BOUNDARY-CONTACT PROBLEMS OF DYNAMICS FOR MIXTURES OF TWO ISOTROPIC ELASTIC BODIES

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Abstract

By means of the Laplace transformation and the potential and the singular integral equation theories the uniqueness and existence theorems are proved.

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This paper is dedicated to the investigation of dynamic boundary-contact problems of the linear theory of a mixture of two isotropic elastic materials. As a model problem we investigate here the so-called basic contact problem whose investigation is fraught with difficulties typical of other problems. Note that the basic boundary value problems of dynamics, statics and oscillations are investigated in [2].

Let $\mathbb{R}^3$ be a three-dimensional Euclidean space and let $D_1 \subset \mathbb{R}^3$ be the domain $D_2 = \mathbb{R}^3 \setminus \overline{\mathbb{R}}_1$, $\overline{D}_i = D_i \cup S$, $i = 1, 2$, bounded by a smooth surface S. It is assumed that $S \in \mathcal{L}_k(\gamma)$, $k \geq 1$, $\gamma \in \,]0, 1[\,$.

In the theory of mixtures of two isotropic elastic bodies we consider two displacement vectors $u' = (u_1', u_2', u_3')$ and $u'' = (u_1'', u_2'', u_3'')$ at each point of the domain filled with the mixture.

Dynamic equations in terms of displacement vectors have the form

$$a_1 \Delta u' + b_1 \operatorname{grad} \operatorname{div} u' + c \Delta u'' + d \operatorname{grad} \operatorname{div} u'' - \rho_1 \frac{\partial^2 u'}{\partial t^2} = -\rho_1 F', \qquad (1)$$

$$c \Delta u' + d \operatorname{grad} \operatorname{div} u' + a_2 \Delta u'' + b_2 \operatorname{grad} \operatorname{div} u'' - \rho_2 \frac{\partial^2 u''}{\partial t^2} = -\rho_2 F'', \qquad (2)$$

where Δ is the Laplace operator,

$$\begin{gathered}\rho=\rho_1+\rho_2,\quad \lambda_3-\lambda_4=\alpha_2,\\ a_1=\mu_1-\lambda_5,\quad b_1=\mu_1+\lambda_5+\lambda_1-\frac{\rho_1}{\rho}\,\alpha_2,\\ a_2=\mu_2-\lambda_5,\;\; c=\mu_3+\lambda_5,\quad b_2=\mu_2+\lambda_2+\lambda_5+\frac{\rho_1}{\rho}\,\alpha_2,\\ d=\mu_3+\lambda_3-\lambda_5-\frac{\rho_1}{\rho}\,\alpha_2=\mu_3+\lambda_4-\lambda_5+\frac{\rho_2}{\rho}\,\alpha_2,\end{gathered}\tag{3}$$

$F'=(F'_1,F'_2,F'_3)$ and $F''=(F''_1,F''_2,F''_3)$ are mass forces.

If we introduce the differential matrix operator [2]

$$A(D)=\begin{Vmatrix}A^{(1)}(D) & A^{(2)}(D)\\ A^{(3)}(D) & A^{(4)}(D)\end{Vmatrix}_{6\times 6},\tag{4}$$

where

$$\begin{gathered}A^{(j)}(D)=\|A^{(j)}_{kp}\|_{3\times 3}\;\; j=\overline{1,4},\;\; k,p=\overline{1,3},\\ A^{(1)}_{kp}(\xi)=a_1|\xi|^2\delta_{kp}+b_1\xi_k\xi_p,\\ A^{(2)}_{kp}(\xi)=A^{(3)}_{kp}(\xi)=c|\xi|^2\delta_{kp}+d\xi_k\xi_p,\;\; \xi=(\xi_1,\xi_2,\xi_3),\\ A^{(4)}_{kp}(\xi)=a_2|\xi|^2\delta_{kp}+b_2\xi_k\xi_p\end{gathered}$$

and denote $U=(u',u'')$, then (1), (2) can be written in the matrix form as follows:

$$A(D)U-ED_t^2U=F,\tag{5}$$

where

$$E=\begin{Vmatrix}\rho_1\mathcal{J} & 0\\ 0 & \rho_2\mathcal{J}\end{Vmatrix}_{6\times 6},\quad \mathcal{J}=\|\delta_{kj}\|_{3\times 3},\;\; F=(F',F''),\;\; D_t^2=\frac{\partial^2}{\partial t^2}\,.$$

