DIFFERENTIAL SCHEME OF HIGH DEGREE PRECISION DECOMPOSITION OF NONHOMOGENOUS EVOLUTION PROBLEM

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Abstract

In the present work symmetrized sequential-parallel decomposition method of the third degree precision for the solution of Cauchy abstract problem for the nonhomogenous evolution equation is offered. Third degree precision is reached by introducing a complex coefficient. For the error of approximated solution the explicit a priori estimation is obtained.

Key words and phrases: Decomposition Method, Semigroup, Trotter formula, Cauchy abstract problem.

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1. *Introduction*

The study of the approximated schemes of solving of evolution problems leads to the conclusion that to each approximated scheme there corresponds a definite operator (solving operator of a discrete problem), which approximates a solving operator (semigroup) of a source continuous problem. The opposite is also true: Constructing approximation of a continuous semigroup, we build an approximated scheme of a solution of an evolution problem.

For example, if we apply Rotte's method for a solution of an evolution problem, a solving operator of the obtained difference problem will be a discrete semigroup and we come to a problem of approximating a continuous semigroup with the help of discrete semigroups (in this case see T. Kato [16], Ch. IX).

In case of applying a decomposition method, the solving operator of the applicable decomposed problem generates the Trotter formula, (see Trotter H. [25]) or the Chernoff formula, (see Chernoff P. R. [1,2]) or a formula, which is a combination of these formulas. Therefore, the error estimation of a decomposition method is equivalent to a problem of approximating of a continuous semigroup using Trotter type formulas. Papers [13,14] (see also [21], Ch. II) are dedicated to the error estimations of Trotter type formulas.

The scheme of decomposition, associated with the Trotter formula, allows us to split Cauchy problem for an evolution equation with an operator $A = A_1 + A_2 + ... + A_m$ to m problems correspondingly with operators $A_1, A_2, ..., A_m$, which are solved sequentially on each time interval with the length t/n.

The decomposition scheme, associated with the Chernoff formula, is known as a method of fractional steps (see N. N. Ianenko [13]).

As it is known, the decomposition method is sufficiently general for obtaining economical schemes for the solution of the multidimensional problems of mathematical physics. They can be divided into two groups: the schemes of sequential account (N. N. Ianenko [13], A.A. Samarskii [22], Marchuk G. I. [19], A. A. Samarskii, P. N. Vabishchevich [23], I. V. Fryazinov [5], E. G. Diakonov [4], Temam R. [24], D.G. Gordeziani [9]) and the schemes of parallel account (D. G. Gordeziani and A. A. Samarskii [12], D. G. Gordeziani and H. V. Meladze [9,10], A. M. Kuzyk and V. L. Makarov [18]). In [21] (see Ch. II) the explicit estimations for decomposition schemes of the parallel account are obtained, which were considered in [11]. At present, there are many works dedicated to the decomposition method (see references [18,22]).

In the above-stated works the schemes considered are of the first or second degree precision. As far as we know, high degree precision decomposition formulas in case of two addends ($A = A_1 + A_2$) for the first time were obtained in [3].

In the present work a symmetrized sequential-parallel decomposition method of the third degree precision for the solution of the Cauchy abstract problem with operator $A = A_1 + A_2 + ... + A_m$ is presented. For the considered scheme the explicit a priori estimation is obtained. Under explicit estimations we understand such a priori estimations for an error of solution, where the constants of a right member do not depend on a solution of an initial continuous problem, i.e. are absolute.

2. *Setting of the problem*

Let us consider the Cauchy abstract problem in the Banach space X :

$$\frac{du(t)}{dt} + Au(t) = f(t), \quad t > 0, \quad u(0) = \varphi. \tag{2.1}$$

Here A is a closed linear operator with the domain $D(A)$, which is everywhere dense in X, φ is a given element from $D(A)$, $f(t) \in C^1([0;\infty);X)$ and for every fixed t, $f(t) \in D(A)$.

Suppose that $(-A)$ operator generates a strongly continuous semigroup $\{\exp(-tA)\}_{t\geq 0}$, then the solution of the problem (2.1) is given by the following formula (see [14,16]):

$$u(t) = U(t,A)\varphi + \int_0^t U(t-s,A)f(s)ds, \tag{2.2}$$

where $U(t,A) = exp(-tA)$.

Let $A = A_1 + A_2 + ... + A_m$, where $A_i \quad (i = 1,2,...,m)$ are compactly defined, closed linear operators in X.

Let us introduce a difference net domain:

$$\overline{\omega}_\tau = \{t_k = k\tau, k = 1,2,..., \tau > 0\}.$$

Suppose that $f(t) \in C^2([0,\infty);X)$. Along with problem (2.1) we consider two sequences of the following problems on each interval $[t_{k-1}, t_k]$:

$$\begin{aligned} \frac{dv_k^1(t)}{dt} + \alpha A_1 v_k^1(t) &= \frac{\alpha}{m} f(t_k) - 2\sigma_1(t_k - t)f'(t_k), \\ v_k^1(t_{k-1}) &= u_{k-1}(t_{k-1}), \end{aligned}$$

$$\begin{aligned} \frac{dv_k^{i+1}(t)}{dt} + \alpha A_{i+1} v_k^{i+1}(t) &= \frac{\alpha}{m} f(t_k) - 2\sigma_{i+1}(t_k - t)f'(t_k), \\ v_k^{i+1}(t_{k-1}) &= v_k^i(t_k), \qquad i = 1,...,m-2, \end{aligned}$$

$$\begin{aligned} \frac{dv_k^m(t)}{dt} + A_m v_k^m(t) &= \frac{1}{m} f(t_k) - 2\sigma_m(t_k - t)f'(t_k), \\ v_k^m(t_{k-1}) &= v_k^{m-1}(t_k), \end{aligned}$$

$$\begin{aligned} \frac{dv_k^{m+i}(t)}{dt} + \overline{\alpha} A_{m-i} v_k^{m+i}(t) &= \frac{\overline{\alpha}}{m} f(t_k) - 2\sigma_{m+i}(t_k - t)f'(t_k), \\ v_k^{m+i}(t_{k-1}) &= v_k^{m+i-1}(t_k), \qquad i = 1,...,m-2, \end{aligned}$$

$$\begin{aligned} \frac{dv_k^{2m-1}(t)}{dt} + \overline{\alpha} A_1 v_k^{2m-1}(t) &= \frac{\overline{\alpha}}{m} f(t_k) - 2\sigma_{2m-1}(t_k - t)f'(t_k) \\ &\quad + \frac{(t_k - t)^2}{2} f''(t_k), \\ v_k^{2m-1}(t_{k-1}) &= v_k^{2m-2}(t_k); \end{aligned}$$

