INVESTIGATION OF THE FIRST AND THE SECOND EXTERIOR PLANE BOUNDARY VALUE PROBLEMS OF STEADY STATE OSCILLATIONS IN THE LINEAR THEORY OF ELASTIC MIXTURES

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Abstract

The displacement vectors are represented in the form of combinations of special potentials and singular integral equations of the normal type with zero index are obtained for the first and second boundary value problem of the steady oscillations in the theory of elastic mixtures. It is proved that the corresponding homogenous singular integral equations in the case of positive frequencies have only the trivial solution.

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1^0. The homogeneous equations of steady state oscillations in the linear theory of an isotropic mixture of two elastic solids in the matrix form can be written as [1]

$$A(\partial x,\sigma)u = A(\partial x)u + \sigma^2 E(\rho)u = 0, \tag{1.1}$$

where

$$A(\partial x) = \begin{bmatrix} A^{(1)}(\partial x) & A^{(2)}(\partial x) \\ A^{(2)}(\partial x) & A^{(3)}(\partial x) \end{bmatrix}_{4\times 4}, \qquad A^{(p)}(\partial x) = \left[A^{(p)}_{kj}(\partial x)\right]_{2\times 2}, \quad p = 1,2,3, \tag{1.2}$$

$$A^{(1)}_{kj}(\partial x) = a_1\delta_{kj}\Delta + b_1\frac{\partial^2}{\partial x_k \partial x_j}, \quad A^{(2)}_{kj}(\partial x) = c\delta_{kj}\Delta + d\frac{\partial^2}{\partial x_k \partial x_j},$$
$$A^{(3)}_{kj}(\partial x) = a_2\delta_{kj}\Delta + b_2\frac{\partial^2}{\partial x_k \partial x_j}, \quad \delta_{kj} = \begin{cases} 1, & k = j, \\ 0, & k = j; \end{cases} \tag{1.3}$$

$$E(\rho) = \begin{bmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_1 & 0 & 0 \\ 0 & 0 & \rho_2 & 0 \\ 0 & 0 & 0 & \rho_2 \end{bmatrix}, \tag{1.4}$$

$u = \{u', u''\}^\top = \{u'_1, u'_2, u''_1, u''_2\}^\top = \{u_1, u_2, u_3, u_4\}^\top$, u' and u'' are partial displacements, Δ is the Laplace operator, $x = (x_1, x_2) \in R^2$;

$$\begin{gathered} a_1 = \mu_1 - \lambda_5, \quad a_2 = \mu_2 - \lambda_5, \quad b_1 = \mu_1 + \lambda_1 + \lambda_5 - \rho_2\alpha_2\rho^{-1}, \\ b_2 = \mu_2 + \lambda_2 + \lambda_5 + \rho_1\alpha_2\rho^{-1}, \quad \alpha_2 = \lambda_3 - \lambda_4, \\ \rho = \rho_1 + \rho_2, \quad c = \mu_3 + \lambda_5, \\ d = \mu_3 + \lambda_3 - \lambda_5 - \rho_1\alpha_2\rho^{-1} \equiv \mu_3 + \lambda_4 - \lambda_5 + \rho_2\alpha_2\rho^{-1}. \end{gathered} \tag{1.5}$$

μ_1, μ_2, μ_3, λ_p, $p = \overline{1,5}$, are elastic constants, $\sigma > 0$ is the frequency parameter, ρ_1 and ρ_2 are partial densities. We assume that [1]

$$\begin{gathered} \mu_1 > 0, \quad \lambda_5 < 0, \quad \Delta_1 = \mu_1\mu_2 - \mu_3^2 > 0, \quad \lambda_1 - \rho_2\alpha_2\rho^{-1} + \frac{2}{3}\mu_1 > 0, \\ (\lambda_1 - \rho_2\alpha_2\rho^{-1} + \frac{2}{3}\mu_1)(\lambda_2 + \rho_1\alpha_2\rho^{-1} + \frac{2}{3}\mu_2) > (\lambda_3 - \rho_1\alpha_2 + \frac{2}{3}\mu_3)^2. \end{gathered} \tag{1.6}$$

A homogeneous system of equations of statics of the theory of elastic mixtures is written as

$$A(\partial x)u = 0. \tag{1.7}$$

By

$$T(\partial x, n(x))u(x) = M_1\frac{\partial u}{\partial n(x)} + M_2\frac{\partial u}{\partial s(x)} + M_3 u \tag{1.8}$$

we denote the stress vector, where [2]

$$M_1 = \begin{bmatrix} a & 0 & c_0 & 0 \\ 0 & a & 0 & c_0 \\ c_0 & 0 & b & 0 \\ 0 & c_0 & 0 & b \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0 & a - 2\mu_1 & 0 & c_0 - 2\mu_3 \\ 2\mu_1 - a & 0 & 2\mu_3 - c_0 & 0 \\ 0 & c_0 - 2\mu_3 & 0 & b - 2\mu_2 \\ 2\mu_3 - c_0 & 0 & 2\mu_2 - b & 0 \end{bmatrix}, \tag{1.9}$$

$$M_3 = \begin{bmatrix} -b_1 n_2\frac{\partial}{\partial x_2} & b_1 n_2\frac{\partial}{\partial x_1} & -dn_2\frac{\partial}{\partial x_2} & -dn_2\frac{\partial}{\partial x_1} \\ b_1 n_1\frac{\partial}{\partial x_2} & -b_1 n_2\frac{\partial}{\partial x_1} & dn_1\frac{\partial}{\partial x_2} & -dn_1\frac{\partial}{\partial x_1} \\ -dn_2\frac{\partial}{\partial x_2} & dn_2\frac{\partial}{\partial x_1} & -b_2 n_2\frac{\partial}{\partial x_2} & b_2 n_2\frac{\partial}{\partial x_1} \\ dn_1\frac{\partial}{\partial x_2} & -dn_1\frac{\partial}{\partial x_1} & b_2 n_1\frac{\partial}{\partial x_2} & -b_2 n_1\frac{\partial}{\partial x_1} \end{bmatrix}$$