We have

$$\begin{aligned}W(U,V)=&\left[\left(\lambda_1-\frac{\rho_2}{\rho}\,\alpha_2\right)\delta_{rs}\delta_{ij}+\mu_1(\delta_{ri}\delta_{sj}+\delta_{si}\delta_{rj})\right]\varepsilon'_{rs}\overline{e}'_{ij}+\\ &+\left[\left(\lambda_2+\frac{\rho_1}{\rho}\,\alpha_2\right)\delta_{rs}\delta_{ij}+\mu_2(\delta_{ri}\delta_{ij}+\delta_{si}\delta_{rj})\right]\varepsilon''_{rs}\overline{e}''_{ij}+\\ &+\left[\left(\lambda_4+\frac{\rho_2}{\rho}\,\alpha_2\right)\delta_{rs}\delta_{ij}+\mu_3(\delta_{ri}\delta_{sj}+\delta_{si}\delta_{rj})\right]\varepsilon'_{rs}\overline{e}''_{ij}+\\ &+\left[\left(\lambda_3-\frac{\rho_1}{\rho}\,\alpha_2\right)\delta_{rs}\delta_{ij}+\mu_3(\delta_{ri}\delta_{sj}+\delta_{si}\delta_{rj})\right]\varepsilon''_{rs}\overline{e}'_{ij}-\\ &-2\lambda_5h_{ij}\overline{h}^{*}_{ij},\end{aligned}\tag{6}$$

where $V=(v',v'')$, ε'_{ij}, ε''_{ij}, h_{ij} and e'_{ij}, e''_{ij}, h^*_{ij} are the partial rotations corresponding to the vectors (u',u'') and (v',v''). In general, $W(U,V)=W(\overline{V},\overline{U})$. From (6) it obviously follows that if U and V are real vectors, then

$$W(U,V)=W(V,U). \tag{7}$$

The generalized stress operator has the form [2]

$$\mathcal{P}=\begin{Vmatrix}\mathcal{P}^{(1)} & \mathcal{P}^{(2)}\\ \mathcal{P}^{(3)} & \mathcal{P}^{(4)}\end{Vmatrix}_{6\times 6},\quad \mathcal{P}^{(k)}=\|\mathcal{P}^{(k)}_{ij}\|,\ \ k=\overline{1,k},$$

where

$$\begin{aligned}
\mathcal{P}^{(1)}_{ij}(D,n)&=(\mu_1-\lambda_5)\delta_{ij}\frac{\partial}{\partial n}+(\mu_1+\lambda_5)n_jD_i+\\
&\quad+\left(\lambda_1-\frac{\rho_2}{\rho}\,\alpha_2\right)n_iD_j,\\
\mathcal{P}^{(2)}_{ij}(D,n)&=(\mu_3+\lambda_5)\delta_{ij}\frac{\partial}{\partial n}+(\mu_3-\lambda_5)n_jD_i+\\
&\quad+\left(\lambda_3-\frac{\rho_1}{\rho}\,\alpha_2\right)n_iD_j,\\
\mathcal{P}^{(3)}_{ij}(D,n)&=(\mu_3+\lambda_5)\delta_{ij}\frac{\partial}{\partial n}+(\mu_3-\lambda_5)n_jD_i+\\
&\quad+\left(\lambda_4+\frac{\rho_2}{\rho}\,\alpha_2\right)n_iD_j,\\
\mathcal{P}^{(4)}_{ij}(D,n)&=(\mu_2-\lambda_5)\delta_{ij}\frac{\partial}{\partial n}+(\mu_2+\lambda_5)n_jD_i+\\
&\quad+\left(\lambda_2+\frac{\rho_1}{\rho}\,\alpha_2\right)n_iD_j,
\end{aligned}\tag{8}$$

where n is the outward normal unit vector with respect to D_1, $\partial/\partial n$ is the operator of differentiation along the normal, $D_k=\partial/\partial x_k$.

From (8) it obviously follows that $\mathcal{P}^{(2)}_{ij}=\mathcal{P}^{(3)}_{ij}$, $\mathcal{P}\neq\mathcal{P}'$ and $\mathcal{P}(\xi,\xi)=A(\xi)$.

