ON THE SOLVABILITY OF ONE BOUNDARY VALUE PROBLEM GEOMETRICALLY NONLINEAR THEORY OF PLATES

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Abstract

In the present paper a geometrical non-linear plates is considered. One problem of stretch-press of plate is solved by the method of I. Vekua. For solving this problem is used the small parameter method and complex variable functions theory.

Key words and phrases: Plate, Small parameter, Complex variable, Stretch-press.

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In the present paper a boundary value problem of plates is considered. This three-dimensional problem is reduced to two-dimensional problem by I. Vekua's method [1],[2]. Then we consider the case of geometrically non-linear plate for approximation $N = 0$. Our aim is to solve concrete problem using these theories.

The three-dimensional equilibrium equation has following form:

$$-\partial_j(\sigma_{ij} + \sigma_{kj}\partial_k u_i) = f_i, \tag{1}$$

where σ_{ij} - are the components of the stress tensor, $\mathbf{u} = (u_1, u_2, u_3)$ - is the vector of displacement, f_i - is the given density per unit volume of the applied body forces.

Under repeating indexes we mean summation, the Latin letters taking the values 1, 2, 3 and the Greek one - 1, 2.

The tensors stresses and strains are related as follows

$$\sigma_{ij} = \lambda E_{pp}(\mathbf{u})\delta_{ij} + 2\mu E_{ij}(\mathbf{u}),$$

where

$$E_{ij}(\mathbf{u}) = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i + \partial_i \mathbf{u}\partial_j \mathbf{u}\right),$$

$\lambda,\ \mu$ - are the Lame's constants, δ_{ij} - is Kroneker symbol.

The equilibrium equation (1) may be written as follows

$$-(\partial_\alpha T_{i\alpha} + \partial_3 T_{i3}) = f_i,$$

where components T_{ij} are connected with σ_{ij} by the following form

$$T_{ij} = \sigma_{kj}(\delta_{ik} + \partial_k u_i) = \sigma_{ij} + \sigma_{kj}\partial_k u_i.$$

The three-dimensional system will be reduced to two-dimensional one by I. Vekua's method on the midsurface of the plate [1], [2].

This method in case of geometrically and physically nonlinear theory of plates and shells was studied by T. Meunargia [3].

The obtained system for approximate $N = 0$ we rewrite in complex form and use method of Signorini [4]. We assume that volume forces and components of stress tensor are analytical functions of small parameter ε. Therefore, we can find solution in the form of the asymptotic series as follows

$$\mathbf{u}(x_1, x_2) = \sum_{n=1}^{\infty} \overset{(n)}{\boldsymbol{u}} \varepsilon^n.$$

$$T_{ij}(x_1, x_2) = \sum_{n=1}^{\infty} \overset{(n)}{T}_{ij} \varepsilon^n.$$

The system of equilibrium equations has the form

$$\begin{cases} \frac{\partial}{\partial z}\left[\overset{(n)}{T}_{11} - \overset{(n)}{T}_{22} + i(\overset{(n)}{T}_{12} + \overset{(n)}{T}_{21})\right] \\ +\frac{\partial}{\partial \bar{z}}\left[\overset{(n)}{T}_{11} + \overset{(n)}{T}_{22} + i(\overset{(n)}{T}_{21} - \overset{(n)}{T}_{12})\right] + \overset{(n)}{F}_{+} = 0, \\ \frac{\partial \overset{(n)}{T}_{3+}}{\partial z} + \frac{\partial \overline{\overset{(n)}{T}_{3+}}}{\partial \bar{z}} + \overset{(n)}{F}_{3} = 0, \end{cases}$$

where $z = z_1 + iz_2$ is a complex variable

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}\right),$$
$$\overset{(n)}{F}_{+} = \overset{(n)}{F}_{1} + i\,\overset{(n)}{F}_{2}, \quad \overset{(n)}{T}_{3+} = \overset{(n)}{T}_{31} + i\,\overset{(n)}{T}_{32}.$$

Complex combinations of $\overset{(n)}{T}_{ij}$ and $\overset{(n)}{\sigma}_{ij}$ are related as follows

$$\overset{(n)}{T}_{11} - \overset{(n)}{T}_{22} + i\left(\overset{(n)}{T}_{12} + \overset{(n)}{T}_{21}\right) = \left(\overset{(n)}{\sigma}_{11} - \overset{(n)}{\sigma}_{22} + 2i\,\overset{(n)}{\sigma}_{12}\right) +$$
$$+\sum_{k=1}^{n-1}\left[\left(\overset{(k)}{\sigma}_{11} - \overset{(k)}{\sigma}_{22} + 2i\,\overset{(k)}{\sigma}_{12}\right)\frac{\partial \overset{(n-k)}{u}_{+}}{\partial z} + \left(\overset{(k)}{\sigma}_{11} + \overset{(k)}{\sigma}_{22}\right)\frac{\partial \overset{(n-k)}{u}_{+}}{\partial \bar{z}}\right],$$

