SOME PROBLEMS OF THE STRESSES CONCENTRATION FOR NON-SHALLOW CYLINDRICAL SHELLS ON THE BASIS OF I. VEKUA'S THEORY

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Abstract

In the present paper on the basis of I. Vekua' s theory we consider well-known problem of stress concentration for non-shallow cylindrical shell. To solve the problems of plate and cylindrical shell algorithm of full automation is devised by means of the net method. The program named VEKMUS is constructed. By means of the program the problems of stress concentration shallow and non-shallow cylindrical shells are solved.

Key words and phrases: Shallow and non-shallow shells, Stress concentration.

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1 Introduction

In his studies I. Vekua, by means method of the reduction of three-dimensional problems of elasticity to two-dimensional ones, constructed several versions of the refined theory of thin and shallow shells, containing the regular process [1].

Under thin and shallow shells I.Vekua means three-dimensional shell-type elastic bodies, satisfying the following conditions

$$a_\alpha^\beta - x_3\, b_\alpha^\beta \cong a_\alpha^\beta \Rightarrow x_3\, b_\alpha^\beta \cong 0, \quad -h(x^1,x^2) \le x_3 \le h(x^1,x^2)(\alpha,\beta=1,2), \tag{1.1}$$

where a_α^β and b_α^β are mixed components of the metric and curvature tensors of the middle surface S of the shell Ω, x_3 is the thickness coordinate, varying in the interval $(-h,h)$, $2h$ is the shell thickness. Further, by expanding the unknown three-dimensional displacement and stress fields into the Furier-Legendre series and satisfying the boundary conditions on face surfaces $x_3 = \pm h$ I. Vekua obtained the sequence of two-dimensional differential equations, containing the regular process. Besides, it is evident every sequence will contain the unremovable error which is generated by the assumption the form (1.1). Therefore it is of great importance to get rid of this assumption.

The assumption of the type (1.1) means that the interior geometry of the shell does not vary in thickness and therefore such kind of shells are usually called the shells with non-varying geometry.

Under non-shallow shells will be meant elastic bodies tree from the assumption of the type (1.1), or more exactly

$$|x_3\, b_\alpha^\beta| \leq q < 1 \quad (\alpha, \beta = 1, 2). \tag{1.2}$$

Such kind of shells are called shells with varying in thickness geometry or non-shallow shells [2].

In the present paper we consider well-known problem of stress concentration for non-shallow cylindrical shell. To solve the problems of plate and cylindrical shell algorithm of full automation is devised by means of the net method. The programme named VEKMUS is constructed [3]. By means of the programme the problems of stress concentration for shallow and non-shallow cylindrical shells are solved.

2 The Coordinate System Connected Normally with the Surface Shallow and Non-Shallow Cylindrical Shells

Let Ω denote a Cylindrical Shell and domain of the space occupied by this shell. Inside the Cylindrical Shell we consider a Cylindrical surface S with respect to which shell Ω lies symmetrically. The surface S is called a midsurface of the shell Ω. The radius $\vec{R}$ of any point of the domain Ω can be represented in the form [1]

$$\vec{R}(x^1, x^2, x^3) = \vec{r}(x^1, x^2) + x^3\, \vec{n}(x^1, x^2),$$

where $\vec{r}$ and $\vec{n}$ are the radius-vector and the unit vector of the normal of the midsurface S $(x_3 = 0)$, respectively. (x^1, x^2) - are the Gaussian parameters of the surface S, and x^3 (or x_3) is thickness coordinate, where

$$-h \leq x_3 = x^3 \leq h,$$

when S is a circular cylindrical surface we have

$$\vec{R} = \vec{r} + x^3\, \vec{n} = R\left(\cos\frac{x^2}{R}\, \vec{e}_1 + \sin\frac{x^2}{R}\, \vec{e}_2\right) + x^1\, \vec{e}_3 + x^3\, \vec{n},$$

where R is the radius of cylinder, $x^2 = R\varphi$ and φ is the polar angle, $\vec{e}_i$ $(i = 1, 2, 3)$ are unit vectors of the cartesian coordinate system.

Covariant and contrvariant basis vectors $\vec{R}_i$ and $\vec{R}^i$ of the surface $\vec{S}$: x^3=const and corresponding basis vector $\vec{r}_i$ and $\vec{r}^{\,i}$ of the midsurface S : $x^3 = 0$ are connected by the following relations:

$$\vec{R}_1 = \frac{\partial \vec{R}}{\partial x^1} = \vec{r}_1 = \vec{e}_3, \quad \vec{R}^{\,1} = \vec{r}^{\,1} = \vec{e}_3, \vec{R}_2 = \frac{\partial \vec{R}}{\partial x^2} = \Big(1 + \frac{x^3}{R}\Big)\vec{r}_2,$$

$$\vec{R}^{\,2} = \frac{\vec{r}^{\,2}}{1 + \frac{x^3}{R}} = \frac{1}{1 + \frac{x^3}{R}} \frac{\vec{r}_2}{R^2}, \vec{R}_3 = \frac{\partial \vec{R}}{\partial x^3} = \vec{r}_3 = \vec{n}, \quad \vec{R}^{\,3} = \vec{r}^{\,3} = \vec{n}, \tag{2.1}$$

where $\vec{r}_2 = \frac{\partial \vec{r}}{\partial x^2} = -\sin\frac{x^2}{R}\,\vec{e}_1 + \cos\frac{x^2}{R}\,\vec{e}_2$, $\vec{n} = \cos\frac{x^2}{R}\,\vec{e}_1 + \sin\frac{x^2}{R}\,\vec{e}_2$.