The Green formula has the form [2]

$$\int\limits_{D_1}\big[A(D)U\cdot V+W(U,V)\big]\,dx=\int\limits_{S}\mathcal{P}(D,n)U\cdot V\,ds. \tag{9}$$

Analogously, for $\overline{U}$ and $\overline{V}$ we have

$$\int\limits_{D_1}\big[A(D)\overline{V}\cdot\overline{U}+W(\overline{V},\overline{U})\big]\,dx=\int\limits_{S}\mathcal{P}(D,n)\overline{V}\cdot\overline{U}\,ds. \tag{10}$$

From (9) and (10) we obtain

$$\int_{D_1} \big\{A(D)U \cdot V - U \cdot A(D)\big\}\, dx =$$
$$= \int_S \Big\{\mathcal{P}(D,n)U \cdot V - U \cdot \mathcal{P}(D,n) \cdot V\Big\}\, ds. \tag{11}$$

If, in addition to the smoothness conditions, U and V satisfy, in the neighborhood of the point at infinity, the conditions

$$\big|D^\alpha U_j(x)\big| + \big|D^\alpha V_j(x)\big| < C|x|^{-1+|\alpha_0|}, \quad |\alpha_0| = 0,1,2, \quad j = \overline{1,6},$$

then we obtain

$$\begin{gathered}\int_{D_2} \big[A(D)U \cdot V + W(U,V)\big]\, dx = -\int_S \mathcal{P}(D,n)U \cdot V\, ds,\\ \int_{D_2} \big[A(D)U \cdot V - U \cdot A(D)V\big]\, dx = \\ = -\int_S \big[\mathcal{P}(D,n)U \cdot V - U \cdot \mathcal{P}V\big]\, ds.\end{gathered} \tag{12}$$

Statement of the problem. Find a solution of the equations

$$\begin{gathered}\overset{(l)}{A}(D_k)\overset{(l)}{U}(x,t) - ED_t^2 U(x,t) = \overset{(l)}{\Phi}{}^*(x,t),\\ (x,t) \in D_l \times I, \quad I = [0,\infty[\,,\\ \overset{(l)}{U} \in C^1(\overline{D}_l \times I) \cap C^2(D_l \times I), \quad l = 1,2,\end{gathered} \tag{13}$$

that satisfies the initial conditions

$$\forall\, x \in \overline{D}_l : \quad \overset{(l)}{U}(x,0) = 0, \quad \left.\frac{\partial \overset{(l)}{U}(x,t)}{\partial t}\right|_{t=0} = 0, \quad l = 1,2, \tag{14}$$

and the contact conditions

$$\begin{gathered}\forall\,(z,t) \in S_l \times I : \left[\overset{(1)}{U}(z,t)\right]^+ - \left[\overset{(2)}{U}(z,t)\right]^- = f^*(z,t),\\ \left[\overset{(1)}{\mathcal{P}}(D_z,n)\overset{(1)}{U}(z,t)\right]^+ - \left[\overset{(2)}{\mathcal{P}}(D_z,n)\overset{(2)}{U}(z,t)\right]^- = \\ = F^*(z,t)\end{gathered} \tag{15}$$

and for $|x| \to \infty$:

$$\overset{(2)}{U}_j(x,t) = O(|x|^{-1}), \quad \frac{\partial \overset{(2)}{U}_j(x,t)}{\partial x_i} = o(|x|^{-1}).$$

For the problem posed, the following uniqueness theorem is valid.

Theorem 1 *The homogeneous problem* (13)–(15) ($\overset{(l)}{\Phi} = f^* = F^* = 0$, $\ell = 1,2$) *has only a trivial solution in the class of regular vectors.*

Proof. Let us assume that the homogeneous problem has a nontrivial solution. We write the Green formula for the vectors $\overset{(l)}{U}$ and $\partial \overset{(l)}{U}/\partial t$ in the domains D_1 and D_2:

$$\int_{D_1} \frac{\partial \overset{(1)}{U}(x,t)}{\partial t} \overset{(1)}{A}(D_x) \overset{(1)}{U}(x,t)\, dx =$$

$$= \int_S \left(\frac{\partial \overset{(1)}{U}(z,t)}{\partial t} \right)^+ \left(\overset{(1)}{\mathcal{P}}(D_z,n) \overset{(1)}{U}(z,t) \right)^+ d_z S -$$

$$- \int_{D_1} \overset{(1)}{W} \left(\frac{\partial \overset{(1)}{U}(x,t)}{\partial t}, \overset{(1)}{U}(x,t) \right) dx, \tag{16}$$

$$\int_{D_2} \frac{\partial \overset{(2)}{U}(x,t)}{\partial t} \overset{(2)}{A}(D_x) \overset{(2)}{U}(x,t)\, dx =$$

$$= - \int_S \left(\frac{\partial \overset{(2)}{U}(z,t)}{\partial t} \right)^- \left(\overset{(2)}{\mathcal{P}}(D_z,n) \overset{(2)}{U}(z,t) \right)^- d_z S -$$

$$- \int_{D_2} \overset{(2)}{W} \left(\frac{\partial \overset{(2)}{U}(x,t)}{\partial t}, \overset{(2)}{U}(x,t) \right) dx. \tag{17}$$