$\frac{\partial}{\partial n(x)} = n_1\frac{\partial}{\partial x_1} + n_2\frac{\partial}{\partial x_2}$, $\frac{\partial}{\partial s(x)} = -n_2\frac{\partial}{\partial x_1} + n_1\frac{\partial}{\partial x_2}$, here $n = (n_1, n_2)$ is a unit vector

$$a = a_1 + b_1 > 0, \quad b = a_2 + b_2 > 0, \quad c_0 = c + d. \tag{1.10}$$

The following assertion is true [4]

Theorem 1.1 *If* $u = \{u', u''\}^\top = \{u_1, u_2, u_3, u_4\}^\top$ *is solution of equation* (1.1) *then*

$$u = \sum_{p=1}^{4} \overset{(p)}{V}, \quad (\Delta + k_p^2)\overset{(p)}{V} = 0, \tag{1.11}$$

$$\overset{(p)}{V} = \{\overset{(p)}{V}{}', \overset{(p)}{V}{}''\}^\top = \{\overset{(p)}{V}_1, \overset{(p)}{V}_2, \overset{(p)}{V}_3, \overset{(p)}{V}_4\}^\top, \quad p = \overline{1,4},$$

where

$$\overset{(e)}{V} = -c_e(\Delta + k_{3-l})(\Delta + k_3^2)(\Delta + k_4^2)u, \quad e = 1, 2$$
$$\overset{(e+2)}{V} = -c_{e+2}(\Delta + k_1^2)(\Delta + k_2^2)(\Delta + k_{5-e}^2)u, \quad e = 1, 2 \tag{1.12}$$

$$\overset{(e)}{V}{}'' = A_e \overset{(e)}{V}{}', \quad \overset{(e+2)}{V}{}'' = A_{e+2} \overset{(e+2)}{V}{}', \quad e = 1, 2, \tag{1.13}$$

$$A_e = \frac{\rho_1\sigma^2 - ak_e^2}{c_0 k_e^2} = \frac{c_0 k_e^2}{\rho_2\sigma^2 - bk_e^2}, \quad e = 1, 2$$
$$A_{e+2} = \frac{\rho_1\sigma^2 - a_1 k_{e+2}^2}{ck_{e+2}^2} = \frac{ck_{e+2}^2}{\rho_2\sigma^2 - a_2 k_{e+2}^2}, \quad e = 1, 2, \tag{1.14}$$

$$\mathrm{rot}\, \overset{(e)}{V}{}' = \mathrm{rot}\, \overset{(e)}{V}{}'' = 0, \quad \mathrm{div}\, \overset{(e+2)}{V}{}' = \mathrm{div}\, \overset{(e+2)}{V}{}'' = 0,$$

$$k_j^2 = -\eta_j\sigma^2, \quad k_j = \sqrt{-\eta_j}\sigma > 0, \quad \eta_j < 0, \quad j = \overline{1,4}. \tag{1.15}$$

η_1*,* η_2 *and* η_3*,* η_4 *are real numbers and represent roots of the quadratic equations, respectively:*

$$d_1\eta^2 + (a\rho_2 + b\rho_1)\eta + \rho_1\rho_2 = 0, \quad d_2\eta^2 + (a_1\rho_2 + a_2\rho_1)\eta + \rho_1\rho_2 = 0, \tag{1.16}$$

$$d_2 = a_1 a_2 - c^2 > 0, \quad d_1 = ab - c_0^2 > 0,$$

$$c_p = \prod_{j=1}^{4} (k_p^2 - k_j^2)^{-1}, \quad j \neq p, \quad p = 1, 4. \tag{1.17}$$

The coefficients c_p*,* $p = \overline{1,4}$*, satisfy the following conditions:*

$$\sum_{p=1}^{4} c_p = \sum_{p=1}^{4} c_p k_p^2 = \sum_{p=1}^{4} c_p k_p^4 = 0, \quad \sum_{p=1}^{4} c_p k_p^6 = 1,$$
$$\sum_{j=1}^{3} (k_j^2 - k_4^2)c_j = 0, \quad \sum_{j=1}^{2} (k_j^2 - k_4^2)(k_j^2 - k_3^2)c_j = 0, \tag{1.18}$$
$$c_1 k_2^2 k_3^2 k_4^2 + c_2 k_1^2 k_3^2 k_4^2 + c_3 k_1^2 k_2^2 k_4^2 + c_4 k_1^2 k_2^2 k_3^2 = -1.$$