The main quadratic forms of the surfaces S and $\widehat{S}$ have the forms

$$I = ds^2 = a_{\alpha\beta}\,dx^\alpha dx^\beta, \quad II = k_s\,ds^2 = b_{\alpha\beta}\,dx^\alpha\,dx^\beta,$$

$$\widehat{I} = d\,\widehat{s}^{\,2} = g_{\alpha\beta}\,dx^\alpha\,dx^\beta, \quad \widehat{II} = \widehat{k}_s\,d\,\widehat{s}^{\,2} = \widehat{b}_{\alpha\beta}\,dx^\alpha dx^\beta, \tag{2.2}$$

where k_s and $\widehat{k}_s$ are the curvatures of the surfaces S and $\widehat{S}$,

$$a_{11} = \vec{r}_1\,\vec{r}_1 = 1, \quad a_{22} = \vec{r}_2\,\vec{r}_2 = 1, \quad a_{12} = 0, \Big(a^{11} = 1, \quad a^{22} = 1, \quad a^{12} = 0\Big),$$

$$g_{11} = 1, \quad g_{22} = \Big(1 + \frac{x^3}{R}\Big)^2, \quad g_{12} = 0, \Big(g^{11} = 1, \quad g^{22} = \frac{1}{\big(1 + \frac{x^3}{R}\big)^2},$$

$$g^{12} = 0\Big), \Big(\sqrt{a} = 1, \quad \sqrt{g} = \sqrt{a}\Big(1 + \frac{x^3}{R}\Big)\Big),$$

$$b_{11} = 0, \quad b_{12} = b_{21} = 0, \quad b_{22} = b_2^2 = -\frac{1}{R}$$

$$\widehat{b}_{11} = 0, \quad \widehat{b}_{12} = \widehat{b}_{21} = 0, \quad \widehat{b}_{22} = -R\Big(1 + \frac{x_3}{R}\Big), \widehat{b}_2^2 = -\frac{1}{R}\,\frac{1}{1 + \frac{x_3}{R}}\,. \tag{2.3}$$

For shallow shells the relation of the type (1.1)

$$\vec{R}_1 = \vec{r}_1, \quad \vec{R}_2 \cong \vec{r}_2, \quad \vec{R}^{\,1} = \vec{r}^{\,1}, \quad \vec{R}^{\,2} \cong \vec{r}^{\,2}, \quad \sqrt{g} \cong \sqrt{a}$$

$$\widehat{b}_1^1 = b_1^1 = 0, \quad \widehat{b}_2^1 = \widehat{b}_1^2 = 0, \quad \widehat{b}_2^2 \cong -\frac{1}{R}\,,$$

are valid.

3 A System of Equations of Equilibrium and Hooke's Law for Non-Shallow Cylindrical Shells

The equilibrium equation and Hooke's law in a tensor notation takes the form

$$\begin{cases} \nabla_\alpha\left(\sqrt{\frac{g}{a}}\,\sigma_1^\alpha\right) + \dfrac{\partial\sqrt{\frac{g}{a}}\,\sigma_1^3}{\partial x^3} + \sqrt{\frac{g}{a}}\,\phi_1 = 0, \\ \nabla_\alpha\left(\sqrt{\frac{g}{a}}\,\sigma_2^\alpha\right) + \dfrac{1}{R}\left(\sqrt{\frac{g}{a}}\,\sigma_2^2\right) + \dfrac{\partial\left(\sqrt{\frac{g}{a}}\,\sigma_2^3\right)}{\partial x^3} + \sqrt{\frac{g}{a}}\,\phi_2 = 0, \\ \qquad\qquad \left(\sqrt{\frac{g}{a}} = 1 + \dfrac{x^3}{R}\right), \\ \nabla_\alpha\left(\sqrt{\frac{g}{a}}\,\sigma_3^\alpha\right) - \dfrac{1}{R}\left(\sqrt{\frac{g}{a}}\,\sigma_2^2\right) + \dfrac{\partial\left(\sqrt{\frac{g}{a}}\,\sigma_3^3\right)}{\partial x^3} + \sqrt{\frac{g}{a}}\,\phi^3 = 0, \end{cases} \tag{3.1}$$

$$\left(\sqrt{\frac{g}{a}} = 1 + \frac{x_3}{R}, \quad \sigma_j^i = \vec{\sigma}^i \vec{r}_j, \quad \vec{r}_3 = \vec{n}\right)$$

where

$$\begin{cases} \sigma_1^1 = (\lambda + 2\mu)(\vec{r}^{\,1}\,\partial_1\,\vec{u}) + \lambda\left(\dfrac{\vec{r}^{\,2}\,\partial_2\vec{u}}{1+\frac{x_3}{R}} + \partial_3 u_3\right), \\ \sigma_2^1 = \mu\left(\vec{r}_2\,\partial^1\,\vec{u} + \dfrac{\vec{r}^{\,1}\,\partial_2\vec{u}}{1+\frac{x_3}{R}}\right), \\ \sigma_3^1 = \mu(\vec{r}^{\,1}\,\partial_3\vec{u} + \vec{n}\,\partial^1\,\vec{u}), \end{cases} \tag{3.2}$$

$$\begin{cases} \sigma_1^2 = \dfrac{\mu}{1+\frac{x_3}{R}}\left(\vec{r}^{\,2}\,\partial_1\vec{u} + \dfrac{\vec{r}_1\,\partial_2\vec{u}}{1+\frac{x_3}{R}}\right), \\ \sigma_2^2 = (\lambda+2\mu)\dfrac{\vec{r}^{\,2}\,\partial_2\vec{u}}{\left(1+\frac{x_3}{R}\right)^2} + \dfrac{\lambda}{1+\frac{x_3}{R}}\left(\vec{r}^{\,1}\,\partial_1\vec{u} + \partial_3 u_3\right), \\ \sigma_3^2 = \dfrac{\mu}{1+\frac{x_3}{R}}\left(\vec{r}^{\,2}\,\partial_3\vec{u} + \dfrac{\vec{n}\,\partial_2\vec{u}}{1+\frac{x_3}{R}}\right), \end{cases} \tag{3.3}$$

$$\begin{cases} \sigma_1^3 = \mu\left(\vec{n}\,\partial_1\vec{u} + \partial_3 u_1\right), \\ \sigma_2^3 = \mu\left(\dfrac{\vec{n}\,\partial_2\vec{u}}{1+\frac{x_3}{R}} + \partial_3 u_2\right), \\ \sigma_3^3 = \lambda\left(\vec{r}^{\,1}\,\partial_1\vec{u} + \dfrac{\vec{r}^{\,2}\,\partial_2\vec{u}}{1+\frac{x_3}{R}}\right) + (\lambda+2\mu)(\vec{n}\,\partial_3\vec{u}). \end{cases} \tag{3.4}$$

