(z)\Big]e^{2i\theta}
\end{aligned}
$$

$$
\begin{aligned}
&\Big[i\partial_{\bar z}\chi_2(z,\bar z)-\tfrac{\nu h}{5}\partial_{\bar z}\chi_1(z,\bar z)+\tfrac{8h^2}{3E}\overline{\varphi''(z)}\\
&+\tfrac{1-\nu}{E}(z\overline{\varphi'(z)}+\varphi(z))-2h\overline{\psi(z)}\Big]e^{-i\theta}+\\
&+\Big[-i\partial_z\chi_2(z,\bar z)-\tfrac{\nu h}{5}\partial_z\chi_1(z,\bar z)+\tfrac{8h^2}{3E}\varphi''(z)+\\
&\tfrac{1-\nu}{E}(\bar z\varphi'(z)+\overline{\varphi(z)})-2h\psi(z)\Big]e^{i\theta}=\\
&=\Big[i\partial_{\bar z}\chi^0_2(z,\bar z)-\tfrac{\nu^0 h}{5}\partial_{\bar z}\chi^0_1(z,\bar z)+\tfrac{8h^2}{3E^0}\overline{\varphi^{0\prime\prime}(z)}\\
&+\tfrac{1-\nu^0}{E}(z\overline{\varphi^{0\prime}(z)}+\varphi^0(z))-2h\overline{\psi^0(z)}\Big]e^{-i\theta}+\\
&+\Big[-i\partial_z\chi^0_2(z,\bar z)-\tfrac{\nu^0 h}{5}\partial_z\chi^0_1(z,\bar z)+\tfrac{8h^2}{3E^0}\varphi^{0\prime\prime}(z)\\
&+\tfrac{1-\nu^0}{E^0}(\bar z\varphi^{0\prime}(z)+\overline{\varphi^0(z)})-2h\psi^0(z)\Big]e^{i\theta}
\end{aligned}
$$

$$
\begin{aligned}
&\Big[i\partial_{\bar z}\chi_2(z,\bar z)-\tfrac{\nu h}{5}\partial_{\bar z}\chi_1(z,\bar z)+\tfrac{8h^2}{3E}\overline{\varphi''(z)}\\
&+\tfrac{1-\nu}{E}(z\overline{\varphi'(z)}+\varphi(z))-2h\overline{\psi(z)}\Big]e^{-i\theta}-\\
&-\Big[-i\partial_z\chi_2(z,\bar z)-\tfrac{\nu h}{5}\partial_z\chi_1(z,\bar z)+\tfrac{8h^2}{3E}\varphi''(z)\\
&+\tfrac{1-\nu}{E}(\bar z\varphi'(z)+\overline{\varphi(z)})-2h\psi(z)\Big]e^{i\theta}=\\
&=\Big[i\partial_{\bar z}\chi^0_2(z,\bar z)-\tfrac{\nu^0 h}{5}\partial_{\bar z}\chi^0_1(z,\bar z)+\tfrac{8h^2}{3E^0}\overline{\varphi^{0\prime\prime}(z)}\\
&+\tfrac{1-\nu^0}{E^0}(z\overline{\varphi^{0\prime}(z)}+\varphi^0(z))-2h\overline{\psi^0(z)}\Big]e^{-i\theta}-\\
&-\Big[-i\partial_z\chi^0_2(z,\bar z)-\tfrac{\nu^0 h}{5}\partial_z\chi^0_1(z,\bar z)+\tfrac{8h^2}{3E^0}\varphi^{0\prime\prime}(z)\\
&+\tfrac{1-\nu^0}{E^0}(\bar z\varphi^{0\prime}(z)+\overline{\varphi^0(z)})-2h\psi^0(z)\Big]e^{i\theta}
\end{aligned}
$$

$$
\begin{aligned}
&\tfrac{\nu}{10}\chi_1(z,\bar z)-\tfrac{1-\nu}{2Eh}(z\overline{\varphi(z)}+\bar z\varphi(z))+\psi(z)+\overline{\psi(z)}=\\
&=\tfrac{\nu^0}{10}\chi^0_1(z,\bar z)-\tfrac{1-\nu^0}{2E^0h}(z\overline{\varphi^0(z)}+\bar z\varphi^0(z))+\psi^0(z)+\overline{\psi^0(z)}\\
&\chi_1(z,\bar z)-\tfrac{2\nu h}{3E}(\varphi'(z)+\overline{\varphi'(z)})=\chi^0_1(z,\bar z)-\tfrac{2\nu^0 h}{3E^0}(\varphi^{0\prime}(z)+\overline{\varphi^{0\prime}(z)})
\end{aligned}
$$