$$\begin{aligned}\frac{dw_k^1(t)}{dt}+\alpha A_m w_k^1(t) &= \frac{\alpha}{m}f(t_k)-2\sigma_1(t_k-t)f'(t_k),\\ w_k^1(t_{k-1}) &= u_{k-1}(t_{k-1}),\end{aligned}$$

$$\begin{aligned}\frac{dw_k^{i+1}(t)}{dt}+\alpha A_{m-i} w_k^{i+1}(t) &= \frac{\alpha}{m}f(t_k)-2\sigma_{i+1}(t_k-t)f'(t_k),\\ w_k^{i+1}(t_{k-1}) &= w_k^i(t_k), \qquad i=1,...,m-2,\end{aligned}$$

$$\begin{aligned}\frac{dw_k^m(t)}{dt}+A_1 v_k^m(t) &= \frac{1}{m}f(t_k)-2\sigma_m(t_k-t)f'(t_k),\\ w_k^m(t_{k-1}) &= w_k^{m-1}(t_k),\end{aligned}$$

$$\begin{aligned}\frac{dw_k^{m+i}(t)}{dt}+\overline{\alpha} A_{i+1} w_k^{m+i}(t) &= \frac{\overline{\alpha}}{m}f(t_k)-2\sigma_{m+i}(t_k-t)f'(t_k),\\ v_k^{m+i}(t_{k-1}) &= v_k^{m+i-1}(t_k), \qquad i=1,...,m-2,\end{aligned}$$

$$\begin{aligned}\frac{dw_k^{2m-1}(t)}{dt}+\overline{\alpha} A_m w_k^{2m-1}(t) &= \frac{\overline{\alpha}}{m}f(t_k)-2\sigma_{2m-1}(t_k-t)f'(t_k)\\ &\quad+\frac{(t_k-t)^2}{2}f''(t_k),\\ w_k^{2m-1}(t_{k-1}) &= w_k^{2m-2}(t_k).\end{aligned}$$

Here $\sigma_1,\ \sigma_2,...,\ \sigma_{2m-1}$ and α are numerical complex parameters, $Re(\alpha)>0, \quad u_0(0)=\varphi$.

Suppose that $(-A_j)$, $(-\alpha A_j)$ and $(-\overline{\alpha}A_j) \quad (j=1,2,...,m)$ operators generate strongly continuous semigroups.

On each $[t_{k-1},t_k]$ $(k=1,2,...)$ interval $u_k(t)$ are defined as follows:

$$u_k(t)=\frac{1}{2}[v_k^{2m-1}(t)+w_k^{2m-1}(t)].$$

We consider the function $u_k(t)$ as an approximate solution of the problem (2.1) on the interval $[t_{k-1},t_k]$.

The above-stated scheme in case of $m=2$ addends for the homogenous equation is considered in [7] and for the non-homogenous equation in [6].

We will need natural degrees of the operator $A=A_1+A_2+...+A_m \quad (A^s, \quad s=2,3,4)$. In case of two addends $(m=2)$ they are defined as follows:

$$A^2=\left(A_1^2+A_2^2\right)+\left(A_1A_2+A_2A_1\right),$$

$$A^3=\left(A_1^3+A_2^3\right)+\left(A_1^2A_2+...+A_2^2A_1\right)+\left(A_1A_2A_1+A_2A_1A_2\right),$$

$$A^4 = \left(A_1^4 + A_2^4\right) + \left(A_1^3 A_2 + ... + A_2^3 A_1\right)$$
$$+ \left(A_1^2 A_2 A_1 + ... + A_2^2 A_1 A_2\right) + \left(A_1 A_2 A_1 A_2 + A_2 A_1 A_2 A_1\right).$$

Analogously are defined A^s $(s = 2, 3, 4)$ when $m > 2$. Obviously, the domain $D(A^s)$ of the operator A^s is the intersection of the domains of its addends.

Let us introduce the following definitions:

$$\|\varphi\|_A = \|A_1\varphi\| + ... + \|A_m\varphi\|, \quad \varphi \in D(A),$$

$$\|\varphi\|_{A^2} = \sum_{i,j=1}^{m} \|A_i A_j \varphi\|, \quad \varphi \in D\left(A^2\right),$$

where $\|\cdot\|$ is a norm in X, similarly are defined $\|\varphi\|_{A^s} \quad (s = 3, 4)$.

Theorem. *Let the following conditions be satisfied:*

a) $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}} \quad \left(i = \sqrt{-1}\right)$;

b) $(-\gamma A_j)$, $\gamma = 1, \alpha, \overline{\alpha}$ $(j = 1, 2, ..., m)$ *and* $(-A)$ *operators generate strongly continuous semigroups, for which the following estimations hold correspondingly:*

$$\|U(t, \gamma A_j)\| \leq e^{\omega t},$$

$$\|U(t, A)\| \leq M e^{\omega t}, \quad M, \omega = const > 0;$$

c) $U(s, A)\varphi \in D\left(A^4\right)$ *for every fixed* $s \geq 0$;

d) $f(t) \in C^3([0, \infty); X)$; $f(t) \in D\left(A^3\right)$, $f^k(t) \in D\left(A^{3-k}\right)$, $k = 1, 2$, *and* $U(s, A) f(t) \in D\left(A^4\right)$ *for every fixed* t *and* s $(t, s \geq 0)$;

e) $m > 2$, $\sigma_j = \widetilde{\sigma}_1$, $\sigma_m = \widetilde{\sigma}_2$, $\sigma_{m+j} = \widetilde{\sigma}_3$ $(j = 1, 2, ..., m-1)$,

$$\widetilde{\sigma}_3 = \frac{1}{2\left[(9m+1)\alpha - 3m\right]}, \quad \widetilde{\sigma}_1 = (9\alpha - 4)\widetilde{\sigma}_3, \quad \widetilde{\sigma}_2 = \frac{1}{2} - \widetilde{\sigma}_1 - \widetilde{\sigma}_3.$$

Then the following estimation holds:

$$\|u(t_k) - u_k(t_k)\| \leq c e^{\omega_0 t_k} t_k \tau^3 \Bigg(\sup_{s \in [0, t_k]} \|U(s, A)\varphi\|_{A^4}$$
$$+ t_k \sup_{s,t \in [0, t_k]} \|U(s, A) f(t)\|_{A^4} + \sup_{t \in [0, t_k]} \|f(t)\|_{A^3}$$
$$+ \sup_{t \in [0, t_k]} \left\|f'(t)\right\|_{A^2} + \sup_{t \in [0, t_k]} \left\|f''(t)\right\|_A + \sup_{t \in [0, t_k]} \left\|f'''(t)\right\| \Bigg),$$

where c, ω_0 *are positive constants.*

3. *Auxiliary Lemmas*

Let us prove the auxiliary Lemmas on which the proof of the theorem is based.