4 I. Vekua's Method of Reduction

Since the system of Legendre polynomials $\{P_m(\frac{x^3}{h})\}$ is complete in the interval $[-h,h]$, for equation (3.1) we obtain the equivalent infinite system of two-dimensional equations:

$$\begin{cases} \nabla_\alpha \overset{(m)}{\sigma_1^\alpha} - \dfrac{2m+1}{h}\Big(\overset{(m-1)}{\sigma_1^3} + \overset{(m-3)}{\sigma_1^3} + \cdots\Big) + \overset{(m)}{F_1} = 0, \\ \nabla_\alpha \overset{(m)}{\sigma_2^\alpha} + \dfrac{1}{R}\overset{(m)}{\sigma_2^3} - \dfrac{2m+1}{h}\Big(\overset{(m-1)}{\sigma_2^3} + \overset{(m-3)}{\sigma_2^3} + \cdots\Big) + \overset{(m)}{F_2} = 0, \\ \nabla_\alpha \overset{(m)}{\sigma_3^\alpha} - \dfrac{1}{R}\overset{(m)}{\sigma_2^2} - \dfrac{2m+1}{h}\Big(\overset{(m-1)}{\sigma_3^3} + \overset{(m-3)}{\sigma_3^3} + \cdots\Big) + \overset{(m)}{F_3} = 0, \end{cases} \tag{4.1}$$

where

$$\Big(\overset{(m)}{\sigma_j^i}, \overset{(m)}{\phi_j}\Big) = \frac{2m+1}{2h}\int_{-h}^{h}\Big(\sqrt{\frac{g}{a}}\,\sigma_j^i, \sqrt{\frac{g}{a}}\,\phi_j\Big) P_m\Big(\frac{x_3}{h}\Big) dx_3$$
$$(i,j=1,2,3),$$

$$\overset{(m)}{F_i} = \overset{(m)}{\phi_i} + \frac{2m+1}{2h}\Big[\sqrt{\frac{g_+}{a}}\,\overset{(+)}{\sigma_i^3} - (-1)^m\sqrt{\frac{g_-}{a}}\,\overset{(-)}{\sigma_i^3}\Big], \quad \sqrt{\frac{g_\pm}{a}} = 1 \pm \frac{h}{R}.$$

Thus we have obtained the infinite system of two-dimensional equation of the theory of shells for which the boundary conditions on the face surfaces $(x_3 = \pm h)$ are satisfied, i,e $\overset{(\pm)}{\sigma^3} = \sigma^3(x^1, x^2, \pm h)$ is the preassigned vector field.

For the Hooke's law we have [2]

$$\begin{aligned} \overset{(m)}{\sigma}_{11} &= (\lambda+2\mu)\Big[\partial_1 \overset{(m)}{u_1} + \frac{h}{R}\Big(\frac{m}{2m-1}\partial_1 \overset{(m-1)}{u_1} + \frac{m+1}{2m+3}\partial_1 \overset{(m+1)}{u_1}\Big)\Big] \\ &\quad + \lambda\Big[\Big(\partial_2 \overset{(m)}{u_2} + \frac{1}{R}\overset{(m)}{u_3}\Big) + \overset{(m)}{u_3'} + \frac{h}{R}\overset{(m)}{u_3''}\Big], \\ \overset{(m)}{\sigma}_{12} &= \mu\Big[\partial_2 \overset{(m)}{u_1} + \partial_1 \overset{(m)}{u_2} + \frac{h}{R}\Big(\frac{m}{2m-1}\partial_1 \overset{(m-1)}{u_2} + \frac{m+1}{2m+3}\partial_1 \overset{(m+1)}{u_2}\Big)\Big], \\ \overset{(m)}{\sigma}_{21} &= \mu\Big[\partial_1 \overset{(m)}{u_2} + \sum_{s=0}^{\infty} A_{ms}\partial_2 \overset{(s)}{u}_1\Big], \\ \overset{(m)}{\sigma}_{22} &= (\lambda+2\mu)\sum_{s=0}^{\infty} A_{ms}\Big(\partial_2 \overset{(s)}{u_3} + \frac{1}{R}\overset{(s)}{u_3}\Big) + \lambda\Big(\partial_1 \overset{(m)}{u_1} + \overset{(m)}{u_3'}\Big), \qquad (4.2) \\ \overset{(m)}{\sigma}_{13} &= \mu\Big[\partial_1 \overset{(m)}{u_3} + \overset{(m)}{u_1'} + \frac{h}{R}\Big(\frac{m}{2m-1}\partial_1 \overset{(m-1)}{u_3} + \frac{m+1}{2m+3}\partial_1 \overset{(m+1)}{u_3} + \overset{(m)}{u_1''}\Big)\Big], \end{aligned}$$

$$\overset{(m)}{\sigma}_{23} = \mu\left[\overset{(m)}{u'_2} + \sum_{s=0}^{\infty} A_{ms}\Big(\partial_2 \overset{(m)}{u_3} - \frac{1}{R}\overset{(s)}{u_2}\Big)\right],$$

$$\overset{(m)}{\sigma}_{31} = \mu\left[\overset{(m)}{u'_1} + \partial_1 \overset{(m)}{u_3} + \frac{h}{R}\Big(\overset{(m)}{u''_1} + \frac{m}{2m-1}\,\partial_1 \overset{(m-1)}{u_3} + \frac{m+1}{2m+3}\,\partial_1 \overset{(m+1)}{u_3}\Big)\right],$$