Since

$$2W\left(\frac{\partial U}{\partial t}, U\right) = \frac{\partial}{\partial t} W(U,U) = W\left(\frac{\partial U}{\partial t}, U\right) + W\left(U, \frac{\partial U}{\partial t}\right),$$

$$\frac{\partial U}{\partial t} A(D)U = E\frac{\partial^2 U}{\partial t^2} \cdot U = \frac{1}{2}\frac{\partial}{\partial t}\left(\rho_1 |\dot{u}'|^2 + \rho_2 |\dot{u}''|^2\right),$$

where

$$\dot{u}' = \frac{\partial u}{\partial t}, \quad \dot{u}'' = \frac{\partial^2 u}{\partial t^2},$$

we have

$$\frac{\partial}{\partial t} W(U,U) = W(\dot{U},U) + W(U,\dot{U}),$$

$$\frac{\partial}{\partial t}\int\limits_{D_1}\left[\rho_1|\overset{(1)}{\dot{u}}{}'|^2 + \rho_2|\overset{(1)}{\dot{u}}{}''|^2 + \overset{(1)}{W}(\overset{(1)}{U},\overset{(1)}{U})\right]dx =$$

$$= \int\limits_{S}\left(\frac{\partial \overset{(1)}{U}}{\partial t}\right)^{+}(\overset{(1)}{\mathcal{P}}\overset{(1)}{U})^{+}\,d_zS, \tag{18}$$

$$\frac{\partial}{\partial t}\int\limits_{D_1}\left[\rho_1|\overset{(2)}{\dot{u}}{}'|^2 + \rho_2|\overset{(2)}{\dot{u}}{}''|^2 + \overset{(2)}{W}(\overset{(2)}{U},\overset{(2)}{U})\right]dx =$$

$$= -\int\limits_{S}\left(\frac{\partial \overset{(2)}{U}}{\partial t}\right)^{-}(\overset{(2)}{\mathcal{P}}\overset{(2)}{U})^{-}\,d_zS. \tag{19}$$

Adding (18) and (19) and taking into account that the surface integral vanishes, we obtain by virtue of the homogeneity of the contact conditions

$$\sum_{l=1}^{2}\left\{\int\limits_{D_l}\rho_1\left|\frac{\partial \overset{(l)}{u}{}'}{\partial t}\right|^2 + \rho_2\left|\frac{\partial \overset{(l)}{u}{}''}{\partial t}\right|^2 + \overset{(l)}{W}(\overset{(l)}{U},\overset{(l)}{U})\right\}dx = \mathrm{const}\,.$$

But at the initial moment this constant is equal to zero and therefore we obtain $\overset{(l)}{U}(x,t) \equiv 0$, $(x,t) \in D_l \times I$, $l = 1,2$, which contradicts the assumption that the homogeneous problem has a nontrivial solution. This contradiction proves the theorem.

Now let us consider the following elliptic problem of pseudo-oscillation: Find the regular solution $\overset{(l)}{V}(x,\tau) = (\overset{(l)}{v}{}'(x,\tau), \overset{(l)}{v}{}''(x,\tau))$ of the equations

$$\forall\, x \in D_l: \quad \overset{(l)}{A}(D_x)\overset{(l)}{V}(x,\tau) - r_l\tau^2\overset{(l)}{V}(x,\tau) = \overset{(l)}{\Phi}(x,\tau), \;\; l = 1,2, \tag{20}$$

which satisfies the contact conditions

$$\forall z \in S: \quad \left[\overset{(1)}{V}(z,\tau)\right]^{+} - \left[\overset{(2)}{V}(z,\tau)\right]^{+} = f(z,\tau),$$
$$\left[\overset{(1)}{\mathcal{P}}(D_z,n)\overset{(1)}{V}(z,\tau)\right]^{+} - \left[\overset{(2)}{\mathcal{P}}(D_z,n)\overset{(2)}{V}(z,\tau)\right]^{-} = \tag{21}$$
$$= F(z,\tau),$$

where

$$\overset{(l)}{\Phi}(x,\tau) = \int_0^\infty e^{-\tau t}\overset{(l)}{\Phi}{}^*(x,t)\,dt, \quad l=1,2,$$
$$f(z,\tau) = \int_0^\infty e^{-\tau t} f^*(z,t)\,dt,$$
$$F(z,\tau) = \int_0^\infty e^{-\tau t} F^*(z,t)\,dt,$$