$$\overset{(n)}{T}_{11} + \overset{(n)}{T}_{22} + i\left(\overset{(n)}{T}_{21} - \overset{(n)}{T}_{12}\right) = \overset{(n)}{\sigma}_{11} + \overset{(n)}{\sigma}_{22} +$$
$$+\sum_{k=1}^{n-1}\left[\left(\overset{(k)}{\sigma}_{11} + \overset{(k)}{\sigma}_{22}\right)\frac{\partial \overset{(n-k)}{u}_{+}}{\partial z} + \left(\overset{(k)}{\sigma}_{11} - \overset{(k)}{\sigma}_{22} - 2i\,\overset{(k)}{\sigma}_{12}\right)\frac{\partial \overset{(n-k)}{u}_{+}}{\partial \bar{z}}\right],$$

$$
\begin{aligned}
&\overset{(n)}{T}_{3+} = \overset{(n)}{\sigma}_{+} \\
&+\sum_{k=1}^{n-1}\left[\left(\overset{(k)}{\sigma}_{11} - \overset{(k)}{\sigma}_{22} + 2i\overset{(k)}{\sigma}_{12}\right)\frac{\partial \overset{(n-k)}{u}_{3}}{\partial z} + \left(\overset{(k)}{\sigma}_{11} + \overset{(k)}{\sigma}_{22}\right)\frac{\partial \overset{(n-k)}{u}_{3}}{\partial \bar z}\right], \\
&\overset{(n)}{T}_{+3} = \overset{(n)}{\sigma}_{+} + \sum_{k=1}^{n-1}\left[\overset{(k)}{\sigma}_{+}\frac{\partial \overset{(n-k)}{u}_{3}}{\partial z} + \overline{\overset{(k)}{\sigma}_{+}}\frac{\partial \overset{(n-k)}{u}_{3}}{\partial \bar z}\right], \\
&\overset{(n)}{T}_{33} = \overset{(n)}{\sigma}_{33} + \sum_{k=1}^{n-1}\left[\overset{(k)}{\sigma}_{+}\frac{\partial \overset{(n-k)}{u}_{3}}{\partial z} + \overline{\overset{(k)}{\sigma}_{+}}\frac{\partial \overset{(n-k)}{u}_{3}}{\partial \bar z}\right],
\end{aligned}
$$

where

$$
\overset{(n)}{u}_{+} = \overset{(n)}{u}_{1} + i\overset{(n)}{u}_{2}, \quad \overset{(n)}{\sigma}_{+} = \overset{(n)}{\sigma}_{13} + i\overset{(n)}{\sigma}_{23}.
$$

For the complex combination of $\overset{(n)}{\sigma}_{ij}$ we get the following expressions

$$
\begin{aligned}
\overset{(n)}{\sigma}_{11} &- \overset{(n)}{\sigma}_{22} + 2i\overset{(n)}{\sigma}_{12} = 4\mu\left(\frac{\partial \overset{(n)}{u}_{+}}{\partial \bar z} + \sum_{k=1}^{n}\left(\frac{\partial \overset{(k)}{u}_{+}}{\partial \bar z}\frac{\partial \overline{\overset{(n-k)}{u}_{+}}}{\partial \bar z} + \frac{\partial \overset{(k)}{u}_{3}}{\partial \bar z}\frac{\partial \overset{(n-k)}{u}_{3}}{\partial \bar z}\right)\right), \\
\overset{(n)}{\sigma}_{11} &+ \overset{(n)}{\sigma}_{22} = 2(\lambda+\mu)\left[\overset{(n)}{\theta} + \sum_{k=1}^{n}\left(\frac{\partial \overset{(k)}{u}_{+}}{\partial z}\frac{\partial \overline{\overset{(n-k)}{u}_{+}}}{\partial \bar z} + \right.\right. \\
&+ \left.\left.\frac{\partial \overline{\overset{(k)}{u}_{+}}}{\partial z}\frac{\partial \overset{(n-k)}{u}_{+}}{\partial \bar z} + 2\frac{\partial \overset{(k)}{u}_{3}}{\partial z}\frac{\partial \overset{(n-k)}{u}_{3}}{\partial \bar z}\right)\right], \\
\overset{(n)}{\sigma}_{+} &= 2\mu\frac{\partial \overset{(n)}{u}_{3}}{\partial \bar z}, \\
\overset{(n)}{\sigma}_{33} &= \lambda\left[\overset{(n)}{\theta} + \sum_{k=1}^{n}\left(\frac{\partial \overset{(k)}{u}_{+}}{\partial z}\frac{\partial \overline{\overset{(n-k)}{u}_{+}}}{\partial \bar z} + \frac{\partial \overset{(k)}{u}_{+}}{\partial z}\frac{\partial \overset{(n-k)}{u}_{+}}{\partial \bar z} + 2\frac{\partial \overset{(k)}{u}_{3}}{\partial z}\frac{\partial \overset{(n-k)}{u}_{3}}{\partial \bar z}\right)\right],
\end{aligned}
$$