$$\overset{(m)}{\sigma}_{32} = \mu\left[\overset{(m)}{u'_2} + \frac{h}{R}\overset{(m)}{u''_2} + \partial_2 \overset{(m)}{u_3} - \frac{1}{R}\overset{(m)}{u_2}\right],$$

$$\overset{(m)}{\sigma}_{33} = \lambda\bigg[\partial_1 \overset{(m)}{u_1} + \partial_2 \overset{(m)}{u_2} + \frac{1}{R}\overset{(m)}{u_3}$$
$$+ \frac{h}{R}\Big(\frac{m}{2m-1}\,\partial_1 \overset{(m-1)}{u_1} + \frac{m+1}{2m+3}\,\partial_1 \overset{(m+1)}{u_1}\Big)\bigg]$$
$$+ (\lambda+2\mu)\Big(\overset{(m)}{u'_3} + \frac{h}{R}\overset{(m)}{u''_3}\Big),$$

where

$$\overset{(m)}{u'_i} = \frac{2m+1}{h}\Big(\overset{(m+1)}{u} + \overset{(m+3)}{u}\Big),$$

$$\overset{(m)}{u''_i} = \frac{1}{h}\left[m\overset{(m)}{u} + (2m+1)\Big(\overset{(m+2)}{u} + \overset{(m+4)}{u} + \cdots\Big)\right],$$

$$\overset{(m)}{u_i} = \frac{2m+1}{h}\int_{-h}^{h} u_i\, P_m\left(\frac{x_3}{h}\right) dx_3,$$

$$A_{ms} = \frac{2m+1}{2h}\int_{-h}^{h} \frac{P_m\big(\frac{x_3}{h}\big)\, P_s\big(\frac{x_3}{h}\big) dx_3}{1+\frac{x_3}{R}}$$
$$= (-1)^{m+s}(2m+1)\,\frac{R}{h}\left[\begin{array}{ll} P_m\left(\dfrac{R}{h}\right) Q_s\left(\dfrac{R}{h}\right), & m \le s, \\ Q_m\left(\dfrac{R}{h}\right) P_s\left(\dfrac{R}{h}\right), & m \ge s. \end{array}\right.$$

Here $Q_s(x)$ is the Legendre function of second order.

For the system (4.1) and (4.2) we consider the following basic boundary value problems:

Problem I. Find a solution of the system (4.1) and (4.2) consistent with the physical condition of the type

$$\overset{(m)}{\vec{\sigma}}_{(l)} = \overset{(m)}{\sigma}_{(ll)}\vec{l} + \overset{(m)}{\sigma}_{(ls)}\vec{s} + \overset{(m)}{\sigma}_{(ln)}\vec{n} = \overset{(m)}{\vec{f}}_{(l)} \quad (\text{on } \partial S),$$

where $\overset{(m)}{f}_{(l)}$ is the given vector function on the contour ∂S. By $\sigma_{(ll)}, \sigma_{(ls)}, \sigma_{(ln)}$ we denote respectively the normal, tangential and transversal tangential stress acting on the area with the normal $\vec{l}$.

Problem II. Find a solution of the system (3.1) and (3.2) consistent with the kinematic boundary condition of the type

$$\overset{(m)}{\vec{U}} = \overset{(m)}{u}_{(l)}\vec{l} + \overset{(m)}{u}_{(s)}\vec{s} + \overset{(m)}{u}_{(3)}\vec{n} = \vec{g} \quad \text{on } \partial S,$$

where $\vec{g}$ is the given vector function on ∂S, and by $u_{(l)}, u_{(s)}, u_{(3)}$ are denoted respectively the normal, tangential and transversal displacements of the vector $\vec{U}$.

5 Problems of this stress concentration

On bases of I. Vekua's approximate N=1 an automatic numerical program named VEKMUS is compiled to calculate the stress concentration for the cylindrical shells and plates weakened by rectangular holes. The calculation of shells is possible by means of corresponding mathematical models of shallow and non-shallow shell theory.

The corresponding models for shallow shells with thickness $2h$ =const have the following form

$$\begin{aligned}
&\frac{\partial \overset{(0)}{\sigma}_{11}}{\partial x_1} + \frac{\partial \overset{(0)}{\sigma}_{21}}{\partial x_2} = -\frac{1}{2h}\Big(\overset{(+)}{P_1} - \overset{(-)}{P_1}\Big),\\
&\frac{\partial \overset{(0)}{\sigma}_{12}}{\partial x_1} + \frac{\partial \overset{(0)}{\sigma}_{22}}{\partial x_2} + \frac{1}{R}\overset{(0)}{\sigma}_{23} = -\frac{1}{2h}\Big(\overset{(+)}{P_2} - \overset{(-)}{P_2}\Big),\\
&\frac{\partial \overset{(0)}{\sigma}_{13}}{\partial x_1} + \frac{\partial \overset{(0)}{\sigma}_{23}}{\partial x_2} - \frac{1}{R}\overset{(0)}{\sigma}_{22} = -\frac{1}{2h}\Big(\overset{(+)}{P_3} - \overset{(-)}{P_3}\Big),\\
&\frac{\partial \overset{(1)}{\sigma}_{11}}{\partial x_1} + \frac{\partial \overset{(1)}{\sigma}_{21}}{\partial x_2} - \frac{3}{h}\overset{(0)}{\sigma}_{13} = -\frac{3}{2h}\Big(\overset{(+)}{P_1} + \overset{(-)}{P_1}\Big),\\
&\frac{\partial \overset{(1)}{\sigma}_{12}}{\partial x_1} + \frac{\partial \overset{(1)}{\sigma}_{22}}{\partial x_2} + \frac{1}{R}\overset{(1)}{\sigma}_{23} - \frac{3}{h}\overset{(0)}{\sigma}_{23} = -\frac{3}{2h}\Big(\overset{(+)}{P_2} + \overset{(-)}{P_2}\Big),\\
&\frac{\partial \overset{(1)}{\sigma}_{13}}{\partial x_1} + \frac{\partial \overset{(1)}{\sigma}_{23}}{\partial x_2} - \frac{1}{R}\overset{(1)}{\sigma}_{22} - \frac{3}{h}\overset{(0)}{\sigma}_{33} = -\frac{3}{2h}\Big(\overset{(+)}{P_3} + \overset{(-)}{P_3}\Big),
\end{aligned} \tag{5.1}$$