Let D^+ be a bounded domain surrounded by a closed curve $S \in C^{2,\beta}$, $0 < \beta \le 1$, $\overline{D}^+ = D^+ \cup S$, $D^- \equiv R^2 \setminus \overline{D}^+$, $\overline{D}^- = D^- \cup S$. In what follows we provide $u \in C^2(D^+) \cap C^1(\overline{D}^+)$, $[u \in C^2(D^-) \cap C^1(\overline{D}^-)]$ moreover, in the case of an unbounded domain D^- we assume that $u = \{u_1, u_2, u_3, u_4\}^\top$ satisfies the Sommerfeld–Kupradze type radiation conditions:

$$\overset{(p)}{V}(x) = O(|x|^{-\frac{1}{2}}), \quad \frac{\partial \overset{(p)}{V}(x)}{\partial (x)} - ik_p \overset{(p)}{V} = O(|x|^{-\frac{3}{2}}), \qquad (1.19)$$
$$|x|^2 = x_1^2 + x_2^2, \quad p = \overline{1,4}.$$

The first and the second exterior BVP_s are formulated as follows: find a regular solution to equation (1.1) in D^- satisfying one of the boundary conditions:

$$1) \quad u^+(z) = \lim_{D^- \ni x \to z \in S} u(x) = f(z), \quad (\text{Problem} \quad (\overset{\sigma}{I})^-_{0,f}), \qquad (1.20)$$

$$2) \quad \{T(\partial z_1 n(z))u(z)\}^- = F(z), \;\; z \in S, \quad (\text{Problem} \quad (\overset{\sigma}{II})^-_{0,f}), \qquad (1.21)$$

where $f \in \{f_1, f_2, f_3, f_4\}^\top$ and $F = \{F_1, F_2, F_3, F_4\}^\top$ are sufficiently smooth vector-functions. Throughout this paper $n(z)$ denotes the exterior to D^+ unit normal vector at the point $z \in S$.

Just in the some way as in the three-dimensional case (which is considered in [2]) in two-dimensional case we can prove the following

Theorem 1.2 *The homogeneous boundary value problems* $(\overset{\sigma}{I})^-_{0,0}$ *and* $(\overset{\sigma}{II})^-_{0,0}$ *have only the trivial solution in the class of regular vectors.*

2^0. The matrix of fundamental solutions of equation (1.1) has the form [4]

$$\phi(x-y,\sigma)$$
$$= \frac{\pi}{2i}\Big\{H_1 \frac{k_3^2 H_0^{(1)}(k_3 r) - k_4^2 H_0(k_4 r)}{k_3^2 - k_4^2} - \frac{\sigma^2}{d_2} H_2 \frac{H_0^{(1)}(k_3 r) - H_0(k_4 r)}{k_3^2 - k_4^2}$$
$$-H_3 \sum_{p=1}^{4} c_p k_p^4 H_0^{(1)}(k_p r) + \sigma^2 H_4 \sum_{p=1}^{4} k_p^2 c_p H_0^{(1)}(k_p r) + \sigma^4 H_5 \sum_{p=1}^{4} c_p H_0^{(1)}(k_p r)\Big\}, \qquad (2.1)$$

where $r = |x-y| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)}$,

$$H_1 = \begin{bmatrix} e_1 & 0 & e_2 & 0 \\ 0 & e_1 & 0 & e_0 \\ e_2 & 0 & e_3 & 0 \\ 0 & e_2 & 0 & e_3 \end{bmatrix} \quad H_2 = \begin{bmatrix} \rho_2 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & 0 \\ 0 & 0 & \rho_1 & 0 \\ 0 & 0 & 0 & \rho_1 \end{bmatrix}, \qquad (2.2)$$

$$H_{2+j} = \begin{bmatrix} R_j \frac{\partial^2}{\partial x_1^2} & R_j \frac{\partial^2}{\partial x_1 \partial x_2} & P_j \frac{\partial^2}{\partial x_1^2} & P_j \frac{\partial^2}{\partial x_1 x_2} \\ R_j \frac{\partial^2}{\partial x_1 \partial x_2} & R_j \frac{\partial^2}{\partial x_2^2} & P_j \frac{\partial^2}{\partial x_1 \partial x_2} & P_j \frac{\partial^2}{\partial x_2^2} \\ P_j \frac{\partial^2}{\partial x_1^2} & P_j \frac{\partial^2}{\partial x_1 \partial x_2} & Q_j \frac{\partial^2}{\partial x_1^2} & Q_j \frac{\partial^2}{\partial x_1 x_2} \\ P_j \frac{\partial^2}{\partial x_1 \partial x_2} & P_j \frac{\partial^2}{\partial x_2^2} & Q_j \frac{\partial^2}{\partial x_1 \partial x_2} & Q_j \frac{\partial^2}{\partial x_2^2} \end{bmatrix}, \quad j = 1, 2, 3; \tag{2.3}$$

$$e_1 = \frac{a_2}{d_2}, \quad e_2 = -\frac{c}{d_2}, \quad e_3 = \frac{a_1}{d_2}, \quad R_1 = e_4, \quad P_1 = e_5, \quad Q_1 = e_6,$$

$$R_2 = d_3, \quad P_2 = d_4, \quad Q_2 = d_5, \quad R_3 = \nu_1, \quad P_3 = \nu_2, \quad Q_3 = \nu_3$$

$$e_4 = \frac{c_0(a_2 d - c b_2) + b(cd - a_2 b_1)}{d_1 d_2}, \quad e_5 = \frac{c_0(a_2 b_1 - cd) + a(c b_2 - a_2 d)}{d_1 d_2}$$