$\tau = \sigma + i\omega$, $\sigma \geq \sigma_0' > \sigma_0$. This half-plane is denoted by π_{σ_0}.

Problem (20)–(21) is obtained from problem (13)–(15) by the formal Laplace transformation:

$$\overset{(l)}{V}(x,\tau) = \int_0^\infty e^{-\tau t}\overset{(l)}{U}(x,t)\,dt, \quad l=1,2,$$
$$\overset{(l)}{V} \in C^1(\overline{D}_l)\cap C^2(D_l), \quad \overset{(2)}{V}_j = O(|x|^{-1}),$$
$$\frac{\partial \overset{(2)}{V}_j}{\partial x_j} = o(|x|^{-1}) \ \ (|x|\to\infty).$$

The following uniqueness theorem is true.

Theorem 2 *The homogeneous problem* (20)–(21) ($\overset{(l)}{\Phi} = f = F = 0$) *has only a trivial solution.*

Green tensor. Denote by $G(y,x;\eta_0)$ the Green tensor of problem

(20)–(21) which is defined as follows:

$$G(y,x;\eta_0)=\begin{cases}\overset{(1)}{G}(y,x;\eta_0), & y\in D_1,\ \ x\in D_1\cup D_2,\ \ x\neq y,\\ \overset{(2)}{G}(y,x;\eta_0), & y\in D_2,\ \ x\in D_1\cup D_2,\ \ x\neq y,\end{cases}, \tag{22}$$
$$\overset{(l)}{G}(y,x;\eta_0)=\overset{(l)}{\Psi}(y-x;\eta_0)+\overset{(l)}{\psi}(y,x;\eta_0),\ \ l=1,2,$$

where $\overset{(l)}{\psi}$ is a regular solution of the equation

$$\begin{gathered}\overset{(l)}{A}(D)\overset{(l)}{\psi}(y,;\eta_0)-r_l\eta_0^2\overset{(l)}{\psi}(y,x;\eta_0)=0,\\ y\in D_l,\ \ x\in D_1\cup D_2,\ \ l=1,2,\ \ \eta_0>0\end{gathered} \tag{23}$$

and $\overset{(l)}{\Psi}(y,x;\eta_0)$ is the matrix of fundamental solutions of the same equation [2].

Besides equation (23), $\overset{(l)}{\psi}(y,x;\eta_0)$ also satisfies the conditions

$$\begin{gathered}\left(\overset{(1)}{\psi}(z,x;\eta_0)\right)^+-\left(\overset{(2)}{\psi}(z,x;\eta_0)\right)^-=\overset{(2)}{\Psi}(z-x;\eta_0)-\overset{(1)}{\Psi}(z-x;\eta_0),\\ \left[\overset{(1)}{\mathcal{P}}(D_z,n)\overset{(1)}{\psi}(z,x;\eta_0)\right]^+-\left[\overset{(2)}{\mathcal{P}}(D_z,n)\overset{(2)}{\psi}(z,x;\eta_0)\right]^-=\\ =\overset{(2)}{\mathcal{P}}(D_z,n)\overset{(2)}{\Psi}(z-x;\eta_0)-\overset{(1)}{\mathcal{P}}(D_z,n)\overset{(1)}{\Psi}(z-x;\eta_0).\end{gathered} \tag{24}$$

Problem (23)–(24) has a unique solution which is to be sought for in the form of single-layer potentials

$$\overset{(1)}{\psi}(y,x;\eta_0)=\int\limits_S\overset{(1)}{\Psi}(y-\xi;\eta_0)\overset{(1)}{g}(\xi)\,d_\xi S,\ \ y\in D_1,\ \ x\in D_1\cup D_2, \tag{25}$$

$$\overset{(2)}{\psi}(y,x;\eta_0)=\int\limits_S\overset{(2)}{\Psi}(y-\xi;\eta_0)\overset{(2)}{g}(\xi)\,d_\xi S,\ \ y\in D_2,\ \ x\in D_1\cup D_2. \tag{26}$$