where

$$\overset{(0)}{\sigma}_{11} = (\lambda + 2\mu)\frac{\partial \overset{(0)}{u}_1}{\partial x_1} + \lambda\left(\frac{\partial \overset{(0)}{u}_2}{\partial x_2} + \frac{1}{R}\overset{(0)}{u}_3 + \frac{1}{h}\overset{(1)}{u}_3\right),$$

$$\overset{(0)}{\sigma}_{12} = \overset{(0)}{\sigma}_{21} = \mu\left(\frac{\partial \overset{(0)}{u}_1}{\partial x_2} + \frac{\partial \overset{(0)}{u}_2}{\partial x_1}\right),$$

$$
\begin{aligned}
&\overset{(0)}{\sigma}_{13} = \overset{(0)}{\sigma}_{31} = \mu\left(\frac{\partial \overset{(0)}{u}_3}{\partial x_1} + \frac{1}{h}\overset{(1)}{u}_1\right),\\
&\overset{(0)}{\sigma}_{22} = (\lambda+2\mu)\left(\frac{\partial \overset{(0)}{u}_2}{\partial x_2} + \frac{1}{R}\overset{(0)}{u}_3\right) + \lambda\left(\frac{\partial \overset{(0)}{u}_1}{\partial x_1} + \frac{1}{h}\overset{(1)}{u}_3\right),\\
&\overset{(0)}{\sigma}_{23} = \overset{(0)}{\sigma}_{32} = \mu\left(\frac{\partial \overset{(0)}{u}_3}{\partial x_2} - \frac{1}{R}\overset{(0)}{u}_2 + \frac{1}{h}\overset{(1)}{u}_2\right),\\
&\overset{(0)}{\sigma}_{33} = \lambda\left(\frac{\partial \overset{(0)}{u}_1}{\partial x_1} + \frac{\partial \overset{(0)}{u}_2}{\partial x_2} + \frac{1}{R}\overset{(0)}{u}_3\right) + \frac{\lambda+2\mu}{h}\overset{(1)}{u}_3,\\
&\overset{(1)}{\sigma}_{11} = (\lambda+2\mu)\frac{\partial \overset{(1)}{u}_1}{\partial x_1} + \lambda\left(\frac{\partial \overset{(1)}{u}_2}{\partial x_2} + \frac{1}{R}\overset{(1)}{u}_3\right),\\
&\overset{(1)}{\sigma}_{12} = \overset{(1)}{\sigma}_{21} = \mu\left(\frac{\partial \overset{(1)}{u}_1}{\partial x_2} + \frac{\partial \overset{(1)}{u}_2}{\partial x_1}\right),\\
&\overset{(1)}{\sigma}_{13} = \overset{(1)}{\sigma}_{31} = \mu\frac{\partial \overset{(1)}{u}_3}{\partial x_1},\\
&\overset{(1)}{\sigma}_{22} = \lambda\frac{\partial \overset{(1)}{u}_1}{\partial x_1} + (\lambda+2\mu)\left(\frac{\partial \overset{(1)}{u}_2}{\partial x_2} + \frac{1}{R}\overset{(1)}{u}_3\right),\\
&\overset{(1)}{\sigma}_{23} = \overset{(1)}{\sigma}_{32} = \mu\left(\frac{\partial \overset{(1)}{u}_3}{\partial x_2} - \frac{1}{R}\overset{(1)}{u}_2\right),\\
&\overset{(1)}{\sigma}_{33} = \lambda\left(\frac{\partial \overset{(1)}{u}_1}{\partial x_1} + \frac{\partial \overset{(1)}{u}_2}{\partial x_2} + \frac{1}{R}\overset{(1)}{u}_3\right).
\end{aligned}
\tag{5.2}
$$

In the case non-shallow shells we have

$$
\begin{aligned}
&\frac{\partial \overset{(0)}{\sigma}_{11}}{\partial x_1} + \frac{\partial \overset{(0)}{\sigma}_{21}}{\partial x_2} = -\frac{1}{2h}\left[\left(1+\frac{h}{R}\right)\overset{(+)}{P}_1 - \left(1-\frac{h}{R}\right)\overset{(-)}{P}_1\right],\\
&\frac{\partial \overset{(0)}{\sigma}_{12}}{\partial x_1} + \frac{\partial \overset{(0)}{\sigma}_{22}}{\partial x_2} + \frac{1}{R}\overset{(0)}{\sigma}_{23} = -\frac{1}{2h}\left[\left(1+\frac{h}{R}\right)\overset{(+)}{P}_2 - \left(1-\frac{h}{R}\right)\overset{(-)}{P}_2\right],\\
&\frac{\partial \overset{(0)}{\sigma}_{13}}{\partial x_1} + \frac{\partial \overset{(0)}{\sigma}_{23}}{\partial x_2} - \frac{1}{R}\overset{(0)}{\sigma}_{22} = -\frac{1}{2h}\left[\left(1+\frac{h}{R}\right)\overset{(+)}{P}_3 - \left(1-\frac{h}{R}\right)\overset{(-)}{P}_3\right],\\
&\frac{\partial \overset{(1)}{\sigma}_{11}}{\partial x_1} + \frac{\partial \overset{(1)}{\sigma}_{21}}{\partial x_2} - \frac{3}{h}\overset{(0)}{\sigma}_{31} = -\frac{3}{2h}\left[\left(1+\frac{h}{R}\right)\overset{(+)}{P}_1 + \left(1-\frac{h}{R}\right)\overset{(-)}{P}_1\right],
\end{aligned}
\tag{5.3}
$$