$$= \frac{c_0(a_1 b_2 - cd) + b(c b_1 - a_1 d)}{d_1 d_2}, \quad e_6 = \frac{c_0(a_1 d - c b_1) + a(cd - a_1 b_1)}{d_1 d_2}; \tag{2.4}$$

$$d_3 = \frac{d(2c+d) - b_1(2a_2 + b_2)}{d_1 d_2}\rho_2, \quad d_4 = \frac{c b_2 - a_2 d)\rho_1 + (c b_1 - a_1 d)\rho_2}{d_1 d_2},$$

$$d_5 = \frac{d(2c+d) - b_2(2a_1 + b_1)}{d_1 d_2}\rho_1, \quad \nu_1 = \frac{b_1 \rho_2^2}{d_1 d_2}, \quad \nu_2 = \frac{d \rho_1 \rho_2}{d_1 d_2}, \quad \nu_3 = \frac{b_2 \rho_1^2}{d_1 d_2},$$

$$H_0^{(1)}(k_2) = J_0(kr) + i N_0(kr), \tag{2.5}$$

here $H_0^{(1)}(kr)$, $J_0(kr)$ and $N_0(kr)$ are the first kind Hankel function, Bessel function and Neumann function of zero order, respectively,

$$J_0(kr) = \frac{2}{\pi} \sum_{s=0}^{\infty} \frac{(-1)^s}{(s!)^2} \left(\frac{kr}{2}\right)^{2s}$$

$$N_0(kr) = \frac{2}{\pi} J_0(kr) \ln \frac{kr}{2} - \frac{2}{\pi} \sum_{s=1}^{\infty} \frac{(-1)^s}{(s!)^2} \left(\frac{kr}{2}\right)^{2s} \frac{\Gamma'(s+1)}{\Gamma(s+1)}, \tag{2.6}$$

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx.$$

It is evident that the matrix $\phi(x-y,\sigma)$ is symmetric.

Moreover, on the basis of the equation

$$\frac{\pi}{2i} H_0^{(1)}(kr) = \ln r - \frac{k^2}{4} r^2 \ln r + \text{const} + O(r^2), \tag{2.7}$$

we easily conclude that ϕ has a logarithmic singularity. It can be shown that columns of the matrices $\phi(x-y,\sigma)$ are solutions to equation (1.1) with respect to x for any $x \neq y$; moreover, $\phi(x-y,\sigma) \in C^{\infty}(R^2 | \{x = y\})$.

In what follows we need the fundamental matrix of the operator $A(\partial x)$ [2]

$$\phi(x-y) = \mathrm{Re}(m \ln \sigma_0 + \frac{1}{4} n \frac{\overline{\sigma}_0}{\sigma_0}) \tag{2.8}$$

where $\sigma_0 = z - \zeta$, $\overline{\sigma}_0 = \overline{z} - \overline{\zeta}$, $z = x_1 + ix_2$, $\zeta = y_1 + iy_2$, $\overline{z} = x_1 - ix_2$, $\overline{\zeta} = y_1 - iy_2$;

$$m = \begin{bmatrix} m_1 & 0 & m_2 & 0 \\ 0 & m_1 & 0 & m_2 \\ m_2 & 0 & m_3 & 0 \\ 0 & m_2 & 0 & m_3 \end{bmatrix}, \quad n = \begin{bmatrix} e_4 & ie_4 & e_5 & ie_5 \\ ie_4 & -e_4 & ie_5 & -e_5 \\ e_5 & ie_5 & e_6 & ie_6 \\ ie_5 & -e_5 & ie_6 & -e_6 \end{bmatrix}, \tag{2.9}$$

$$m_k = e_k + \frac{1}{2} e_{k+3}, \quad k = 1, 2, 3. \tag{2.10}$$

By simple calculations we conclude that

$$\overset{(0)}{\phi}(x-y,\sigma) = \phi(x-y,\sigma) - \phi(x-y) = O(r^2 \ln r). \tag{2.11}$$

In solving boundary value problems by the method of potential theory not only the fundamental matrix is of a great importance but also some other matrices of singular solutions to equations (1.7) and (1.1)

$$T(x-y, n(x)) = T(\partial x, n(x))\phi(x-y), \tag{2.12}$$

$$[T(y-x, n(y))]^\top = [T(\partial y, n(y))\phi(y-x)]^\top, \tag{2.13}$$

$$T(x-y, n(x), \sigma) = T(\partial x, n(x))\phi(x-y,\sigma), \tag{2.14}$$

$$[T(y-x, n(y), \sigma)]^\top = [T(\partial y, n(y))\phi(y-x,\sigma)]^\top, \tag{2.15}$$

where the symbol "T" denotes the transposition of a matrix.