$$
\overset{(n)}{\theta} = \frac{\partial \overset{(n)}{u}_{+}}{\partial z} + \frac{\partial \overline{\overset{(n)}{u}_{+}}}{\partial \bar z}.
$$

The boundary conditions can be written as follows

$$\begin{cases} \overset{(n)}{T}_{(ll)} + i\overset{(n)}{T}_{(ls)} = \frac{1}{2}\Big\{\overset{(n)}{T}_{11} + \overset{(n)}{T}_{22} + i(\overset{(n)}{T}_{21} - \overset{(n)}{T}_{12}) - \\ -[\overset{(n)}{T}_{11} - \overset{(n)}{T}_{22} + i(\overset{(n)}{T}_{12} + \overset{(n)}{T}_{21})]\left(\frac{d\bar{z}}{ds}\right)^2\Big\}, \\ \overset{(n)}{T}_{(l3)} = -Im\left(\overset{(n)}{T}_{3+}\frac{d\bar{z}}{ds}\right), \end{cases}$$

where **l** - is a unit vector of the tangential normal of the middle surface

$$\mathbf{l} \times \mathbf{s} = \mathbf{e}_3.$$

For the first approximation of small parameter we get linear system of equations for the plane elastic theory, we obtain $\overset{(n)}{\boldsymbol{u}}$ after solving n recurrent problems. The right hand side are nonlinear combinations of the solutions $\overset{(1)}{\boldsymbol{u}}, \overset{(2)}{\boldsymbol{u}}, ..., \overset{(n-1)}{\boldsymbol{u}}$.

We consider stretch problem of infinite plate with circular hole, when there put rigid body [6]. It means that in the bound is given the following conditions

$$u_r = 0, \;\; T_{r\theta} = 0, \;\; u_3 = 0, \;\; (z = re^{i\theta}, \;\; \mid z \mid = R).$$

and at infinity we have

$$T_{11}^{\infty} = \varepsilon p_1, \;\; T_{22}^{\infty} = \varepsilon p_2, \;\; p_1 = const, \;\; p_2 = const,$$

$$T_{12}^{\infty} = T_{21}^{\infty} = T_{31}^{\infty} = T_{32}^{\infty} = 0.$$

For this problem approximate $n = 1$ has the form:

$$\begin{cases} \mu\Delta \overset{(1)}{u}_{+} + (2\lambda + \mu)\frac{\partial \overset{(1)}{\theta}}{\partial \bar{z}} = 0, \\ \mu\Delta \overset{(1)}{u}_{3} = 0, \;\; \Delta = 4\dfrac{\partial^2}{\partial z \partial \bar{z}} \end{cases}$$

This is a well-known case of the linear plate for which we have [5]

$$\overset{(1)}{u}_{+} = æ\overset{(1)}{\varphi}(z) - z\overline{\overset{(1)}{\varphi}{}'(z)} - \overline{\overset{(1)}{\psi}(z)},$$
$$\overset{(1)}{u}_{3} = g(z) + \overline{g(z)},$$

where $æ = \frac{\lambda+3\mu}{\lambda+\mu}$, $\overset{(1)}{\varphi}(z)$, $\overset{(1)}{\psi}(z)$ and $g(z)$ are analytic functions of complex variable z

$$\overset{(1)}{\varphi}{}'(z) = a_0 + \frac{a_2}{z^2}, \quad \overset{(1)}{\psi}{}'(z) = a_0^{'} + \frac{a_2^{'}}{z^2} + \frac{a_4^{'}}{z^4},$$

$$a_0 = \frac{p_1 + p_2}{8\mu}, \quad a_2 = \frac{\lambda + \mu}{4\mu(2\lambda + 5\mu)}(p_1 - p_2)R^2,$$

R is radius of hole.

$$a_0^{'} = -\tfrac{p_1 - p_2}{4\mu}, \quad a_2^{'} = -\tfrac{p_1 + p_2}{4(\lambda+\mu)}R^2,$$
$$a_4^{'} = -\tfrac{3(p_1 - p_2)}{4(2\lambda+5\mu)}R^4.$$