$$\frac{\partial \overset{(1)}{\sigma}_{12}}{\partial x_1} + \frac{\partial \overset{(1)}{\sigma}_{22}}{\partial x_2} + \frac{1}{R}\overset{(1)}{\sigma}_{23} - \frac{3}{h}\overset{(0)}{\sigma}_{32} = -\frac{3}{2h}\left[\left(1+\frac{h}{R}\right)\overset{(+)}{P_2} + \left(1-\frac{h}{R}\right)\overset{(-)}{P_2}\right],$$
$$\frac{\partial \overset{(1)}{\sigma}_{13}}{\partial x_1} + \frac{\partial \overset{(1)}{\sigma}_{23}}{\partial x_2} - \frac{1}{R}\overset{(1)}{\sigma}_{22} - \frac{3}{h}\overset{(0)}{\sigma}_{33} = -\frac{3}{2h}\left[\left(1+\frac{h}{R}\right)\overset{(+)}{P_3} + \left(1-\frac{h}{R}\right)\overset{(-)}{P_3}\right],$$

where

$$\begin{aligned}
\overset{(0)}{\sigma}_{11} =&(\lambda+2\mu)\left(\frac{\partial \overset{(0)}{u}_1}{\partial x_1} + \frac{h}{3R}\frac{\partial \overset{(1)}{u}_1}{\partial x_1}\right) + \lambda\left(\frac{\partial \overset{(0)}{u}_2}{\partial x_2} + \frac{1}{R}\overset{(0)}{u}_3 + \frac{1}{h}\overset{(1)}{u}_3\right),\\
\overset{(0)}{\sigma}_{22} =&\lambda\left(\frac{\partial \overset{(0)}{u}_1}{\partial x_1} + \frac{1}{h}\overset{(1)}{u}_3\right) + (\lambda+2\mu)\sum_{s=0}^{1} A_{0s}\left(\frac{\partial \overset{(s)}{u}_2}{\partial x_2} + \frac{1}{R}\overset{(s)}{u}_3\right),\\
\overset{(0)}{\sigma}_{21} =&\mu\left(\frac{\partial \overset{(0)}{u}_2}{\partial x_1} + \sum_{s=0}^{1} A_{0s}\frac{\partial \overset{(s)}{u}_1}{\partial x_2}\right),\\
\overset{(0)}{\sigma}_{12} =&\mu\left(\frac{\partial \overset{(0)}{u}_1}{\partial x_2} + \frac{\partial \overset{(0)}{u}_2}{\partial x_1} + \frac{h}{3R}\frac{\partial \overset{(1)}{u}_2}{\partial x_1}\right),\\
\overset{(0)}{\sigma}_{13} =&\mu\left(\frac{1}{h}\overset{(1)}{u}_1 + \frac{\partial \overset{(0)}{u}_3}{\partial x_1} + \frac{h}{3R}\frac{\partial \overset{(1)}{u}_3}{\partial x_1}\right),\\
\overset{(0)}{\sigma}_{23} =&\mu\left(\frac{1}{h}\overset{(1)}{u}_2 + \sum_{s=0}^{1} A_{0s}\left(\frac{\partial \overset{(s)}{u}_3}{\partial x_2} - \frac{1}{R}\overset{(s)}{u}_2\right)\right),\\
\overset{(0)}{\sigma}_{31} =&\mu\left(\frac{\partial \overset{(0)}{u}_3}{\partial x_1} + \frac{h}{3R}\frac{\partial \overset{(1)}{u}_3}{\partial x_1} + \frac{1}{h}\overset{(1)}{u}_1\right),\\
\overset{(0)}{\sigma}_{32} =&\mu\left(\frac{\partial \overset{(0)}{u}_3}{\partial x_2} + \frac{1}{h}\overset{(0)}{u}_2 - \frac{\mu}{R}\overset{(0)}{u}_2\right),\\
\overset{(0)}{\sigma}_{33} =&\lambda\left(\frac{\partial \overset{(0)}{u}_1}{\partial x_1} + \frac{\partial \overset{(0)}{u}_2}{\partial x_2} + \frac{1}{R}\overset{(0)}{u}_3 + \frac{h}{3R}\frac{\partial \overset{(1)}{u}_1}{\partial x_1}\right) + \frac{\lambda+2\mu}{h}\overset{(1)}{u}_3,\\
\overset{(1)}{\sigma}_{11} =&(\lambda+2\mu)\left(\frac{\partial \overset{(1)}{u}_1}{\partial x_1} + \frac{h}{R}\frac{\partial \overset{(0)}{u}_1}{\partial x_1}\right) + \lambda\left(\frac{\partial \overset{(1)}{u}_2}{\partial x_2} + \frac{2}{R}\overset{(1)}{u}_3\right),\\
\overset{(1)}{\sigma}_{22} =&\lambda\frac{\partial \overset{(1)}{u}_1}{\partial x_1} + (\lambda+2\mu)\sum_{s=0}^{1} A_{1s}\left(\frac{\partial \overset{(s)}{u}_2}{\partial x_2} + \frac{1}{R}\overset{(s)}{u}_3\right),
\end{aligned}\tag{5.4}$$

$$\overset{(1)}{\sigma}_{21} = \mu \left(\frac{\partial \overset{(1)}{u}_2}{\partial x_1} + \sum_{s=0}^{1} A_{1s} \frac{\partial \overset{(s)}{u}_1}{\partial x_1} \right),$$

$$\overset{(1)}{\sigma}_{12} = \mu \left(\frac{\partial \overset{(1)}{u}_1}{\partial x_2} + \frac{\partial \overset{(1)}{u}_2}{\partial x_1} + \frac{h}{R} \frac{\partial \overset{(0)}{u}_2}{\partial x_1} \right),$$

$$\overset{(1)}{\sigma}_{13} = \mu \left(\frac{\partial \overset{(1)}{u}_3}{\partial x_1} + \frac{h}{R} \frac{\partial \overset{(0)}{u}_3}{\partial x_1} \right),$$