Lemma 3.1. *Let the conditions a), b) and c) of the theorem be satisfied, then the following estimation holds:*

$$\left\| \left[U\left(t_k, A\right) - V^k\left(\tau\right) \right] \varphi \right\| \leq c e^{\omega_0 t_k} t_k \tau^3 \sup_{s\in[0,t_k]} \left\| U\left(s, A\right) \varphi \right\|_{A^4},$$

where

$$V\left(\tau\right) = \frac{1}{2}\left[U\left(\tau, \overline{\alpha}A_1\right) ... U\left(\tau, \overline{\alpha}A_{m-1}\right) U\left(\tau, A_m\right) U\left(\tau, \alpha A_{m-1}\right) ... U\left(\tau, \alpha A_1\right)\right.$$
$$\left. + U\left(\tau, \overline{\alpha}A_m\right) ... U\left(\tau, \overline{\alpha}A_2\right) U\left(\tau, A_1\right) U\left(\tau, \alpha A_2\right) ... U\left(\tau, \alpha A_m\right)\right],$$

here c *and* ω_0 *are positive constants.*

Proof. According to the formula (see Kato. T. [1], p. 603):

$$A \int_r^t U\left(s, A\right) ds = U\left(r, A\right) - U\left(t, A\right), \quad 0 \leq r \leq t,$$

we can get the following expansion:

$$U(t, A) = \sum_{i=0}^{k-1} (-1)^i \frac{t^i}{i!} A^i + R_k(t, A), \tag{3.1}$$

where

$$R_k(t, A) = (-A)^k \int_0^t \int_0^{s_1} ... \int_0^{s_{k-1}} U(s, A) ds ds_{k-1} ... ds_1. \tag{3.2}$$

Let us suppose that $T\left(\tau\right)$ is a combination (sum, product) of semi-groups, generated by operators $\left(-\gamma A_i\right)$ $(i = 1, 2, ..., m)$. Let us decompose all semigroups included in the operator $T\left(\tau\right)$ according to the formula (3.1), multiply these decompositions, group together similar members and define the coefficients of the members $\left(-\tau A_i\right), \left(\tau^2 A_i A_j\right)$ and $\left(-\tau^3 A_i A_j A_k\right)$ $(i, j, k = 1, 2, ..., m)$ to be correspondingly $\left[T\left(\tau\right)\right]_i, \left[T\left(\tau\right)\right]_{i,j}$ and $\left[T\left(\tau\right)\right]_{i,j,k}$ in the obtained decomposition.

If we decompose all semigroups in the $V(\tau)$ from right to left according to the formula (3.1) so that each residual member is of the fourth degree, we get the following formula:

$$V\left(\tau\right) = I - \tau \sum_{i=1}^{m} \left[V\left(\tau\right)\right]_i A_i + \tau^2 \sum_{i,j=1}^{m} \left[V\left(\tau\right)\right]_{i,j} A_i A_j$$

$$-\tau^3 \sum_{i,j,k=1}^{m} [V(\tau)]_{i,j,k} A_i A_j A_k + R_4^{(m)}(\tau), \tag{3.3}$$

where according to the first inequality of the condition b)) of the theorem the following estimation is true for $R_4^{(m)}(\tau)$:

$$\left\| R_4^{(m)}(\tau)\varphi \right\| \le c e^{\omega_2 \tau} \tau^4 \|\varphi\|_{A^4}, \quad \varphi \in D\left(A^4\right). \tag{3.4}$$

It is easy to show that if $m = 2$, then

$$[V(\tau)]_i = 1, \tag{3.5}$$

$$[V(\tau)]_{i,j} = 1/2, \tag{3.6}$$

$$[V(\tau)]_{i,j,k} = 1/6 \tag{3.7}$$

and

$$R_4^{(2)}(\tau) = \frac{1}{2}\left[R_{1,2}(\tau) + R_{2,1}(\tau)\right],$$

where

$$R_{i,j}(\tau) = R_4(\tau,\overline{\alpha}A_i) - \tau R_3(\tau,\overline{\alpha}A_i)A_j + \frac{1}{2}\tau^2 R_2(\tau,\overline{\alpha}A_i)A_j^2$$

$$-\frac{1}{6}\tau^3 R_1(\tau,\overline{\alpha}A_i)A_j^3 + +U(\tau,\overline{\alpha}A_i)R_4(\tau,A_j) - \alpha\tau R_3(\tau,\overline{\alpha}A_i)A_i$$

$$+\alpha\tau^2 R_2(\tau,\overline{\alpha}A_i)A_jA_i - \frac{1}{2}\alpha\tau^3 R_1(\tau,\overline{\alpha}A_i)A_j^2A_i - \alpha\tau U(\tau,\overline{\alpha}A_i)R_3(\tau,A_j)A_i$$

$$+\frac{1}{2}\alpha^2\tau^2 R_2(\tau,\overline{\alpha}A_i)A_i^2 - \frac{1}{2}\alpha^2\tau^3 R_1(\tau,\overline{\alpha}A_i)A_jA_i^2$$

$$+\frac{1}{2}\alpha^2\tau^2 U(\tau,\overline{\alpha}A_i)R_2(\tau,A_j)A_i^2 - \frac{1}{6}\alpha^3\tau^3 R_1(t,\overline{\alpha}A_i)A_i^3$$

$$-\frac{1}{6}\alpha^3\tau^3 U(\tau,\overline{\alpha}A_i)R_1(\tau,A_j)A_i^3 + U(\tau,\overline{\alpha}A_i)U(\tau,A_j)R_4(\tau,\alpha A_i), \quad i,j = 1,2.$$

Now let us consider the case, when $m > 2$.

Let us introduce the following definitions:

$$V_1(\tau) = U(\tau,\overline{\alpha}A_1)...U(\tau,\overline{\alpha}A_{m-1})U(\tau,A_m)U(\tau,\alpha A_{m-1})...U(\tau,\alpha A_1),$$

$$V_2(\tau) = U(\tau,\overline{\alpha}A_m)...U(\tau,\overline{\alpha}A_2)U(\tau,A_1)U(\tau,\alpha A_2)...U(\tau,\alpha A_m).$$

Then it is obvious that:

$$[V(\tau)]_i = \frac{1}{2}\left([V_1(\tau)]_i + [V_2(\tau)]_i\right), \quad i = 1,2,...,m,$$

$$[V(\tau)]_{i,j} = \frac{1}{2}\left([V_1(\tau)]_{i,j} + [V_2(\tau)]_{i,j}\right), \quad i,j = 1,2,...,m,$$

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\left([V_1(\tau)]_{i,j,k} + [V_2(\tau)]_{i,j,k}\right), \quad i,j,k = 1,2,...,m.$$