Consider the approximate $n = 2$. In this case we have:

$$\begin{cases} \mu\Delta \overset{(2)}{u}_+ + 2(\lambda+\mu)\frac{\partial \overset{(2)}{\theta}}{\partial \bar{z}} = F_+, \\ \mu\Delta \overset{(2)}{u}_3 = 0, \end{cases} \qquad (2)$$

where

$$\begin{aligned} F_+ &= \left(A_1 z + \frac{A_2}{z}\right)\frac{1}{\bar{z}^4} + \left(A_3 + \frac{A_4}{\bar{z}^4} + \frac{A_5}{\bar{z}^4}\right)\frac{1}{z^3} \\ &+ \left(A_6 + \frac{A_7}{z^2} + \frac{A_8}{z^4}\right)\frac{1}{\bar{z}^3} + \left(A_9 + \frac{A_{10}}{z^2} + \frac{A_{11}}{z^4}\right)\frac{1}{\bar{z}^5}, \end{aligned}$$

$$\begin{aligned}
A_1 &= \frac{3(\lambda^2 + 4\lambda\mu + 3\mu^2)}{4\mu^2(2\lambda+5\mu)}(p_1 - p_2)^2 R^2, \\
A_2 &= -\frac{3(\lambda+3\mu)}{4\mu(2\lambda+5\mu)}(p_1^2 - p_2^2)R^4, \\
A_3 &= \frac{(\mu-\lambda)(\lambda^2 + 4\lambda\mu + 3\mu^2)}{4\mu^2(2\lambda+5\mu)(\lambda+\mu)}(p_1 - p_2)^2 R^2, \\
A_4 &= -\frac{\lambda^2 + 6\lambda\mu + 9\mu^2}{4\mu(\lambda+\mu)(2\lambda+5\mu)}(p_1^2 - p_2^2)R^4, \\
A_5 &= \frac{6(\lambda+3\mu)(3\lambda+4\mu)}{\mu(2\lambda+5\mu)^2}(p_1 - p_2)^2 R^6, \\
A_6 &= -\frac{6\lambda^2 + 19\lambda\mu + 19\mu^2}{4\mu(\lambda+\mu)(2\lambda+5\mu)}(p_1^2 - p_2^2)R^2, \\
A_7 &= \frac{(\lambda+\mu)(3\lambda + 13\lambda\mu + 16\mu^2)}{2\mu^2(2\lambda+5\mu)^2}(p_1 - p_2)^2 + \frac{(\lambda+3\mu)}{4(\lambda+\mu)^2}(p_1 + p_2)^2, \\
A_8 &= -\frac{3(\lambda+3\mu)}{4(\lambda+\mu)(2\lambda+5\mu)}(p_1^2 - p_2^2)R^6, \\
A_9 &= \frac{37\mu^4 + 55\lambda\mu^3 + 16\lambda^2\mu 2 - 3\lambda^3\mu - \lambda^4}{2\mu^2(\lambda+\mu)(2\lambda+5\mu)^2}(p_1 - p_2)^2 R^4, \\
A_{10} &= -\frac{3(\lambda+3\mu)}{2(\lambda+\mu)(2\lambda+5\mu)}(p_1^2 - p_2^2)R^6, \\
A_{11} &= \frac{9(\lambda+3\mu)}{2(2\lambda+5\mu)^2}(p_1 - p_2)^2 R^8.
\end{aligned}$$

The boundary conditions are take as

$$\overset{(2)}{u}_{r} = 0, \quad \overset{(2)}{T}_{r\theta} = 0, \quad \overset{(2)}{u}_{3} = 0, \tag{3}$$

the stresses are bounded at the infinity

$$\overset{(2)}{T}{}^{\infty}_{11} = \overset{(2)}{T}{}^{\infty}_{22} = \overset{(2)}{T}{}^{\infty}_{12} = \overset{(2)}{T}{}^{\infty}_{21} = \overset{(2)}{T}{}^{\infty}_{31} = \overset{(2)}{T}{}^{\infty}_{32} = 0.$$