We have (see [2]):

$$T(x-y, n(x)) = \mathrm{Im} \frac{\partial}{\partial s(x)}[(E + iA)\ln \sigma_0 + \frac{1}{2} B \frac{\overline{\sigma}_0}{\sigma_0}], \tag{2.16}$$

$$[T(y-x, n(y))]^\top = \mathrm{Im} \frac{\partial}{\partial s(x)}[m \ln \sigma_0 + \frac{1}{4} n \frac{\overline{\sigma}_0}{\sigma_0}](m^{-1} + i\varkappa_N), \tag{2.17}$$

where E is the 4×4 unit matrix,

$$A = \begin{bmatrix} 0 & 1-A_1 & 0 & -A_2 \\ A_1 - 1 & 0 & A_2 & 0 \\ 0 & -A_3 & 0 & 1-A_4 \\ A_3 & 0 & A_4 - 1 & 0 \end{bmatrix},$$
$$B = \begin{bmatrix} B_1 & iB_1 & B_2 & iB_2 \\ iB_1 & -B_1 & iB_2 & -B_2 \\ B_3 & iB_3 & B_4 & iB_4 \\ iB_3 & -B_3 & iB_4 & -B_4 \end{bmatrix}, \tag{2.18}$$

$$A_1 = 2(\mu_1 m_1 + \mu_3 m_2), \quad A_2 = 2(\mu_1 m_2 + \mu_3 m_3), \quad A_3 = 2(\mu_3 m_1 + \mu_2 m_2),$$
$$A_4 = 2(\mu_3 m_2 + \mu_2 m_3), \quad B_1 = \mu_1 e_4 + \mu_3 e_5, \quad B_2 = \mu_1 e_5 + \mu_3 e_6,$$
$$B_3 = \mu_2 e_5 + \mu_3 e_4, \quad B_4 = \mu_2 e_6 + \mu_3 e_5, \tag{2.19}$$

$$m^{-1} = \frac{1}{\Delta_0} \begin{bmatrix} m_3 & 0 & -m_2 & 0 \\ 0 & m_3 & 0 & -m_2 \\ -m_2 & 0 & m_1 & 0 \\ 0 & -m_2 & 0 & m_1 \end{bmatrix}, \quad \Delta_0 = m_1 m_3 - m_2^2, \tag{2.20}$$

$$\varkappa_N = \begin{bmatrix} 0 & 2\mu_1 - \frac{m_3}{\Delta_0} & 0 & 2\mu_3 + \frac{m_2}{\Delta_0} \\ \frac{m_3}{\Delta_0} - 2\mu_1 & 0 & -2\mu_3 - \frac{m_2}{\Delta_0} & 0 \\ 0 & 2\mu_3 + \frac{m_2}{\Delta_0} & 0 & 2\mu_2 - \frac{m_1}{\Delta_0} \\ -2\mu_3 - \frac{m_2}{\Delta_0} & 0 & \frac{m_1}{\Delta_0} - 2\mu_2 & 0 \end{bmatrix}, \tag{2.21}$$

It is easy to check, that columns of the matrices (2.13) and (2.15), respectively, are solutions of equations (1.7) and (1.1) with respect to the variable x for $x \neq y$. It is also clear, that the elements of the matrices (2.11)-(2.15) are singular kernels in the Cauchy Principal Value sense.

On basis of (2.7) and

$$ae_4 + c_0 e_5 + e_1 b_1 + e_2 d = 0, \quad ce_5 + be_6 + e_2 d + b_2 e_3 = 0,$$

we obtain

$$\begin{aligned} &\overset{(0)}{T}(x-y, n(x), \sigma) \\ &\qquad = T(x-y, n(x), \sigma) - T(x-y, n(x)) = O(\ln|x-y|), \\ &[\overset{(0)}{T}(y-x, n(y), \sigma)]^\top \\ &\qquad = [T(y-x, n(y), \sigma)]^\top - [T(y-x, n(y))]^\top = O(\ln|x-y|). \end{aligned} \tag{2.22}$$

Using the method given in [1] (see also [2]) we can also establish

Theorem 2.1 *Let $S \in C^{1,\beta}$ $0 < \beta \leq 1$, and let $u = \{u_1, u_2, u_3, u_4\}^\top$ be a regular solution of the equation* (1.1) *in D^+. Then*

$$\begin{aligned} u(x) = \frac{1}{2\pi} \int_S \Big\{ &[T(y-x, n(y), \sigma)]^\top u^+(y) \\ &-\phi(y-x, \sigma)(T(\partial y, n(y)) u(y))^+ \Big\} d_y s, \end{aligned} \tag{2.23}$$

where $\phi(y-x, \sigma)$ is the basic fundamental matrix and $[T(y-x, n(y), \sigma)]^\top$ is given by (2.15).

3^0. Let us introduce single and double layer potentials.

The vector

$$V(x) = \frac{1}{\pi} \int_S \phi(x-y,\sigma) h(y) d_y S \tag{3.1}$$

where $\phi(x-y,\sigma)$ is given by (2.1) and $h = \{h_1, h_2, h_3, h_4\}^\top$ is a continuous vector, is called a single layer potential, and

$$U(x) = \frac{1}{\pi} \int_S [T(y-x, n(y), \sigma)]^\top g(y) d_y S \tag{3.2}$$

where $[T(y-x, n(y), \sigma)]^\top$ is given by (2.15) and $g = \{g_1, g_2, g_3, g_4\}^\top$ is a Hölder continuous vector, is called a double layer potential.

It is evident that the potentials introduced above are solutions to equation (1.1) in $R^2 \setminus S$. These potentials have certain continuity and jump properties when the point x either crosses the surface S or approaches some point $z = (z_1, z_2) \in S$ from D^+. Those properties can be obtained very easily since the kernel–functions of the above potentials are quite. Similar to the potentials corresponding to the statical case [2], [1].

Therefore we will only formulate results.