Let us compute coefficients $[V_1(\tau)]_i$. Obviously, we get the corresponding members of these coefficients from decomposition of only those multipliers (semigroups) of the operator $V_1(\tau)$ which are generated by operators $(-\gamma A_i)$. Only the first addends (identical operators) will be used from decomposition of other semigroups . So we have:

$$[V_1(\tau)]_i = [U(\tau, A_i)]_i = 1.$$

Analogously

$$[V_2(\tau)]_i = [U(\tau, A_i)]_i = 1.$$

So we have

$$[V(\tau)]_i = 1, \quad i = 1,2,...,m.$$

Let us compute coefficients $[V_1(\tau)]_{i,j}$. Obviously, we get the corresponding members of these coefficients from decomposition of only those multipliers (semigroups) of the operator $V_1(\tau)$ which are generated by operators $(-\gamma A_i)$ and $(-\gamma A_j)$. Only the first addends (identical operators) will be used from decomposition of other semigroups . So we have:

$$[V_1(\tau)]_{i,j} = [U(\tau, \overline{\alpha} A_{i_1}) U(\tau, A_{i_2}) U(\tau, \alpha A_{i_1})]_{i,j}.$$

Analogously

$$[V_2(\tau)]_{i,j} = [U(\tau, \overline{\alpha} A_{i_2}) U(\tau, A_{i_1}) U(\tau, \alpha A_{i_2})]_{i,j},$$

where (i_1, i_2) is a pair of i and j indices, arranged in an increasing order. According to the formula (3.6) we have:

$$[V(\tau)]_{i,j} = \frac{1}{2}, \quad i,j = 1,2,...,m.$$

Let us compute coefficients $[V_1(\tau)]_{i,j,k}$. Obviously, we get the corresponding members of these coefficients from decomposition of only those multipliers (semigroups) of the operator $V_1(\tau)$, which are generated by operators $(-\gamma A_i), (-\gamma A_j)$ and $(-\gamma A_k)$. Only first addends (identical operators) will be used from decomposition of other semigroups . So we have:

$$[V_1(\tau)]_{i,j,k} = [U(\tau, \overline{\alpha} A_{i_1}) U(\tau, \overline{\alpha} A_{i_2}) U(\tau, A_{i_3}) U(\tau, \alpha A_{i_2}) U(\tau, \alpha A_{i_1})]_{i,j,k}.$$

Analogously

$$[V_2(\tau)]_{i,j,k} = [U(\tau, \overline{\alpha} A_{i_3}) U(\tau, \overline{\alpha} A_{i_2}) U(\tau, A_{i_1}) U(\tau, \alpha A_{i_2}) U(\tau, \alpha A_{i_3})]_{i,j,k},$$

where (i_1, i_2, i_3) is a triple of i, j and k indices, arranged in an increasing order.

First of all, let us consider the case when $i = j = k$,we have:

$$[V_1(\tau)]_{i,j,k} = [U(\tau, A_i)]_{i,i,i} = \frac{1}{6}$$

and

$$[V_2(\tau)]_{i,j,k} = [U(\tau, A_i)]_{i,i,i} = \frac{1}{6}.$$

Now let us consider the case when only two of i, j, k indices are different. In this case we have:

$$[V_1(\tau)]_{i,j,k} = [U(\tau, \overline{\alpha} A_{i_1}) \, U(\tau, A_{i_2}) \, U(\tau, \alpha A_{i_1})]_{i,j,k}$$

and

$$[V_2(\tau)]_{i,j,k} = [U(\tau, \overline{\alpha} A_{i_2}) \, U(\tau, A_{i_1}) \, U(\tau, \alpha A_{i_2})]_{i,j,k}.$$

where (i_1, i_2) is pair of different indices of i, j and k triple, arranged in an increasing order. According to the formula (3.7) we have**:**

$$[V(\tau)]_{i,j,k} = \frac{1}{6}.$$

Now let us consider the case when i, j, k indices are different. We have six variants. Let us consider each one separately:

Case 1. If $i < j < k$, then

$$[V_1(\tau)]_{i,j,k} = [U(\tau, \overline{\alpha} A_i) \, U(\tau, \overline{\alpha} A_j) \, U(\tau, A_k) \, U(\tau, \alpha A_j) \, U(\tau, \alpha A_i)]_{i,j,k}$$

$$= [U(\tau, \overline{\alpha} A_i)]_i \, [U(\tau, \overline{\alpha} A_j)]_j \, [U(\tau, A_k)]_k = \overline{\alpha}^2$$

and

$$[V_2(\tau)]_{i,j,k} = [U(\tau, \overline{\alpha} A_k) \, U(\tau, \overline{\alpha} A_j) \, U(\tau, A_i) \, U(\tau, \alpha A_j) \, U(\tau, \alpha A_k)]_{i,j,k}$$

$$= [U(\tau, A_i)]_i \, [U(\tau, \alpha A_j)]_j \, [U(\tau, \alpha A_k)]_k = \alpha^2.$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2} \left(\alpha^2 + \overline{\alpha}^2 \right) = \frac{1}{6}.$$

Case 2. If $i < k < j$, then

$$[V_1(\tau)]_{i,j,k} = [U(\tau, \overline{\alpha} A_i) \, U(\tau, \overline{\alpha} A_k) \, U(\tau, A_j) \, U(\tau, \alpha A_k) \, U(\tau, \alpha A_i)]_{i,j,k}$$

$$= [U(\tau, \overline{\alpha} A_i)]_i \, [U(\tau, A_j)]_j \, [U(\tau, \alpha A_k)]_k = \alpha \overline{\alpha}$$

and

$$[V_2(\tau)]_{i,j,k} = [U(\tau,\overline{\alpha}A_j)\,U(\tau,\overline{\alpha}A_k)\,U(\tau,A_i)\,U(\tau,\alpha A_k)\,U(\tau,\alpha A_j)]_{i,j,k} = 0.$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\alpha\overline{\alpha} = \frac{1}{6}.$$

Case 3. If $j < i < k$, then

$$[V_1(\tau)]_{i,j,k} = [U(\tau,\overline{\alpha}A_j)\,U(\tau,\overline{\alpha}A_i)\,U(\tau,A_k)\,U(\tau,\alpha A_i)\,U(\tau,\alpha A_j)]_{i,j,k} = 0$$

and

$$[V_2(\tau)]_{i,j,k} = [U(\tau,\overline{\alpha}A_k)\,U(\tau,\overline{\alpha}A_i)\,U(\tau,A_j)\,U(\tau,\alpha A_i)\,U(\tau,\alpha A_k)]_{i,j,k}$$
$$= [U(\tau,\overline{\alpha}A_i)]_i\,[U(\tau,A_j)]_j\,[U(\tau,\alpha A_k)]_k = \alpha\overline{\alpha}.$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\alpha\overline{\alpha} = \frac{1}{6}.$$