The general solution of system (15) has the form

$$\overset{(2)}{u}_{+} = æ\,\overset{(2)}{\varphi}(z) - z\overline{\overset{(2)}{\varphi'}(z)} - \overline{\overset{(2)}{\psi}(z)} + \widehat{u},$$
$$\overset{(2)}{u}_{3} = 0,$$

where $\widehat{u}$ is the particular solution of the non-homogeneous equation

$$\begin{aligned}\widehat{u} &= \left(B_0 z^2 + \frac{B_1}{z^2}\right)\frac{1}{\bar{z}^3} + \left(B_2\bar{z} + \frac{B_3}{\bar{z}}\right)\frac{1}{z^2} + \left(\frac{B_4}{\bar{z}^2} + \frac{B_5}{\bar{z}^4}\right)z + \\ &+ \left(B_6 + \frac{B_7}{\bar{z}^2} + \frac{B_8}{\bar{z}^4}\right)\frac{1}{z} + \left(B_9 + \frac{B_{10}}{\bar{z}^2} + \frac{B_{11}}{\bar{z}^4}\right)\frac{1}{z^3}.\end{aligned}$$

$$\begin{aligned}
B_0 &= \frac{21\mu^5 - 25\lambda\mu^4 - 26\lambda^2\mu^3 - 22\lambda^3\mu^2 - 9\lambda^4\mu - \lambda^5}{32\mu^3(\lambda+\mu)^2(\lambda+2\mu)(2\lambda+5\mu)}(p_1-p_2)^2R^2,\\
B_1 &= \frac{(\lambda+3\mu)(3\lambda+4\mu)}{4\mu(\lambda+2\mu)(2\lambda+5\mu)^2}(p_1-p_2)^2R^6,\\
B_2 &= \frac{3\mu^4 - 2\lambda\mu^3 - 10\lambda^2\mu^2 - 6\lambda^3\mu - \lambda^4}{32\mu^3(\lambda+\mu)(\lambda+2\mu)(2\lambda+5\mu)}(p_1-p_2)^2R^2,\\
B_3 &= -\frac{\lambda+3\mu}{16\mu(\lambda+\mu)(2\lambda+5\mu)}(p_1^2-p_2^2)R^4,\\
B_4 &= \frac{6\lambda^2+19\lambda\mu+19\mu^2}{32\mu(\lambda+\mu)(\lambda+2\mu)(2\lambda+5\mu)}(p_1^2-p_2^2)R^2,\\
B_5 &= -\frac{37\mu^4+55\lambda\mu^3+16\lambda^2\mu^2-3\lambda^3\mu-\lambda^4}{32\mu^2(\lambda+\mu)(\lambda+2\mu)^2(2\lambda+5\mu)}(p_1-p_2)^2R^4,\\
B_6 = - & \quad \frac{6\lambda^2+19\lambda\mu+19\mu^2}{32\mu(\lambda+\mu)(\lambda+2\mu)(2\lambda+5\mu)}(p_1^2-p_2^2)R^2,\\
B_7 &= \frac{R^4}{16(\lambda+2\mu)}\left[\frac{(\lambda+\mu)(3\lambda+13\lambda\mu+16\mu^2)}{\mu^2(2\lambda+5\mu)^2}(p_1-p_2)^2\right.\\
&\left.+\frac{\lambda+3\mu}{2(\lambda+\mu)^2}(p_1+p_2)^2\right]\\
B_8 &= -\frac{3\mu(\lambda+3\mu)}{16(\lambda+\mu)^2(\lambda+2\mu)(2\lambda+5\mu)}(p_1^2-p_2^2)R^6,
\end{aligned}$$

$$
\begin{aligned}
B_9 &= \frac{37\mu^4+55\lambda\mu^3+16\lambda^2\mu^2-3\lambda^3\mu-\lambda^4}{96\mu^2(\lambda+\mu)(\lambda+2\mu)(2\lambda+5\mu)^2}(p_1-p_2)^2R^4,\\
B_{10} &= -\frac{\lambda+3\mu}{16(\lambda+\mu)(\lambda+2\mu)(2\lambda+5\mu)}(p_1^2-p_2^2)R^6,\\
B_{11} &= \frac{3(\lambda+3\mu)}{32(\lambda+2\mu)(2\lambda+5\mu)^2}(p_1-p_2)^2R^8.
\end{aligned}
$$

Let us introduced following $\overset{(2)}{\varphi}{}'(z)$ and $\overset{(2)}{\psi}{}'(z)$ by series

$$
\overset{(2)}{\varphi}{}'(z)=\sum_{k=0}^{\infty}\alpha_k z^{-k},\quad \overset{(2)}{\psi}{}'(z)=\sum_{k=0}^{\infty}\beta_k z^{-k}. \tag{4}
$$

coefficients α_0 and β_0 are defined from the conditions at infinity

$$
\left[\overset{(2)}{T}_{11}+\overset{(2)}{T}_{22}+i\left(\overset{(2)}{T}_{21}-\overset{(2)}{T}_{12}\right)\right]^{\infty}=\left[4\mu\left(\overset{(2)}{\varphi}{}'(z)+\overline{\overset{(2)}{\varphi}{}'(z)}\right)\right.
$$
$$
\left.+d_0(r)+d_1(r)e^{2i\theta}+d_2(r)e^{-2i\theta}+d_3(r)e^{4i\theta}+d_4(r)e^{-4i\theta}\right]^{\infty}=0,
$$