$$\overset{(1)}{\sigma}_{23} = \mu \sum_{s=0}^{1} A_{1s} \left(\frac{\partial \overset{(s)}{u}_3}{\partial x_2} - \frac{1}{R} \overset{(s)}{u}_2 \right),$$

$$\overset{(1)}{\sigma}_{31} = \mu \left(\frac{\partial \overset{(1)}{u}_3}{\partial x_1} + \frac{h}{R} \frac{\partial \overset{(0)}{u}_3}{\partial x_1} \right), \quad \overset{(1)}{\sigma}_{32} = \mu \frac{\partial \overset{(1)}{u}_3}{\partial x_2},$$

$$\overset{(1)}{\sigma}_{33} = \lambda \left(\frac{\partial \overset{(1)}{u}_1}{\partial x_1} + \frac{\partial \overset{(1)}{u}_2}{\partial x_2} + \frac{h}{R} \frac{\partial \overset{(0)}{u}_1}{\partial x_1} \right) + \frac{2(\lambda + \mu)}{R} \overset{(1)}{u}_3,$$

$$A_{00} = \frac{1}{2} \frac{R}{h} \ln \frac{1 + \frac{h}{R}}{1 - \frac{h}{R}}, \quad A_{01} = -\frac{R}{h}(A_{00} - 1),$$

$$A_{10} = -3\frac{R}{h}(A_{00} - 1), \quad A_{11} = 3\left(\frac{R}{h}\right)^2 (A_{00} - 1),$$

$$u_i \approx \overset{(0)}{u}_i + \frac{x_3}{h} \overset{(1)}{u}_i, \quad \sigma_{ij} \approx \overset{(0)}{\sigma}_{ij} + \frac{x_3}{h} \overset{(1)}{\sigma}_{ij}, \quad i, j = 1, 2, 3.$$

u_i are the components of displacement vector, and σ_{ij} are components of the stress tensor. In the program VEKMUS the algorithm of interchanging the partial differential equations system of general form by the finite-difference scheme is constructed.

$$\begin{aligned} & A_{k11} \frac{\partial^2 u_1}{\partial x_1^2} + A_{k12} \frac{\partial^2 u_1}{\partial x_1 \partial x_2} + A_{k13} \frac{\partial^2 u_1}{\partial x_2^2} + A_{k14} \frac{\partial u_1}{\partial x_1} + A_{k15} \frac{\partial u_1}{\partial x_2} + A_{k16} u_1 \\ & + A_{k21} \frac{\partial^2 u_2}{\partial x_1^2} + A_{k22} \frac{\partial^2 u_2}{\partial x_1 \partial x_2} + A_{k23} \frac{\partial^2 u_2}{\partial x_2^2} + A_{k24} \frac{\partial u_2}{\partial x_1} + A_{k25} \frac{\partial u_2}{\partial x_2} + A_{k26} u_2 \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + A_{kn1} \frac{\partial^2 u_n}{\partial x_1^2} + A_{kn2} \frac{\partial^2 u_n}{\partial x_1 \partial x_2} + A_{kn3} \frac{\partial^2 u_n}{\partial x_2^2} + A_{kn4} \frac{\partial u_n}{\partial x_1} + A_{kn5} \frac{\partial u_n}{\partial x_2} + A_{kn6} u_n \\ & = f_k. \end{aligned} \tag{5.5}$$

$u_i = u_i(x_1, x_2)$ are the decided continuous on some domain ω. $A_{klm} = A_{klm}(x_1, x_2)$, $f_k = f_k(x_1, x_2)$ are given continuous functions on the same domain ω, $k,l=\overline{1,n}$, $m=\overline{1,6}$.

It is easy to see that of Vekua's shell theory or plane elasticity theory are particular cases of the system (5.5). By this reason above mentioned algorithm will be served equally by this two theories. The corresponding to (5.1)-(5.4) finite-difference scheme will be constructed automatically in the matrix-vector form

$$\begin{cases} A_0W_0+B_0W_1=F_0, \\ A_1W_0+B_1W_1+C_1W_2=F_1, \\ \qquad A_2W_1+B_2W_2+C_2W_3=F_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \qquad\qquad A_{M-1}W_{M-1}+B_{M-1}W_{M-1}+C_{M-1}W_M=F_{M-1}, \\ \qquad\qquad\qquad A_MW_{M-1}+B_MW_M=F_M, \end{cases} \tag{5.6}$$

where $A_i,\ B_i,\ C_i,\ i=\overline{0,M}$, are the quadratic matrix of order nN of the same structure,

$$\begin{aligned} F_i &= \big[(f_1,f_2,\dots,f_n)_{i0},(f_1,f_2,\dots,f_n)_{i1},\dots,(f_1,f_2,\dots,f_n)_{iN}\big], \\ W_i &= \big[(u_1,u_2,\dots,u_n)_{i0},(u_1,u_2,\dots,u_n)_{i1},\dots,(u_1,u_2,\dots,u_n)_{iN}\big], \end{aligned}$$

are the vectors of order nN, n is the number of unknowns on nodal points of the net, N and M are the discretation parameters of the net.

From the system (5.6) we get easily the algorithm of its solution

$$\begin{aligned} &W_0=B_0^{-1}(F_0-C_0\,W_1), \quad W_1=X_1\,W_2+Y_1, \\ &W_i=X_i\,W_{i+1}+Y_i, \quad i=2,3,\dots,M-1, \\ &W_M=X_M\,Y_M, \end{aligned} \tag{5.7}$$

where

$$\begin{aligned} &X_1=-\big(B_1-A_1\,B_0^{-1}\,C_0\big)^{-1}C_1, \\ &Y_1=\big(B_1-A_1\,B_0^{-1}\,C_0\big)^{-1}\big(F_1-A_1\,B_0^{-1}\,F_0\big), \\ &X_i=-\big(A_i\,X_{i-1}+B_i\big)^{-1}C_i, \\ &Y_i=\big(A_i\,X_{i-1}+B_i\big)^{-1}\big(F_i-A_i\,Y_{i-1}\big), \quad i=2,3,\dots,M-1, \\ &X_M=-\big(A_M\,X_{M-1}+B_M\big)^{-1}, \\ &Y_M=F_M-A_M\,Y_{M-1}. \end{aligned} \tag{5.8}$$

The formulas (5.7) and (5.8) are algorithms of matrix factorization. On the first step the coefficients $X_1,Y_1,\dots,X_M,Y_M$ are calculated (the direct step), and on the second step the unknown vectors $W_M,W_{M-1},\dots,W_0$ are calculated (inverse step).