Case 4. If $j < k < i$, then

$$[V_1(\tau)]_{i,j,k} = [U(\tau,\overline{\alpha}A_j)\,U(\tau,\overline{\alpha}A_k)\,U(\tau,A_i)\,U(\tau,\alpha A_k)\,U(\tau,\alpha A_j)]_{i,j,k} = 0$$

and

$$[V_2(\tau)]_{i,j,k} = [U(\tau,\overline{\alpha}A_i)\,U(\tau,\overline{\alpha}A_k)\,U(\tau,A_j)\,U(\tau,\alpha A_k)\,U(\tau,\alpha A_i)]_{i,j,k}$$
$$= [U(\tau,\overline{\alpha}A_i)]_i\,[U(\tau,A_j)]_j\,[U(\tau,\alpha A_k)]_k = \alpha\overline{\alpha}.$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\alpha\overline{\alpha} = \frac{1}{6}.$$

Case 5. If $k < i < j$, then

$$[V_1(\tau)]_{i,j,k} = [U(\tau,\overline{\alpha}A_k)\,U(\tau,\overline{\alpha}A_i)\,U(\tau,A_j)\,U(\tau,\alpha A_i)\,U(\tau,\alpha A_k)]_{i,j,k}$$
$$= [U(\tau,\overline{\alpha}A_i)]_i\,[U(\tau,A_j)]_j\,[U(\tau,\alpha A_k)]_k = \alpha\overline{\alpha}$$

and

$$[V_2(\tau)]_{i,j,k} = [U(\tau,\overline{\alpha}A_j)\,U(\tau,\overline{\alpha}A_i)\,U(\tau,A_k)\,U(\tau,\alpha A_i)\,U(\tau,\alpha A_j)]_{i,j,k} = 0.$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\alpha\overline{\alpha} = \frac{1}{6}.$$

Case 6. If $k < j < i$, then

$$[V_1(\tau)]_{i,j,k} = [U(\tau,\overline{\alpha}A_k)\,U(\tau,\overline{\alpha}A_j)\,U(\tau,A_i)\,U(\tau,\alpha A_j)\,U(\tau,\alpha A_k)]_{i,j,k}$$

$$= [U(\tau, A_i)]_i \, [U(\tau, \alpha A_j)]_j \, [U(\tau, \alpha A_k)]_k = \alpha^2$$

and

$$[V_2(\tau)]_{i,j,k} = [U(\tau, \overline{\alpha} A_i) \, U(\tau, \overline{\alpha} A_j) \, U(\tau, A_k) \, U(\tau, \alpha A_j) \, U(\tau, \alpha A_i)]_{i,j,k}$$

$$= [U(\tau, \overline{\alpha} A_i)]_i \, [U(\tau, \overline{\alpha} A_j)]_j \, [U(\tau, A_k)]_k = \overline{\alpha}^2.$$

So we have

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\left(\alpha^2 + \overline{\alpha}^2\right) = \frac{1}{6}.$$

Finally, for any triple (i, j, k) we have:

$$[V(\tau)]_{i,j,k} = \frac{1}{6}.$$

Inserting the obtained coefficients into (3.3) we will get:

$$V(\tau) = I - \tau \sum_{i=1}^{m} A_i + \frac{1}{2}\tau^2 \sum_{i,j=1}^{m} A_i A_j - \frac{1}{6}\tau^3 \sum_{i,j,k=1}^{m} A_i A_j A_k + R_4^{(m)}(\tau)$$

$$= I - \tau \sum_{i=1}^{m} A_i + \frac{1}{2}\tau^2 \left(\sum_{i=1}^{m} A_i\right)^2 - \frac{1}{6}\tau^3 \left(\sum_{i=1}^{m} A_i\right)^3 + R_4^{(m)}(\tau)$$

$$= I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + R_4^{(m)}(\tau). \tag{3.8}$$

According to the formula (3.1) we have:

$$U(\tau, A) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + R_4(\tau, A). \tag{3.9}$$

According to the second inequality of the condition b)) of the theorem the following estimation is true for $R_4(\tau, A)$:

$$\|R_4(\tau, A)\varphi\| \le c e^{\omega\tau} \tau^4 \left\|A^4 \varphi\right\| \le c e^{\omega\tau} \tau^4 \|\varphi\|_{A^4}. \tag{3.10}$$

According to the formulas (3.8) and (3.9) we have:

$$U(\tau, A) - V(\tau) = R_4(\tau, A) - R_4^{(m)}(\tau).$$

Hence using inequalities (3.4) and (3.10) we can get the following estimation:

$$\|[U(\tau, A) - V(\tau)]\varphi\| \le c e^{\omega_2 \tau} \tau^4 \|\varphi\|_{A^4}. \tag{3.11}$$

According to the property of a semigroup we have:

$$\left[U(t_k, A) - V^k(\tau)\right]\varphi = \left[U^k(\tau, A) - V^k(\tau)\right]\varphi$$

$$= \sum_{i=1}^{k} V^{k-i}(\tau)\left[U(\tau,A) - V(\tau)\right] U((i-1)\tau, A)\varphi.$$

Hence according to the inequality (3.11) and the condition b) of the theorem we can obtain the following estimation:

$$\left\| \left[U(t_k, A) - V^k(\tau) \right] \varphi \right\|$$

$$\leq \sum_{i=1}^{k} \|V(\tau)\|^{k-i} \left\| \left[U(\tau,A) - V(\tau)\right] U((i-1)\tau, A)\varphi \right\|$$

$$\leq \sum_{i=1}^{k} e^{\omega_1 (k-i)\tau} c e^{\omega_2 \tau} \tau^4 \left\| U((i-1)\tau, A)\varphi \right\|_{A^4}$$

$$\leq c e^{\omega_0 t_k} \tau^4 \sum_{i=1}^{k} \left\| U((i-1)\tau, A)\varphi \right\|_{A^4}$$

$$\leq c e^{\omega_0 t_k} t_k \tau^3 \sup_{s \in [o, t_k]} \left\| U(s, A)\varphi \right\|_{A^4}. \blacksquare$$

Remark: *The operator $V^k(\tau)$ is a solving operator of the above considered decomposed problem. It is obvious that according to the condition of the theorem $\left(U(t,\gamma A_i) \leq e^{\omega t}\right)$*

$$\left\| V^k(\tau) \right\| \leq e^{\omega_1 t_k},$$

where $\omega_1 = (2m-1)\omega$. From here follows the stability of the above-stated decomposition scheme on each finite time interval. $\blacksquare$