where

$$
\begin{aligned}
d_0(r) &= -4(\lambda+\mu)\left(\frac{B_7}{r^4}+\frac{2B_1}{r^6}+\frac{3B_{11}}{r^8}\right)+\frac{24\mu^2}{\lambda+\mu}a_0^2\\
&+ \frac{4(\lambda^2+4\lambda\mu+7\mu^2)}{\lambda+\mu}\frac{a_2^2}{r^4}+2(\lambda+3\mu)\left[(a_0^{'})^2+\frac{a_2^{'}}{r^4}+\left(\frac{a_4^{'}}{r^4}-\frac{2a_2}{r^2}\right)^2\right],\\
d_1(r) &= 2(\lambda+\mu)\left(\frac{B_4-B_6}{r^2}-\frac{2B_3}{r^4}-\frac{B_8+3B_{10}}{r^6}\right)\\
&+ 2(\lambda+3\mu)\frac{a_2^{'}}{r^2}\left(a_0^{'}-\frac{2a_2}{r^2}+\frac{a_4^{'}}{r^4}\right)+\frac{8\mu(\mu-\lambda)}{\lambda+\mu}\frac{a_0a_2}{r^2},\\
d_2(r) &= 2(\lambda+\mu)\left(\frac{B_4-B_6}{r^2}-\frac{2B_3}{r^4}-\frac{B_8+3B_{10}}{r^6}\right)\\
&+ 2(\lambda+3\mu)\frac{a_2^{'}}{r^2}\left(a_0^{'}-\frac{2a_2}{r^2}+\frac{a_4^{'}}{r^4}\right)+\frac{8\mu(5\mu+\lambda)}{\lambda+\mu}\frac{a_0a_2}{r^2},\\
d_3(r) &= 2(\lambda+\mu)\left(2\frac{B_0-B_2}{r^2}+\frac{B_5-3B_9}{r^4}\right)\\
&+ 2(\lambda+3\mu)\left(\frac{a_4^{'}}{r^4}-\frac{2a_2}{r^2}\right)a_0^{'}-2(\lambda+5\mu)\frac{a_2^2}{r^4},
\end{aligned}
$$

$$\begin{aligned} d_4(r) &= 2(\lambda+\mu)\left(2\frac{B_0-B_2}{r^2}+\frac{B_5-3B_9}{r^4}\right) \\ &+ 2(\lambda+3\mu)\left(-\frac{2a_2a_0^{'}}{r^2}+\frac{a_4^{'}a_0}{r^4}\right)+2æ(\mu-\lambda)\frac{a_2^2}{r^4}, \qquad (5) \end{aligned}$$

$$\alpha_0=-\frac{1}{64\mu^2}\left[\frac{3\mu}{\lambda+\mu}(p_1+p_2)^2+\frac{\lambda+3\mu}{\mu}(p_1-p_2)^2\right].$$

$$\left[\overset{(2)}{T}_{11}-\overset{(2)}{T}_{22}+i\left(\overset{(2)}{T}_{12}+\overset{(2)}{T}_{21}\right)\right]^{\infty}=\Big[-4\mu\left(\overline{z\overset{(2)}{\varphi}{}''(z)+\overset{(2)}{\psi}{}'(z)}\right)$$
$$+c_0(r)e^{2i\theta}+c_1(r)e^{4i\theta}+c_2(r)+c_3(r)e^{6i\theta}+c_4(r)e^{-2i\theta}\Big]^{\infty}=0,$$

where

$$\begin{aligned} c_0(r) &= -4\mu\left[\frac{3B_1}{r^6}+\frac{2B_7}{r^4}+\frac{4B_{11}}{r^8}+æ\left(-\frac{2a_0^2}{r^4}+\frac{2a_0a_2^{'}}{r^2}+\frac{a_2a_0^{'}}{r^2}+\frac{a_2a_4^{'}}{r^6}\right)\right], \\ c_1(r) &= -4\mu\left[\frac{2B_4}{r^2}+\frac{4B_8}{r^6}+æ\left(-\frac{4a_0a_2}{r^2}+\frac{2a_0a_4^{'}}{r^4}+\frac{a_2a_2^{'}}{r^4}\right)\right], \\ c_2(r) &= -4\mu\left[\frac{B_3}{r^4}+\frac{2B_{10}}{r^6}+æ\left(2a_0a_0^{'}+\frac{2a_2a_2^{'}}{r^4}\right)\right], \\ c_3(r) &= -4\mu\left[\frac{3B_0}{r^2}+\frac{4B_5}{r^4}+æ\left(-\frac{2a_2^2}{r^4}+\frac{a_2a_4^{'}}{r^6}\right)\right], \\ c_4(r) &= 4\mu\left[\frac{B_2}{r^2}-æ\frac{a_2a_0^{'}}{r^2}\right], \qquad (6) \end{aligned}$$