Taking into account the properties of single-layer potentials from (24), (25), (26), we obtain

$$\int\limits_S\overset{(1)}{\Psi}(z-\xi;\eta_0)\overset{(1)}{g}(\xi)\,d_\xi S-\int\limits_S\overset{(2)}{\Psi}(z-\xi;\eta_0)\overset{(2)}{g}(\xi)\,d_\xi S=$$

$$= \overset{(2)}{\Psi}(z-x;\eta_0) - \overset{(1)}{\Psi}(z-x;\eta_0), \tag{27}$$

$$\overset{(1)}{g}(z) + \int_S \Big[\overset{(1)}{\mathcal{P}}(D_z,n) \overset{(1)}{\Psi}(z-\xi;\eta_0) \Big] \overset{(1)}{g}(\xi)\, d_\xi S +$$

$$+ \overset{(2)}{g}(\xi) - \int_S \Big[\overset{(2)}{\mathcal{P}}(D_z,n) \overset{(2)}{\Psi}(z-\xi;\eta_0) \Big] \overset{(2)}{g}(\xi)\, d_\xi S =$$

$$= \Big[\overset{(2)}{\mathcal{P}}(D_z,n) \overset{(2)}{\Psi}(z-x;\eta_0) - \overset{(1)}{\mathcal{P}}(D_z,n) \overset{(1)}{\Psi}(z-x;\eta_0) \Big] =$$

$$= \Phi(z-x,\eta_0). \tag{28}$$

We introduce the notation [2]

$$\forall\, z \in S: \ \ \overset{(l)}{\mathcal{H}}\varphi(z) = \int_S \overset{(l)}{\Psi}(z-y;\eta_0)\varphi(y)\, d_y S,$$

$$\overset{(l)}{\mathcal{K}}\varphi(z) = \int_S \Big[\overset{(l)}{\mathcal{P}}(D_z,n) \overset{(l)}{\Psi}(z-y;\eta_0) \Big] \varphi(y)\, d_y S,$$

$$\overset{(l)}{\mathcal{K}^*}\varphi(z) = \int_S \Big[\overset{(l)}{\mathcal{P}}(D_z,n) \overset{(l)}{\Psi}(z-y;\eta_0) \Big]^* \varphi(y)\, d_y S, \tag{29}$$

$$\overset{(l)}{\mathcal{L}^{\pm}}\varphi(z) = \lim_{D_{\frac{1}{2}} \ni x \to z \in S} \Big[\overset{(l)}{\mathcal{P}}(D_z,n) \overset{(l)}{V}(x;\varphi) \Big] =$$

$$= \Big[\overset{(l)}{\mathcal{P}}(D_z,n) \overset{(l)}{V}(z;\varphi) \Big]^{\pm},$$

where

$$\overset{(l)}{V}(x;\varphi) = \int_S \Big[\overset{(l)}{\mathcal{P}}(D_z,n) \overset{(l)}{\Psi}(x-y;\eta_0) \Big]^* \varphi(y)\, d_y S.$$

In view of notation (29), equations (27), (28) take the form

$$\overset{(1)}{\mathcal{H}}\overset{(1)}{g}(z) - \overset{(2)}{\mathcal{H}}\overset{(2)}{g}(z) = \Psi(z-x;\eta_0), \tag{30}$$

$$\big(\mathcal{J} + \overset{(1)}{\mathcal{K}}\big)\overset{(1)}{g}(z) - \big(-\mathcal{J} + \overset{(2)}{\mathcal{K}}\big)\overset{(2)}{g}(z) = \Phi(z-x;\eta_0). \tag{31}$$

Applying the operation $\overset{(1)}{B} = \overset{(1)}{\mathcal{L}} + (-\mathcal{J} + \overset{(1)}{\mathcal{K}})$ to both parts of equation (30), we obtain

$$\begin{cases} \overset{(1)}{B}\overset{(1)}{\mathcal{H}}\overset{(1)}{g}(z) - \overset{(2)}{B}\overset{(2)}{\mathcal{H}}\overset{(2)}{g}(z) = \overset{(1)}{B}\Psi(z-x;\eta_0), \\ \big(\mathcal{J} + \overset{(1)}{\mathcal{K}}\big)\overset{(1)}{g}(z) - \big(-\mathcal{J} + \overset{(2)}{\mathcal{K}}\big)\overset{(2)}{g}(z) = \Phi(z-x;\eta_0) \end{cases} \tag{32}$$