Theorem 3.1 *A single layer potential defined by* (3.1) *is continuous on the whole plane and*

$$[T(\partial z, n(z)) V(z)]^\pm = \mp h(z) + \frac{1}{\pi} \int_S T(z-y, n(z), \sigma) h(y) d_y S. \tag{3.3}$$

Theorem 3.2 *Let $U(x)$ be a double layer potential given by* (3.2). *Then for only $z \in S$*

$$U^\pm(z) = \pm g(z) + \frac{1}{\pi} \int_S [T(y-z, n(y), \sigma)]^\top g(y) dyS, \tag{3.4}$$

and

$$[T(\partial z, n(z)) U(z)]^+ = [T(\partial z, n(z)) U(z)]^- \tag{3.5}$$

for $g \in C^1(S)$.

In the case of unbounded domain D^- it is assumed that potentials (3.1) and (3.2) satisfy the Sommerfeld–Kupradze type radiation conditions (see (1.19)).

4^0. To prove the theorems existence of solutions of problems $(\overset{\sigma}{I})^-_{0,f}$ and $(\overset{\sigma}{II})^-_{0,f}$ we use the following lemmata.

Lemma 4.1 *The homogeneous problem*

$$A(\partial x,\sigma)u(x)=0,\quad x\in D^+\quad [T(\partial zn(z))u(z)+iu(z)]^+=0,\quad z\in S,\quad (4.1)$$

has only the trivial solution in the class of regular vectors.

it Proof. Let a complex vectors $u(x)$ be a regular solution of the problem (4.1). Note that for a regular solution of system (1.1) we have the Green formula [2]

$$\int_{D^+}[T(u,\overline{u})-\sigma^2\overline{u}E(\rho)u]dy=\int_S\overline{u}^+\overline{(T(\partial y,n(y))u(y))^+}d_yS,\qquad (4.2)$$

where $T(u,\overline{u})=T(\overline{u},u)\geq 0$, $\sigma^2\overline{u}E(\rho)u\geq 0$.

Using the condition (4.1) we get

$$\int_{D^+}[T(u,\overline{u})-\sigma^2\overline{u}e(\rho)(u)]dy=-i\int_S|u|^2d_yS,$$

and $u^+(z)=0$, $z\in S$. By virtue of $u^+(z)=0$, $z\in S$, from (4.1) we have $(T(\partial z,n(z))u(z))^+=0$, $z\in S$.

Hence, from (2.23) we obtain $u(x)=0$, $x\in D^+$.

Lemma 4.2 *The homogenous problem*

$$A(\partial x,\sigma)u(x)=0,\quad x\in D^+,\qquad (4.3)$$

$$\left[u(z)+i\int_S k_0|y-z|\left(\int_S k_0(|y-t|)T(\partial t,n(t))u(t)d_tS\right)d_yS\right]^+=0,\quad z\in S$$

has only the trivial solution in the class of regular vectors.

Here $k_0(x)=\frac{\pi i}{2}H_0^{(1)}(ix)$ is the Macdonald function.

it Proof. Let a complex vector $u(x)$ be a regular solution of the problem (4.3). From (4.2) and (4.3) we obtain

$$\int_{D^+}[T(u,\overline{u})-\sigma^2uE(\rho)\overline{u}]dx$$

$$=-i\int_S\left\{\int_S k_0(|y-z|)\left(\int_S k_0(|y-t|)(T(\partial t,n(t))u(t))^+d_tS\right)d_yS\right\}\times\overline{(T(\partial z,n(z))u(z))}^+d_zS$$

$$=-i\int_S\left|\int_S k_0(|y-z|)(T(\partial z,n(z))u(z))^+d_zS\right|^2d_yS.\qquad (4.4)$$

From (4.4) we get

$$\int_S k_0(|y-t|)(T(\partial y, n(y))u(y))^+ d_y S = 0 \quad t \in S. \tag{4.5}$$

Now we will show that $(T(\partial y, n(y))u(y) = 0$, $y \in S$. For this purpose we consider a simple layer metaharmonic potential

$$\begin{gathered}\Pi(x) = \int_S k_0(|x-z|)T(\partial z, n(z))U(z)d_z S,\\ x \in D^+, \quad \Pi = (\Pi_1, \Pi_2, \Pi_3, \Pi_4)^\top,\end{gathered} \tag{4.6}$$

From (4.5) it is evident, that $\Pi(x)$ satisfies the following conditions

$$\Delta\Pi(x) - \Pi(x) = 0, \;\; x \in D^+, \;\; (\Pi(y))^+ = 0, \;\; y \in S. \tag{4.7}$$

In this case Green's formula reads as

$$\int_{D^+} (\Pi_p \Delta \Pi_p + |\operatorname{grad} \Pi_p|^2) dx = \int_S (\Pi_p(y))^+ \Big(\frac{\partial \Pi_p}{\partial n(y)}\Big)^+ d_y S, \; p = \overline{1,4}. \tag{4.8}$$

Whence for the solution of the problem (4.7) we get

$$\int_{D^+} (|\Pi_p|^2 + |\operatorname{grad} \Pi_p|^2) dx = 0, \quad p = \overline{1,4}.$$

Therefore $\Pi_p(x) = 0$, $x \in D^+$, $p = \overline{1,4}$.

Thus we have

$$\Pi(x) = 0, \quad x \in D^+. \tag{4.9}$$

Applying the properties of potential (4.6) and equation (4.9) we derive

$$0 = \Big(\frac{\partial \Pi}{\partial n}\Big)^+ - \Big(\frac{\partial \Pi}{\partial n}\Big)^- = 2(Tu)^+$$

and

$$[T(\partial z, n(z))U(z)]^+ = 0, \quad z \in S. \tag{4.10}$$

Substitution of (4.9) and (4.10) into (2.23) leads to $u(x) = 0$, $x \in D^+$.