Problem 1. Let we have the cylindrical shell (Fig.1), the lateral surfaces of which are loading by uniformly distributed tension force Q. Denote by P the intensity of this force. Other surfaces are free. Determine the stress state of the shell. For this problem the boundary conditions are

boundary curves	boundary conditions
AB, DC	$\overset{(0)}{\sigma}_{11} = p,\ \overset{(1)}{\sigma}_{11},\ \overset{(0)}{\sigma}_{21},\ \overset{(1)}{\sigma}_{21},\ \overset{(0)}{\sigma}_{31},\ \overset{(1)}{\sigma}_{31} = 0$
AD, BC, A_1D_1, B_1C_1	$\overset{(0)}{\sigma}_{12},\ \overset{(1)}{\sigma}_{12},\ \overset{(0)}{\sigma}_{22},\ \overset{(1)}{\sigma}_{22},\ \overset{(0)}{\sigma}_{32},\ \overset{(1)}{\sigma}_{32} = 0$
A_1B_1, D_1C_1	$\overset{(0)}{\sigma}_{11},\ \overset{(1)}{\sigma}_{11},\ \overset{(0)}{\sigma}_{21},\ \overset{(1)}{\sigma}_{21},\ \overset{(0)}{\sigma}_{31},\ \overset{(1)}{\sigma}_{31} = 0$

The formulated problem was solved by VEKMUS for shallow and non-shallow shells on the domain $\omega_h^{100}\big(h = \frac{1}{100}\big)$.

From the obtained results we give the short analysis of the solutions.

1) The values of stress tensor components $(\sigma_{11}, \sigma_{12}, \sigma_{22}, \dots)$ are symmetric relative to the axes L_1 and L_2.

2) The values of components σ_{13}, σ_{23}, σ_{31}, σ_{32}, σ_{33} are considerably small relative to other components.

3) The concentration of stress are high valued at the neighborhoods of the points A_1, B_1, C_1, D_1. In the case of shallow shells

$$\max|\sigma_{11}| \approx 18\,\mathrm{p}, \quad \max|\sigma_{12}| \approx 17\ \ \mathrm{p}, \quad \max|\sigma_{22}| \approx 21\,\mathrm{p}.$$

In the case of non-shallow shells

$$\max|\sigma_{11}| \approx 16\,\mathrm{p}, \quad \max|\sigma_{12}| \approx 17\ \ \mathrm{p}, \quad \max|\sigma_{22}| \approx 27\,\mathrm{p}.$$

4) $(\sigma_{11})_k = 4$ in both cases.

Problem 2. For the plate represented on the Figure 2 consider problem with same boundary conditions and with the same data. The solution of the problem is based on the theory of plane elastysity theory.

After the solution we get the following picture:

1) The values σ_{11}, σ_{12} and σ_{22} of the stress tensor components are symmetric relative to the axes L_1 and L_2.

2) The concentration of stress are high valued at the neighborhoods of the points A_1, B_1, C_1, D_1 (see tables 1, 2, 3).

Tab. 1

14	4	1	σ_{11}		
31	42	12			
32	51	83	A_1		
32	60	153	141	125	81
33	43	49	48	46	43
16	22	22	21	21	18

$\max \sigma_{11} = 153$ P

Tab. 2

-7	-11	-1	σ_{12}		
-9	-23	-42			
-9	-24	-86	A_1		
-9	-24	-145	-83	-42	-13
-10	-18	-23	-23	-22	-11
-15	-10	-9	-9	-9	-7

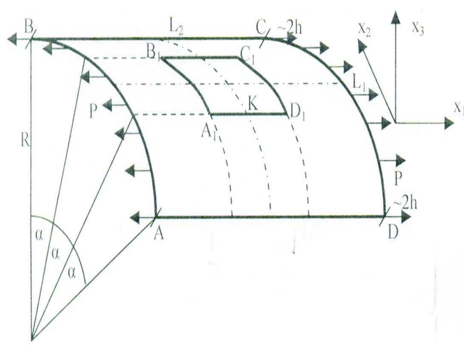
$\max \sigma_{12} = 145$ P

Tab. 3

39	34	79	σ_{22}		
43	39	121			
45	43	139	A_1		
46	46	150	79	13	1
40	40	56	48	39	4
31	31	30	29	28	12

$\max \sigma_{22} = 150$ P

3) $(\sigma_{11})k \approx 2.35$ p.



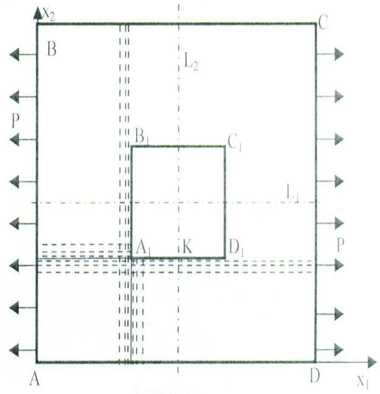


Fig. 1 Fig. 2

$\sigma = 0.3$; $\alpha = \pi/6$; $R = 200$ sm; $2h = 1$ sm; $|AB| = |AD| = 200^{*}\pi/2$ sm, $|A_1B_1| = |A_1D_1| = 200^{*}\pi/6$ sm, P=1.

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