or, taking into account that $\overset{(l)}{\mathcal{L}}\overset{(l)}{\mathcal{H}} = -\mathcal{J} + \overset{(l)}{\mathcal{K}}{}^2$, we rewrite system (32) in the form

$$\begin{cases} \big(-\mathcal{J} + \overset{(1)}{\mathcal{K}}{}^2\big)\overset{(1)}{g}(z) + \big(-\mathcal{J} + \overset{(1)}{\mathcal{K}}{}^2\big)\overset{(1)}{\mathcal{H}}\overset{(1)}{g}(z) - \overset{(1)}{B}\overset{(2)}{\mathcal{H}}\overset{(2)}{g}(z) = \\ \qquad = \overset{(1)}{B}\Psi(z-x;\eta_0), \\ \big(\mathcal{J} + \overset{(1)}{\mathcal{K}}\big)\overset{(1)}{g}(z) - \big(-\mathcal{J} + \overset{(2)}{\mathcal{K}}\big)\overset{(2)}{g}(z) = \Phi(z-x;\eta_0). \end{cases} \tag{33}$$

System (33) is a system of singular integral equations of normal type, with the index equal to zero, while the corresponding homogeneous system has only a trivial solution and therefore the nonhomogeneous system is solvable for an arbitrary right-hand part, i.e. problem (23), (24) has a unique solution representable in form (25), (26). The existence of the Green tensor is thereby proved.

Let us rewrite system (33) in the form

$$Ag + \mathcal{K}g = F, \tag{34}$$

where

$$Ag = \begin{Vmatrix} -\mathcal{J} & 0 \\ 0 & \mathcal{J} \end{Vmatrix} \begin{Vmatrix} \overset{(1)}{g} \\ \overset{(2)}{g} \end{Vmatrix},$$

$$\mathcal{K}g = \begin{Vmatrix} \overset{(1)}{\mathcal{K}}{}^2 + \big(-\mathcal{J} + \overset{(1)}{\mathcal{K}}\big)\overset{(1)}{\mathcal{H}} & -\overset{(1)}{B}\overset{(2)}{\mathcal{H}} \\ \overset{(1)}{\mathcal{K}} & -\overset{(2)}{\mathcal{K}} \end{Vmatrix} \begin{Vmatrix} \overset{(1)}{g} \\ \overset{(2)}{g} \end{Vmatrix},$$

$$F = \begin{Vmatrix} F_1 \\ F_2 \end{Vmatrix}, \quad F_1 = \overset{(1)}{B}\Psi, \quad F_2 = \Phi.$$

From (34) we have

$$g + A^{-1}\mathcal{K}g = A^{-1}F. \tag{35}$$

Since this system is of normal type, there exists its regularizing operator K^*. Applying the operation K^* to both parts of the system and denoting by R the resolvent of the obtained equation, we obtain

$$g = (A + K^*)F + R(\mathcal{J} + K^*)F. \tag{36}$$

Further, we obtain the estimates of the Green tensor

$$G(x,y;\eta_0) < \frac{ce^{-\delta|x-y|}}{|x-y|}, \quad \frac{\partial}{\partial y} G(x,y;\eta_0) < \frac{ce^{-\delta|x-y|}}{|x-y|^2},$$

where

$$(x,y) \in (\overline{D}_1 \cup \overline{D}_2) \times (\overline{D}_1 \cup \overline{D}_2), \quad \delta > 0, \quad c > 0$$

and

$$G(x,y;\eta_0) = \big[G(x,y;\eta_0)\big]^*,$$

where $[\ ]^*$ – transposition, i.e.

$$G_{pq}(x,y;\eta_0) = G_{qp}(x,y;\eta_0).$$

A solution of problem (20), (21) is represented as a sum of two problems

$$\overset{(l)}{V}(x,\tau) = \overset{(l)}{V}_{(0)}(x,\tau) + \overset{(l)}{V}_{(0)}(x,\tau), \quad x \in \overline{D}_l, \quad l = 1,2,$$

where $\overset{(l)}{V}_{(0)}$ is a solution of the problem with a homogeneous equation and nonhomogeneous contact conditions and, vice versa, $\overset{(l)}{V}_{(1)}$ is a solution of the problem with a nonhomogeneous equation and homogeneous contact conditions. The former problem has a unique solution representable in the form