$$\beta_0=\frac{æ(p_1^2-p_2^2)}{16\mu^2}.$$

In virtue of boundary conditions we have

$$æ\left(2\alpha_0R+\sum_{n=2}^{\infty}\frac{\alpha_n}{(1-n)R^{n-1}}e^{-in\theta}+\sum_{n=2}^{\infty}\frac{\overline{\alpha_n}}{(1-n)R^{n-1}}e^{in\theta}\right)$$

$$-\sum_{n=0}^{\infty}\frac{\overline{\alpha_n}}{R^{n-1}}e^{in\theta}-\sum_{n=0}^{\infty}\frac{\alpha_n}{R^{n-1}}e^{-in\theta}-\overline{\beta_0}Re^{-2i\theta}-\beta_0Re^{2i\theta}$$

$$-\sum_{n=2}^{\infty}\frac{\overline{\beta_n}}{(1-n)R^{n-1}}e^{i(n-2)\theta}-\sum_{n=2}^{\infty}\frac{\beta_n}{(1-n)R^{n-1}}e^{-i(n-2)\theta}$$

$$= -\frac{2B_1}{R^5} - \frac{2B_7}{R^3} - \frac{2B_{11}}{R^7} - \left(\frac{B_4+B_6}{R} + \frac{B_8+B_{10}}{R^5} + \frac{B_3}{R^3}\right)e^{2i\theta}$$

$$-\left(\frac{B_4+B_6}{R} + \frac{B_8+B_{10}}{R^5} + \frac{B_3}{R^3}\right)e^{-2i\theta} - \left(\frac{B_0+B_2}{R} + \frac{B_5+B_9}{R^3}\right)e^{4i\theta}$$

$$-\left(\frac{B_0+B_2}{R} + \frac{B_5+B_9}{R^3}\right)e^{-4i\theta} = 0,$$

$$\sum_{n=0}^{\infty}\frac{n\overline{\alpha_n}}{R^n}e^{in\theta} - \sum_{n=0}^{\infty}\frac{\overline{\beta_n}}{R^n}e^{i(n-2)\theta} - \sum_{n=0}^{\infty}\frac{n\alpha_n}{R^n}e^{-in\theta} + \sum_{n=0}^{\infty}\frac{\beta_n}{R^n}e^{-i(n-2)\theta}$$

$$= (E_1(R) - E_2(R))e^{2i\theta} + (E_2(R) - E_2(R))e^{-2i\theta} +$$

$$(E_3(R) - E_4(R))e^{4i\theta} + (E_4(R) - E_3(R))e^{-4i\theta}.$$

where

$$E_n(r) = -\frac{c_n(r) + d_n(r)}{2\mu}, \qquad n = \overline{0,\ 4}.$$

Therefore, functions $\overset{(2)}{\varphi}(z)$ and $\overset{(2)}{\psi}(z)$ have the following forms

$$\overset{(2)}{\varphi}{}'(z) = \alpha_0 + \tfrac{\alpha_2}{z^2} + \tfrac{\alpha_4}{z^4},$$
$$\overset{(2)}{\psi}{}'(z) = \beta_0 + \tfrac{\beta_2}{z^2} + \tfrac{\beta_4}{z^4} + \tfrac{\beta_6}{z^6},$$

$$\begin{aligned}
\alpha_2 &= -\frac{1}{3æ+1}\left(2\beta_0 - E_2(R) + E_1(R)R^2 - 3(B_4+B_6) - \frac{3(B_8+B_{10})}{R^4} + \frac{B_3}{R^2}\right),\\
\alpha_4 &= \frac{3}{5æ+3}((B_5+B_9) + (5(B_0+B_2)R^2 + (E_4(R) - E_3(R))R^4),\\
\beta_2 &= (1-æ)R^2\alpha_0 - \frac{B_1}{R^4} - \frac{B_7}{R^3} - \frac{B_{11}}{R^6},\\
\beta_4 &= \frac{3}{3æ+1}(2B_2 + 2(B_4+B_6)R^2 + (æ-1)R^4\beta_0 +\\
&+ (æ+1)R^4(E_2(R) - E_1(R)) + \frac{2(B_8+B_{10})}{R^2}\Bigg),\\
\beta_6 &= \frac{5}{5æ+3}(12(B_5+B_9)R^2 + 12(B_0+B_2)R^4 + (æ+3)(E_4(R) - E_3(R)R^6).
\end{aligned}$$