5^0. The existence theorems of solutions of problems $(\overset{\sigma}{I})^-_{0,f}$ and $(\overset{\sigma}{II})^-_{0,f}$ we prove by the method given in [3] (see also [1], §10).

We look for a solution to the problem $(\overset{\sigma}{I})_{0,f}$ in the form

$$u(x) = \frac{1}{\pi}\int_S [T(y-x, n(y), \sigma)]^\top g(y) d_y S + \frac{1}{\pi}\int_S \phi(y-x, \sigma) g(y) d_y S, \tag{5.1}$$

where $\phi(y-x,\sigma)$ and $[T(y-x, n(y), \sigma)]^\top$ are given by (2.1) and (2.15), respectively, and the vector g is an unknown Hölder continuous vector.

Due to Theorem 3.1 and 3.2 we get the equation on S

$$-g(z) + \frac{1}{\pi}\int_S [T(y-z,n(y),\sigma)]^\top g(y)dS$$
$$+\frac{1}{\pi}\int_S \phi(y-z,\sigma)g(y)d_yS = f(z), \quad z\in S, \tag{5.2}$$

where $f(z)\in C^{1,\alpha}(s)$, $s\in C^{2,\beta}$, $0<\alpha<\beta\le 1$ is a given vector. If we take into account in (5.2) the singularities of matrices (2.1), (2.8), (2.15), (2.17) and the representations (2.11) and (2.22) then by simple manipulations we can prove that, the index is calculated by the formula

$$\varkappa = \frac{1}{2\pi}\Big[\arg\frac{\det(E+iA^\circ)}{\det(E-iA^\circ)}\Big]_S, \tag{5.3}$$

where

$$A^\circ = -m\varkappa_N = -\begin{bmatrix} 0 & A_1-1 & 0 & A_3 \\ 1-A_1 & 0 & -A_3 & 0 \\ 0 & A_2 & 0 & A_4-1 \\ -A_2 & 0 & 1-A_4 & 0 \end{bmatrix} = -A^\top.$$

By the direct evaluation, we get

$$\det(E-iA_0) = \det(E+iA_0) = 4\Delta_0\Delta_1\Delta_2^\circ, \tag{5.4}$$

here $\Delta_0 = m_1m_3 - m_2^2$, $\Delta_1 = \mu_1\mu_2 - \mu_3^2$, $\Delta_2^\circ = (2-A_1)(2-A_4) - A_2A_3$.

The positive definiteness of the potential energy implies (see [2]) that $\Delta_0 > 0$, $\Delta_1 > 0$ and $\Delta_2^\circ > 0$. Therefore the index (5.3) is equal to zero.

Thus the left-hand side of the equation (5.2) is a singular integral operator of normal type with index equal to zero.

Let us prove that the equation (5.2) is solvable for an arbitrary right-hand side. To this end, let us consider the corresponding homogeneous equation (i,e., $f=0$) (5.2) and show that it has only the trivial solutions.

Let $g_0\in C^{1,\alpha}(S)$ be an arbitrary solution of the homogeneous equation (5.3), i.e.

$$-g_0(z) + \frac{1}{\pi}\int_S [T(y-z,n(y),\sigma)]^\top g_0(y) + \frac{i}{\pi}\int_S \phi(y-z,\sigma)g_0(y)d_yS = 0. \tag{5.5}$$

Let us consider the vector

$$U_0(x) = \frac{1}{\pi}\int_S [T(y-x,n(y),\sigma)]^\top g_0(y)$$

$$+\frac{i}{\pi}\int_S \phi(y-x,\sigma)g_0(y)d_yS, \quad x\in D^{\pm}. \tag{5.6}$$

In this case equation (5.5) corresponds so the boundary condition

$$(U_0(z))^- = 0, \quad x\in D^-. \tag{5.7}$$

Due to (5.7) we obtain

$$(T(\partial z, n(z))U_0(z))^- = 0, \quad z\in S. \tag{5.8}$$

By virtue (3.3), (3.4), and (5.7), (5.8) we can write

$$U_0^+(z) = 2g_0(z),\ (T(\partial z, n(z))U_0(z))^+ = -2ig_0(z), \quad z\in S. \tag{5.9}$$

From (5.9) we have that $U_0(x)$ is a solution of the problem (4.1) and by Lemma 4.1 we get

$$U_0(x) = 0, \quad x\in D^+. \tag{5.10}$$

From (5.10) and (5.9), we have $g_0(z) = 0$, $z\in S$.

Thus, the homogeneous equation $(5.2)_0$ has only the trivial solutions. Consequently the non-homogeneous equation (5.2) has only one solution $g\in C^{1,\alpha}(S)$, $0<\alpha<\beta\le 1$.

Let us now consider problem $(\overset{\sigma}{II})^-_{0,f}$. We look for its solutions as

$$u(x) = \frac{1}{\pi}\int_S \phi(x-y,\sigma)h(y)d_yS$$
$$+\frac{i}{\pi}\int_S [T(y-x,n(y),\sigma)]^\top \varphi(y)d_yS, \quad x\in D^-, \tag{5.11}$$

where $\phi(x-y,\sigma)$ and $[T(y-x,n(y),\sigma)]^\top$ are given by (2.1) and (2.15), respectively

$$\varphi(y) = \int_S k_0(|y-\tau|)\int_{S_0} k_0(|\tau-\zeta|)h(\zeta)d_\zeta S d_\tau S, \quad y\in S, \tag{5.12}$$

$k_0(x)$ is the Macdonald function and h is a Hölder continuous unknown vector.