For the constructing of the solution of the decomposed system we will need to introduce the following operators:

$$\begin{aligned} V_1^{(i)}(\tau, t) &= U(\tau, \overline{\alpha} A_1) \ldots U(\tau, \overline{\alpha} A_{m-1}) U(\tau, A_m) T_1^{(i)} U(t, \alpha A_i), \\ i &= 1, 2, \ldots, m-1, \end{aligned}$$

$$T_1^{(i)} = \prod_{j=1}^{m-i-1} U(\tau, \alpha A_{m-j}), \quad i = 1, \ldots, m-2, \quad T_1^{(m-1)} = I,$$

$$\begin{aligned} V_2^{(i)}(\tau, t) &= U(\tau, \overline{\alpha} A_m) \ldots U(\tau, \overline{\alpha} A_2) U(\tau, A_1) T_2^{(i)} U(t, \alpha A_{m-i+1}), \\ i &= 1, 2, \ldots, m-1, \end{aligned}$$

$$T_2^{(i)} = \prod_{j=2}^{m-i} U(\tau, \alpha A_j), \quad i = 1, \ldots, m-2, \quad T_2^{(m-1)} = I,$$

$$V_1^{(m)}(\tau,t)=U(\tau,\overline{\alpha}A_1)\,U(\tau,\overline{\alpha}A_2)\,...U(\tau,\overline{\alpha}A_{m-1})\,U(t,A_m)\,,$$

$$V_2^{(m)}(\tau,t)=U(\tau,\overline{\alpha}A_m)\,U(\tau,\overline{\alpha}A_{m-1})\,...U(\tau,\overline{\alpha}A_2)\,U(t,A_1)\,,$$

$$V_1^{(m+i)}(\tau,t)=\left(\prod_{j=1}^{m-i-1}U(\tau,\alpha A_j)\right)U(t,\alpha A_{m-i})\,,\quad i=1,2,...,m-2,$$

$$V_2^{(m+i)}(\tau,t)=\left(\prod_{j=0}^{m-i-2}U(\tau,\overline{\alpha}A_{m-j})\right)U(t,\overline{\alpha}A_{i+1})\,,\quad i=1,2,..,m-2,$$

$$V_1^{(2m-1)}(t)=U(t,\overline{\alpha}A_1)\,,\quad V_2^{(2m-1)}(t)=U(t,\overline{\alpha}A_m)\,,$$

$$W(\tau,t;\alpha_1,...,\alpha_{2m-1})=\frac{1}{2}\left[\sum_{j=1}^{2m-2}\alpha_jV_1^{(j)}(\tau,t)\right.$$

$$\left.+\sum_{j=1}^{2m-2}\alpha_jV_2^{(j)}(\tau,t)+\alpha_{2m-1}V_1^{(2m-1)}(t)+\alpha_{2m-1}V_2^{(2m-1)}(t)\right].$$

Lemma 3.2. *If the conditions a) and b) of the theorem are satisfied and $f(t)\in C^3([0,\infty);X)$, then at the point $t=t_k$ the solution of the decomposed problem may be written as follows:*

$$u_k(t_k)=V^k(\tau)\varphi+\sum_{i=1}^{k}V^{k-i}(\tau)F_i^{(2)},\tag{3.12}$$

where

$$F_k^{(2)}=\int_{t_{k-1}}^{t_k}W(\tau,t_k-s,\alpha_1,...,\alpha_{2m-1})f(t_k)ds$$

$$-\int_{t_{k-1}}^{t_k}W(\tau,t_k-s,\sigma_1,...,\sigma_{2m-1})2(t_k-s)f'(t_k)ds$$

$$+\int_{t_{k-1}}^{t_k}\frac{1}{2}\left[V_1^{(2m-1)}(t_k-s)+V_2^{(2m-1)}(t_k-s)\right]\frac{(t_k-s)^2}{2}f''(t_k)ds.\tag{3.13}$$

Proof. It is clear, that according to the formula (2.2) for the decomposed system we have:

$$v_k^j(t_k)=U(\tau,\alpha A_j)\,v_k^{j-1}(t_k)$$

$$+\int\limits_{t_{k-1}}^{t_k} U\left(t_k-s,\alpha A_j\right)\widetilde{f}_{j,k}\left(s\right)ds,\quad j=1,2,...,m-1,$$

$$v_k^m\left(t_k\right)=U\left(\tau,A_m\right)v_k^{m-1}\left(t_k\right)+\int\limits_{t_{k-1}}^{t_k} U\left(t_k-s,A_m\right)\widetilde{f}_{m,k}\left(s\right)ds,$$

$$v_k^{m+j}\left(t_k\right)=U\left(\tau,\overline{\alpha}A_{m-j}\right)v_k^{m+j-1}\left(t_k\right)$$

$$+\int\limits_{t_{k-1}}^{t_k} U\left(t_k-s,\overline{\alpha}A_{m-j}\right)\widetilde{f}_{m+j,k}\left(s\right)ds,\quad j=1,2,...,m-1,$$

where

$$\widetilde{f}_{j,k}\left(s\right)=\frac{\alpha}{m}f(t_k)-2\sigma_j(t_k-s)f'(t_k),\quad j=1,2,...,m-1,$$

$$\widetilde{f}_{m,k}\left(s\right)=\frac{1}{m}f(t_k)-2\sigma_m(t_k-s)f'(t_k),$$

$$\widetilde{f}_{m+j,k}\left(s\right)=\frac{\overline{\alpha}}{m}f(t_k)-2\sigma_{m+j}(t_k-s)f'(t_k),\quad j=1,2,...,m-2,$$

$$\widetilde{f}_{2m-1,k}\left(s\right)=\frac{\overline{\alpha}}{m}f(t_k)-2\sigma_{2m-1}(t_k-s)f'(t_k)+\frac{(t_k-s)^2}{2}f''(t_k),$$

$$v_k^0\left(t_k\right)=u_{k-1}\left(t_{k-1}\right),\quad u_0\left(0\right)=\varphi.$$

Hence we have:

$$v_k^{2m-1}\left(t_k\right)=V_1(\tau)u_{k-1}(t_{k-1})$$

$$+\sum_{j=1}^{2m-2}\int\limits_{t_{k-1}}^{t_k} V_1^{(j)}\left(\tau,t_k-s\right)\widetilde{f}_{j,k}\left(s\right)ds+\int\limits_{t_{k-1}}^{t_k} V_1^{(2m-1)}\left(t_k-s\right)\widetilde{f}_{2m-1,k}\left(s\right)ds$$

$$=V_1(\tau)u_{k-1}(t_{k-1})+\int\limits_{t_{k-1}}^{t_k}\left(\sum_{j=1}^{m-1}\frac{\alpha}{m}V_1^{(j)}\left(\tau,t_k-s\right)+\frac{1}{m}V_1^{(m)}\left(\tau,t_k-s\right)\right.$$

$$\left.+\sum_{j=1}^{m-2}\frac{\overline{\alpha}}{m}V_1^{(m+j)}\left(\tau,t_k-s\right)+\frac{\overline{\alpha}}{m}V_1^{(2m-1)}\left(t_k-s\right)\right)f\left(t_k\right)ds$$

$$+\int\limits_{t_{k-1}}^{t_k}\left(\sum_{j=1}^{m-1}2\sigma_jV_1^{(j)}\left(\tau,t_k-s\right)+2\sigma_mV_1^{(m)}\left(\tau,t_k-s\right)\right.$$

$$+\sum_{j=1}^{m-2} 2\sigma_{m+j} V_1^{(m+j)}(\tau, t_k - s) + 2\sigma_{2m-1} V_1^{(2m-1)}(t_k - s)\Bigg) (t_k - s) f'(t_k)\, ds$$

$$+\int_{t_{k-1}}^{t_k} V_1^{(2m-1)}(t_k - s) \frac{(t_k - s)^2}{2} f''(t_k) ds. \tag{3.14}$$