In case of the second approximation of small parameters components of the stress tensor and displacement vector can be written as follows:

$$
\begin{aligned}
T_{rr} &= 2\mu\left(2a_0 + -\frac{a_2^{'}}{r^2}\left(\frac{4a_2}{r^2} - a_0^{'} - \frac{a_4^{'}}{r^4}\right)cos2\theta\right)\varepsilon \\
&+ 2\mu\left\{2\alpha_0 - \frac{\beta_0}{r^2} - \frac{E_0(r)}{2} + \left(-\beta_0 - \frac{E_1(r)+E_2(r)}{2}\right.\right. \\
&+ \left.\left.\frac{4\alpha_2}{r^2} - \frac{\beta_4}{r^4}\right)cos2\theta + \left(\frac{6\alpha_4}{r^4} - \frac{\beta_6}{r^6} - \frac{E_3(r)+E_4(r)}{2}\right)cos4\theta\right\}\varepsilon^2, \\
T_{r\theta} &= 2\mu\left(2a_0 + \frac{2a_2}{r^2} - \frac{a_4^{'}}{r^4}\right)sin2\theta\varepsilon \\
&+ 2\mu\left\{\left(\beta_0 + \frac{2\alpha_2}{r^2} - \frac{\beta_4}{r^4} + \frac{E_2(r)-E_1(r)}{2}\right)sin2\theta\right. \\
&+ \left.\left(\frac{4\alpha_4}{r^4} - \frac{\beta_6}{r^6} + \frac{E_4(r)-E_3(r)}{2}\right)sin4\theta\right\}\varepsilon^2, \\
T_{\theta\theta} &= 2\mu\left(2a_0 + \frac{a_2^{'}}{r^2} + \left(a_0^{'} + \frac{a_4^{'}}{r^4}\right)cos2\theta\right)\varepsilon \\
&+ \left\{4\mu\alpha_0 + \frac{2\mu\beta_2}{r^2} + 4\mu E_0(r) + d_0(r)\right. \\
&+ \left[2\mu\frac{\beta_4}{r^4} + 2\mu\beta_0 + d_1(r) + d_2(r) + \mu(E_1(r)+E_2(r))\right]cos2\theta \\
&+ \left.\left(\frac{2\mu\beta_6}{r^6} - \frac{4\mu\alpha_4}{r^4} + d_3(r) + d_4(r) + \mu(E_3(r)+E_4(r))\right)cos4\theta\right\}\varepsilon^2, \\
u_r &= \left\{(æ-1)a_0 r + \frac{a_2^{'}}{r} + \left(\frac{a_4^{'}}{3r^3} - \frac{(1+æ)a_2}{r} - a_0^{'}r\right)cos2\theta\right\}\varepsilon \\
&+ \left\{(æ-1)\alpha_0 r + \frac{B_1}{r^5} + \frac{B_7}{r^3} + \frac{B_{11}}{r^7} + \frac{\beta_2}{r}\right. \\
&+ \left(-\beta_0 r - \frac{(æ+1)\alpha_2}{r^2} + \frac{3B_3+\beta_4}{3r^3} + \frac{B_4+B_6}{r} + \frac{B_8+B_{10}}{r^5}\right)cos2\theta \\
&+ \left.\left(-\frac{3-æ}{3r^3}\alpha_4 + \frac{B_0+B_2}{r} + \frac{B_5+B_9}{r^3} + \frac{\beta_6}{5r^5}\right)cos4\theta\right\}\varepsilon^2, \\
u_\theta &= \left(a_0^{'}r + \frac{æ-1}{r}a_2 + \frac{a_4^{'}}{3r^3}\right)sin2\theta\varepsilon \\
&+ \left\{\left(\frac{æ-1}{r}\alpha_2 + \beta_0 r + \frac{\beta_4-3B_3}{3r^3} + \frac{B_4-B_6}{r} + \frac{B_8-B_{10}}{r^5}\right)sin2\theta\right. \\
&+ \left.\left(\frac{æ-3}{3r^3}\alpha^4 + \frac{B_0-B_2}{r} + \frac{B_5-B_9}{r^3} + \frac{\beta_6}{5r^5}\right)sin4\theta\right\}\varepsilon^2.
\end{aligned}
$$

The obtained solutions are compared to the results obtained by two-dimensional linear theory of elasticity. In case of small parameter is equal to $\frac{h}{R}$, the solution of nonlinear problem depends on both the thickness of the plate and on the radius, while linear problem it is depends only radius of the hole.

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