By Theorem (3.1) and (3.2) we get

$$h(z) + \frac{1}{\pi}\int_S T(z-y,n(z),\sigma)h(y)d_yS$$
$$+\frac{1}{\pi}\lim_{D^-\ni x\to z\in S} T(\partial x, n(x))\int_S [T(y-z,n(y),\sigma)]^\top$$
$$\times\int_S k_0(|y-\tau|)\int_S k_0(|\tau-\zeta|)h(\zeta)d_\zeta S d_\tau S d_y S = F(z), \ \ z\in S; \tag{5.13}$$

with F given on S, $F \in C^{1,\alpha}(S)$, $S \in C^{2,\beta}$ $0 < \alpha < \beta \leq 1$.

Rewrite (5.13) by means of the matrices $T(x-y, n(x))$, $[T(y-x, n(y))]^\top$ $\overset{(0)}{T}(y-x), n(x,\sigma)$ and $\overset{(0)}{T}(y-x, n(y),\sigma)]^\top$, in the form

$$h(z) + \frac{1}{\pi}\int_S T(z-y, n(z))h(y)d_yS$$
$$+\frac{1}{\pi}\int_S \overset{0}{T}(z-y, n(z),\sigma)h(y)d_yS + \frac{i}{\pi}\lim_{D^-\ni x\to z\in S} T(\partial x, n(x))$$
$$\times\left\{\int_S [T(y-z, n(y))]^\top \int_S k_0(|y-\tau|)\int_S k_0(|\tau-\zeta|)h(\zeta)d_\zeta S d_\tau S d_y S\right.$$
$$+\int_S [\overset{0}{T}(y-z, n(y),\sigma)]^\top \int_S k_0(|y-\tau|)$$
$$\left.\times\int_S k_0(|y-\zeta|)h(\zeta)d_\zeta S d_\tau S d_y S\right\} = F(z), \quad z \in S. \tag{5.14}$$

If we take into consideration the singularities of the matrices $\overset{0}{T}(x-y, n(x),\sigma)$, $[\overset{0}{T}(y-x, n(x),\sigma)]^\top$, $T(x-y, n(x))$, $[T(y-x), n(y)]^\top$ representation (2.16) and (2.17), and the equation

$$\lim_{D^-\ni x\to z\in S}\frac{\partial}{\partial s(x)}\int_S \frac{\partial \ln|y-x|}{\partial s(y)}\int_S k_0(|y-\tau|)$$
$$\times\int_S k_0(|\tau-\zeta|)h(\zeta)d_\zeta S d_\tau S d_y S = -\int_S \frac{\partial \ln|y-z|}{\partial s(z)}\int_S \frac{\partial k_0(|y-\tau|)}{\partial s(y)}$$
$$\times\int_S k_0(|\tau-\zeta|)h(\zeta)d_\zeta S d_\tau S d_y S, \quad z \in S, \tag{5.15}$$

after elementary calculations we get

$$h(z) + \frac{1}{\pi}\int_S T(z-y), n(z))h(y)d_yS$$
$$+\int_S N(z,y)h(y)d_yS = F(z), \quad z \in S, \tag{5.16}$$

where $N(z,y)$ is Fredholm's Kernel.

By means of (2.16) and (2.18) we can prove that the equation (5.16), i.e. (5.14) is singular integral equation of normal type with index equal to zero.

Now we will show, that (5.16), i.e. (5.14) is solvable for an arbitrary right-hand side. To this and we have to show that the corresponding homogeneous integral equation has no nontrivial solution. In fact, let ho, be some solution to that homogeneous equation (5.14). We have then

$$(T(\partial z, n(z))u_0(z))^- = 0, \quad z \in S, \tag{5.17}$$

where $u_0(x)$ is given be (5.11) with ho instead of h. Further, (5.17) and the uniqueness theorem for the problem $(\overset{\sigma}{II})^-_{0,0}$ yields

$$u_0(x) = 0, \quad x \in D^-. \tag{5.18}$$

From (5.18) we get

$$(u_0(z))^- = 0, \quad z \in S. \tag{5.19}$$

By virtue of (3.1), (3.2), (5.16), and (5.18) we can write

$$u_0^+(z) = 2i\varphi_0(z) = 2i \int_S k_0(|z-\tau|) \int_S k_0(|\tau-\zeta|) h_0(\zeta) d_\zeta S d_\tau S, \quad z \in S, \tag{5.20}$$

$$(T(\partial z, n(z))u_0(z))^+ = -2h_0(z), \quad z \in S. \tag{5.21}$$

From (5.20) and (5.21) it follows that $u_0(z)$ is a solution of the problem (4.3) and by Lemma 4.2 we obtain

$$u_0(x) = 0, \quad x \in D^+. \tag{5.22}$$

From (5.22) and (5.21) it follows that $h_0(z) = 0$, $z \in S$.

Thus the homogeneous integral equation corresponding to (5.14) has no nontrivial solution. Consequently, equation (5.14) has only one solution $h(z) \in C^{1,\alpha}(s)$, $0 < \alpha < \beta \leq 1$, for an arbitrary right-hand side $F \in C^{1,\alpha}(s)$.

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