Clearly the analogous formula is true for the $w_k^{2m-1}(t_k)$:

$$w_k^{2m-1}(t_k) = V_2(\tau) u_{k-1}(t_{k-1})$$

$$+\int_{t_{k-1}}^{t_k} \Bigg(\sum_{i=1}^{m-1} \frac{\alpha}{m} V_2^{(i)}(\tau, t_k - s) + \frac{1}{m} V_2^{(m)}(\tau, t_k - s)$$

$$+\sum_{j=1}^{m-2} \frac{\overline{\alpha}}{m} V_2^{(m+j)}(\tau, t_k - s) + \frac{\overline{\alpha}}{m} V_2^{(2m-1)}(t_k - s)\Bigg) f(t_k)\, ds$$

$$+\int_{t_{k-1}}^{t_k} \Bigg(\sum_{j=1}^{m-1} 2\sigma_j V_2^{(j)}(\tau, t_k - s) + 2\sigma_m V_2^{(m)}(\tau, t_k - s)$$

$$+\sum_{j=1}^{m-2} 2\sigma_{m+j} V_2^{(m+j)}(\tau, t_k - s) + 2\sigma_{2m-1} V_2^{(2m-1)}(t_k - s)\Bigg) (t_k - s) f'(t_k)\, ds$$

$$+\int_{t_{k-1}}^{t_k} V_2^{(2m-1)}(t_k - s) \frac{(t_k - s)^2}{2} f''(t_k) ds. \tag{3.15}$$

From the equalities (3.14) and (3.15) we obtain:

$$u_k(t_k) = V(\tau) u_{k-1}(t_{k-1}) + F_k^{(2)} = V^k(\tau)\varphi + \sum_{i=1}^{k} V^{k-i}(\tau) F_i^{(2)}. \quad \blacksquare$$

Lemma 3.3 *Let the conditions a)) and b)) of the theorem be satisfied, then the following estimation holds:*

$$\left\| \int_0^{\tau} \left[U(t, A) - W(\tau, t; \alpha_1, ..., \alpha_{2m-1}) \right] \varphi dt \right\| \le c e^{\omega_0 \tau} \tau^4 \left\| \varphi \right\|_{A^3}, \quad \varphi \in D\left(A^3\right),$$

where

$$\alpha_j = \begin{cases} \alpha/m, & j < m, \\ 1/m, & j = m, \\ \overline{\alpha}/m, & j > m, \end{cases}$$

here c, ω_0 *are positive constants.*

Proof. If we decompose all semigroups in the $W(\tau, t; \alpha_1, ..., \alpha_{2m-1})$ from right to left according to the formula (3.1) so that each residual member is of the third degree, we get the following formula:

$$W(\tau, t; \alpha_1, ..., \alpha_{2m-1}) = \sum_{i=1}^{2m-1} \alpha_i I - \tau \sum_{i=1}^{m} [W(\tau, t; \alpha_1, ..., \alpha_{2m-1})]_i A_i$$

$$+\tau^2 \sum_{k,i=1}^{m} [W(\tau, t; \alpha_1, ..., \alpha_{2m-1})]_{k,i} A_k A_i + \widetilde{R}_3(\tau, t), \qquad (3.16)$$

where the following estimation is true for $\widetilde{R}_3(\tau, t)$:

$$\left\| \int_0^\tau \widetilde{R}_3(\tau, t) \varphi dt \right\| \leq c e^{\omega_0 \tau} \tau^4 \|\varphi\|_{A^3,} \quad \varphi \in D\left(A^3\right). \qquad (3.17)$$

In case of two addends ($m = 2$) the residual member is explicitly written and estimated in [25].

According to the addition property of $[\cdot]_i$ and $[\cdot]_{k,i}$ operations we have:

$$[W(\tau, t; \alpha_1, ..., \alpha_{2m-1})]_i = \frac{1}{2} \left[\sum_{j=1}^{2m-2} \alpha_j \left[V_1^{(j)}(\tau, t)\right]_i \right.$$

$$\left. + \sum_{j=1}^{2m-2} \alpha_j \left[V_2^{(j)}(\tau, t)\right]_i + \alpha_{2m-1} \left[V_1^{(2m-1)}(t)\right]_i + \alpha_{2m-1} \left[V_2^{(2m-1)}(t)\right]_i \right]. \qquad (3.18)$$

$$[W(\tau, t; \alpha_1, ..., \alpha_{2m-1})]_{k,i} = \frac{1}{2} \left[\sum_{j=1}^{2m-2} \alpha_j \left[V_1^{(j)}(\tau, t)\right]_{k,i} \right.$$

$$\left. + \sum_{j=1}^{2m-2} \alpha_j \left[V_2^{(j)}(\tau, t)\right]_{k,i} + \alpha_{2m-1} \left[V_1^{(2m-1)}(t)\right]_{k,i} + \alpha_{2m-1} \left[V_2^{(2m-1)}(t)\right]_{k,i} \right]. \qquad (3.19)$$

Let us compute the coefficients $\left[V_s^{(j)}(\tau, t)\right]_i$ $(s = 1, 2)$ including in the formula (3.18). Clearly, we have:

$$\left[V_s^{(j)}(\tau, t)\right]_i = \begin{cases} [U(\tau, \overline{\alpha} A_i) U(\tau, \alpha A_i)]_i = [U(\tau, A_i)]_i = 1, & j < l, \\ [U(\tau, \overline{\alpha} A_i) U(t, \alpha A_i)]_i = \overline{\alpha} + \alpha t/\tau, & j = l, \\ [U(\tau, \overline{\alpha} A_i)]_i = \overline{\alpha}, & l < j < 2m - l, \\ [U(t, \overline{\alpha} A_i)]_i = \overline{\alpha} t/\tau, & j = 2m - l, \\ 0, & j > 2m - l, \end{cases}$$

where $i = s, ..., m-2+s$, $j = 1, .., 2m-2$; $l = i$, when $s = 1$ and $l = m-i+1$, when $s = 2$;

$$\left[V_s^{(j)}(\tau,t)\right]_l = \begin{cases} [U(\tau, A_l)]_l = 1, & j < m, \\ [U(t, A_l)]_l = t/\tau, & j = m, \\ 0, & j > m, \end{cases}$$

where $j = 1, ..., 2m-2$; $l = m$, when $s = 1$ and $l = 1$, when $s = 2$;

$$\left[V_s^{(2m-1)}(t)\right]_i = \begin{cases} [U(t, \overline{\alpha} A_l)]_l = \overline{\alpha} t/\tau, & i = l, \\ 0, & i \neq l, \end{cases}$$

where $l = 1$, when $s = 1$ and $l = m$, when $s = 2$.