Let's introduce the functions $\varphi(z),\psi(z),\chi_1(z,\bar z),\chi_2(z,\bar z)$ and $\varphi^0(z),\psi^0(z)$

$\chi_1^0(z,\bar{z})$, $\chi_2^0(z,\bar{z})$ by the series

$$\begin{aligned}
&\varphi^{'}(z)=\sum_{n=0}^{\infty}\frac{a_n}{z^n},\quad \psi^{''}(z)=\sum_{n=0}^{\infty}\frac{a_n^{'}}{z^n},\\
&\chi_1(z,\bar{z})=\sum_{-\infty}^{+\infty}K_n(\eta r)b_ne^{in\theta},\quad \chi_2(z,\bar{z})=\sum_{-\infty}^{+\infty}K_n(\gamma r)c_ne^{in\theta},\\
&\varphi^{0'}(z)=\sum_{n=0}^{\infty}A_nz^n,\quad \psi^{0''}(z)=\sum_{n=0}^{\infty}A_n^{'}z^n,\\
&\chi_1^0(z,\bar{z})=\sum_{-\infty}^{+\infty}I_n(\eta r)B_ne^{in\theta},\quad \chi_2^0(z,\bar{z})=\sum_{-\infty}^{+\infty}I_n(\gamma r)C_n^{'}e^{in\theta}
\end{aligned}\tag{2}$$

By substituting (2) into the boundary conditions and take into account simple conditions of displacement vector we determine all the coefficients and the functions will have the form

$$\begin{aligned}
&\varphi^{'}(z)=a_0+\tfrac{a_2}{z^2},\quad \psi^{''}(z)=a_0^{'}+\tfrac{a_2^{'}}{z^2}+\tfrac{a_4^{'}}{z^4},\\
&\chi_1(z,\bar{z})=b_0K_0(\eta r)+2K_2(\eta r)b_2cos2\theta,\ \ \chi_2(z,\bar{z})=2iK_2(\gamma r)c_2sin2\theta;\\
&\varphi^{0'}(z)=A_0+A_2z^2,\quad \psi^{0''}(z)=A_0^{0'}+A_2^{0'}z^2+A_4^{0'}z^4,\\
&\chi_1^0(z,\bar{z})=B_0I_0(\eta r)+2I_2(\eta r)B_2cos2\theta,\ \ \chi_2^0(z,\bar{z})=2iI_2(\gamma r)C_2^{'0}sin2\theta;
\end{aligned}$$

The components of stress tensor have the form

$$\begin{aligned}
&\overset{(1)}{\sigma}_{rr}=2a_0-\tfrac{4hE}{(1+\nu)R^2}a_2^{'}+\tfrac{3E\nu}{2h(1-\nu^2)}(K_0(\eta r)-2K_2(\eta r))b_0+\tfrac{a_2}{R^2}-\\
&-\Big[\Big(\tfrac{4(1-\nu)}{(1+\nu)R^2}-\tfrac{32h^2}{(1+\nu)R^2}\Big)a_2+\tfrac{4hE}{(1+\nu)R^4}a_4^{'}-\\
&-\tfrac{3E\nu}{h(1-\nu^2)}(K_2(\eta r)-K_4(\eta r)-K_0(\eta r))b_2-i\tfrac{3E}{2h^2(1-\nu)}(K_4(\gamma r)-K_0(\gamma r))c_2\Big]\,cos2\theta;
\end{aligned}$$

$$\begin{aligned}
&\overset{(1)}{\sigma}{}^{0}_{rr}=2A_0+\tfrac{3E^0\nu^0}{2h(1-\nu^{02})}(I_0(\eta r)-2I_2(\eta r))B_0+\\
&+\Big[2A_2R^2+\Big(\tfrac{2(1-\nu^0)R^2}{1+\nu^0}+\tfrac{32h}{3(1+\nu^0)}\Big)A_2-\tfrac{4hE^0}{(1+\nu^0)}A_0^{'}+\\
&+\tfrac{3E^0\nu^0}{h(1-\nu^{02})}(I_2(\eta r)-I_4(\eta r)-I_0(\eta r))B_2+i\tfrac{3E^0}{2h^2(1-\nu^0)}(I_4(\gamma r)-I_0(\gamma r))C_2\Big]\,cos2\theta;
\end{aligned}$$