$$\overset{(1)}{V}_{(0)} = \int_S \overset{(1)}{\Psi}(x-y)\overset{(1)}{g}(y,\tau)\, d_yS, \quad x \in D_1, \tag{37}$$

$$\overset{(2)}{V}_{(0)} = \int_S \overset{(2)}{\Psi}(x-y)\overset{(2)}{g}(y,\tau)\, d_yS, \quad x \in D_2 \tag{38}$$

while the latter problem is equivalent to the integral equation

$$2V_{(1)}(x,\tau) + r\tau^2 \int_D G(x,y)V_{(1)}(y,\tau)\, dy =$$

$$= -\int_D G(x,y)H(y,\tau)\, dy, \quad x \in D = D_1 \cup D_2, \tag{39}$$

where

$$V_{(1)} = \begin{cases} \overset{(1)}{V}_{(1)}, & x \in D_1, \\ \overset{(2)}{V}_{(1)}, & x \in D_1, \end{cases} \quad H = \begin{cases} \overset{(1)}{H} = \overset{(1)}{\Phi} + r_1\tau^2 \overset{(1)}{V}_{(0)}, & x \in D_1, \\ \overset{(2)}{H} = \overset{(2)}{\Phi} + r_2\tau^2 \overset{(2)}{V}_{(0)}, & x \in D_2, \end{cases}$$

$$G=\begin{cases}\overset{(1)}{G}, & x\in D_1,\\ \overset{(2)}{G}, & x\in D_2,\end{cases}\qquad r=\begin{cases}r_1, & x\in D_1,\\ r_2, & x\in D_2.\end{cases}$$

If $S\in\mathcal{L}_2(\alpha)$, $f\in C^{2,\beta}(S)$, $F\in C^{1,\beta}(S)$, $\overset{(1)}{g}$, $\overset{(2)}{g}\in C^{1,\beta}(S)$, $0<\beta<\alpha\le 1$, then we obtain the following estimates with respect to the complex parameter τ:

$$\begin{gathered}\big|\overset{(l)}{V}(x,\tau)\big|\le\frac{C}{|\tau|^4}\,,\quad\left|\frac{\partial\overset{(l)}{V}(x,\tau)}{\partial x_i}\right|\le\frac{C}{|\tau|^{2\frac{2}{5}}}\,,\quad x\in D_l,\ \ l=1,2,\\ \left|\frac{\partial^2\overset{(l)}{V}(x,\tau)}{\partial x_k x_i}\right|\le\frac{C(d)}{|\tau|^{1\frac{7}{40}}}\,,\quad x\in\overline{D}_l^{\,*}\subset D_l,\ \ l=1,2.\end{gathered}\tag{40}$$

Together with the above-given arguments, estimates (40) prove the theorem of the existence of a solution of the initial dynamic problem.

Theorem 3 *If $S\in\mathcal{L}_2(\alpha)$ and the conditions*

$$\begin{aligned}&\forall\, t\in I:\ \ \frac{\partial^p}{\partial t^p}\,f^*(\cdot\,,t)\in C^2(S),\ \ p=\overline{0,7},\\ &\forall\, z\in S:\ \ f^*(z,\cdot)\in C^7(I),\\ &\forall\, t\in I:\ \ \frac{\partial^p}{\partial t^p}\,F^*(\cdot\,,t)\in C^1(S),\ \ p=\overline{0,7},\\ &\forall\, z\in S:\ \ F^*(z,\cdot)\in C^7(I)\end{aligned}\tag{41}$$

and

$$\begin{aligned}&\forall\, x\in D_l:\ \ \frac{\partial^m\overset{(l)}{\Phi}}{\partial t^m}=0,\ \ m=\overline{0,3},\ \ l=1,2,\\ &\forall\, z\in S:\ \ \frac{\partial^m f^*(z,0)}{\partial t^m}=0,\ \ m=\overline{0,5},\\ &\forall\, z\in S:\ \ \frac{\partial^m F^*(z,0)}{\partial t^m}=0,\ \ m=\overline{0,5}\end{aligned}\tag{42}$$

are fulfilled, then problem (13)–(15) *has a unique regular solution of the form*

$$\overset{(l)}{U}(x,t)=\frac{1}{2\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}e^{\tau t}\overset{(l)}{V}(x,\tau)\,d\tau,$$

where $\overset{(l)}{V}(x,\tau)$ is a solution of the pseudo-oscillation problem (20)–(21).

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