$$\begin{aligned}
&\overset{(1)}{\sigma}_{r\theta}=-\tfrac{2h}{R^2}a_2^{'}+\Big[\tfrac{32h^2}{ER^4}a_2-\tfrac{2(1-\nu)}{ER^2}a_2-\tfrac{4h}{R^4}a_4^{'}-\\
&-\tfrac{\eta^2h\nu}{10}(K_4(\eta r)-K_0(\eta r))b_2+i\tfrac{3}{2h^2}(K_4(\gamma r)+K_4(\gamma r)C_2\Big]\,sin2\theta;\\
&\overset{(1)}{\sigma}{}^{0}_{r\theta}=\Big[\tfrac{32h^2}{E^0}A_2-\tfrac{2(1-\nu^0)}{E^0}A_2R^2-4hA_0^{'}-\\
&-\tfrac{\eta^{02}h\nu}{10}(I_4(\eta r)-I_0(\eta r))B_2-i\tfrac{3}{2h^2}(I_4(\gamma r)+I_4(\gamma r)C_2\Big]\,sin2\theta;
\end{aligned}$$

where

$$a_0=\frac{1-2\nu}{4}(M_1+M_2),\quad a_0^{'}=-\frac{M_1-M_2}{4h},$$

$$A_0 = -\frac{12E^0h(1-\nu^{02})}{6R^2lnR[E^0\nu^0(1-\nu^2)(I_0-2I_2)K_0-3E\nu(1-\nu^{02})}$$
$$\frac{1}{(K_0-2K_2)I_0]-2K_0h^2E(1-\nu^{02})(\nu^0I_0-\nu I_0)}$$
$$\times\frac{1}{2k_0(2\nu\nu^0h^2-15(1-\nu^0))+(4\nu\nu^0h^2\eta k_1+60RK_0(1-\nu^0))RlnR}\times$$
$$\times\left[\frac{30R^2lnR[K_0(1-\nu^2)-2\nu^2(K_0-2K_2)]}{15}\right.$$
$$-\frac{-2K_0(1-\nu)[15(1-\nu)E+2\nu^2h^2]}{15}$$
$$-\left[\frac{6R^2lnR[E^0\nu^0(1-\nu^2)(I_0-2I_2)K_0-3E\nu(1-\nu^{02})(K_0-2K_2)I_0]}{4h(1-\nu^0)}\right.$$
$$\left.-\frac{230h^2E(1-\nu)(1-\nu^{02})K_0}{4h(1-\nu^0)}\right]\times$$
$$\left.\times\frac{RlnR(6RK_0(1-\nu)-4\nu^2h^2\eta K_1)+2(15(1-\nu)E+2\nu^2h^2)}{3E[K_0h(\nu^0I_0-\nu I_0)+RlnR(\nu^0\eta hI_1K_0-\nu h\eta K_1I_0)]}\right]a_0$$

$$B_0 = \frac{RlnR(6RK_0(1-\nu)-4\nu^2h^2\eta k_1)+2(15(1-\nu)E+2\nu^2h^2)}{3E[K_0h(\nu^0I_0-\nu I_0)+RlnR(\nu^0\eta hI_1K_0-\nu h\eta K_1I_0)]}a_0$$
$$-\frac{2K_0(2\nu\nu^0h^2-15(1-\nu^0))+(4\nu\nu^02h^2\eta k_1+60RK_0(1-\nu^0))RlnR}{3E[K_0h(\nu^0I_0-\nu I_0)+RlnR(\nu^0\eta hI_1K_0-\nu h\eta K_1I_0)]}A_0$$

$$a_2^{'} = \frac{15(1-\nu)E+2\nu^2h^2}{30EhlnR}a_0-\frac{2\nu\nu^0h^2-15(1-\nu^0)R^2}{30E^0hlnR}A_0-\frac{\nu^0I_0-\nu I_0}{20lnR}B_0$$

$$b_0 = \frac{4\nu h}{3EK_0}a_0-\frac{4\nu^0h}{3E^0K_0}A_0+\frac{I_0}{K_0}B_0$$

The components of stress tensor and displacement vector obtained by classical bending theory depend only radius, while the corresponding components obtained by theory of I. Vekua depended on quantity $\frac{R}{h}$.

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