For $2 \leq i \leq m-1$ according to these formulas from the formula (3.18) we get:

$$2\left[W(\tau, t, \alpha_1, ..., \alpha_{2m-1})\right]_i = \sum_{j=1}^{i-1} \left[V_1^{(j)}(\tau,t)\right]_i \alpha_j + \left[V_1^{(i)}(\tau,t)\right]_i \alpha_i$$

$$+ \sum_{j=i+1}^{2m-i-1} \left[V_1^{(j)}(\tau,t)\right]_i \alpha_j + \left[V_1^{(2m-i)}(\tau,t)\right]_i \alpha_{2m-i}$$

$$+ \sum_{j=1}^{m-i} \left[V_2^{(j)}(\tau,t)\right]_i \alpha_j + \left[V_2^{(m-i+1)}(\tau,t)\right]_i \alpha_{m-i+1}$$

$$+ \sum_{j=m-i+2}^{m+i-2} \left[V_2^{(j)}(\tau,t)\right]_i \alpha_j + \left[V_2^{(m+i)}(\tau,t)\right]_i \alpha_{m+i}$$

$$= \sum_{j=1}^{i-1} \alpha_j + \left(\overline{\alpha} + \alpha \frac{t}{\tau}\right) \alpha_i + \sum_{j=i+1}^{2m-i-1} \overline{\alpha} \alpha_j + \overline{\alpha} \frac{t}{\tau} \alpha_{2m-i}$$

$$+ \sum_{j=1}^{m-i} \alpha_j + \left(\overline{\alpha} + \alpha \frac{t}{\tau}\right) \alpha_{m-i+1} + \sum_{j=m-i+2}^{m+i-2} \overline{\alpha} \alpha_j + \overline{\alpha} \frac{t}{\tau} \alpha_{m+i}$$

$$= \sum_{j=1}^{i-1} \alpha_j + \left(\overline{\alpha} + \alpha \frac{t}{\tau}\right) \alpha_i + \sum_{j=i+1}^{m-1} \overline{\alpha} \alpha_j + \overline{\alpha} \alpha_m + \sum_{j=m+1}^{2m-i-1} \overline{\alpha} \alpha_j + \overline{\alpha} \frac{t}{\tau} \alpha_{2m-i}$$

$$+ \sum_{j=1}^{m-i} \alpha_j + \left(\overline{\alpha} + \alpha \frac{t}{\tau}\right) \alpha_{m-i+1} + \sum_{j=m-i+2}^{m-1} \overline{\alpha} \alpha_j + \overline{\alpha} \alpha_m + \sum_{j=m+1}^{m+i-2} \overline{\alpha} \alpha_j + \overline{\alpha} \frac{t}{\tau} \alpha_{m+i}$$

$$= \sum_{j=1}^{i-1} \frac{\alpha}{m} + \frac{\alpha}{m} \left(\overline{\alpha} + \alpha \frac{t}{\tau}\right) + \sum_{j=i+1}^{m-1} \frac{\alpha}{m} \overline{\alpha} + \frac{1}{m} \overline{\alpha} + \sum_{j=m+1}^{2m-i-1} \frac{\overline{\alpha}}{m} \overline{\alpha} + \frac{\overline{\alpha}}{m} \overline{\alpha} \frac{t}{\tau}$$

$$+\sum_{j=1}^{m-i}\frac{\alpha}{m}+\frac{\alpha}{m}\left(\overline{\alpha}+\alpha\frac{t}{\tau}\right)+\sum_{j=m-i+2}^{m-1}\frac{\alpha}{m}\overline{\alpha}+\frac{1}{m}\overline{\alpha}+\sum_{j=m+1}^{m+i-2}\frac{\overline{\alpha}}{m}\overline{\alpha}+\frac{\overline{\alpha}}{m}\overline{\alpha}\frac{t}{\tau}$$
$$=\frac{(m-1)}{m}+\frac{1}{m}\left(\frac{2}{3}+\frac{2}{3}\frac{t}{\tau}\right).$$

Let us integrate this equality from 0 to τ, we obtain:

$$\int_0^\tau [W(\tau,t;\alpha_1,...,\alpha_{2m-1})]_i\,dt=\frac{1}{2}\tau\left(\frac{(m-1)}{m}+\frac{1}{m}\left(\frac{2}{3}+\frac{1}{3}\right)\right)$$
$$=\frac{1}{2}\tau\left(\frac{(m-1)}{m}+\frac{1}{m}\right)=\frac{1}{2}\tau,\quad 2\le i\le m-1.$$

Let us consider the case, when $i=1$:

$$2[W(\tau,t;\alpha_1,...,\alpha_{2m-1})]_1=\left[V_1^{(1)}(\tau,t)\right]_1\alpha_1+\sum_{j=2}^{2m-2}\left[V_1^{(j)}(\tau,t)\right]_1\alpha_j$$
$$+\left[V_1^{(2m-1)}(\tau,t)\right]_1\alpha_{2m-1}+\sum_{j=1}^{m-1}\left[V_2^{(j)}(\tau,t)\right]_i\alpha_j+\left[V_2^{(m)}(\tau,t)\right]_1\alpha_m$$
$$=\left(\overline{\alpha}+\alpha\frac{t}{\tau}\right)\alpha_1+\sum_{j=2}^{2m-2}\overline{\alpha}\alpha_j+\overline{\alpha}\frac{t}{\tau}\alpha_{2m-1}+\sum_{j=1}^{m-1}\alpha_j+\frac{t}{\tau}\alpha_m$$
$$=\left(\overline{\alpha}+\alpha\frac{t}{\tau}\right)\alpha_1+\sum_{j=2}^{m-1}\overline{\alpha}\alpha_j+\overline{\alpha}\alpha_m+\sum_{j=m+1}^{2m-2}\overline{\alpha}\alpha_j+\overline{\alpha}\frac{t}{\tau}\alpha_{2m-1}+\sum_{j=1}^{m-1}\alpha_j+\frac{t}{\tau}\alpha_m$$
$$=\frac{\alpha}{m}\left(\overline{\alpha}+\alpha\frac{t}{\tau}\right)+\sum_{j=2}^{m-1}\alpha\overline{\alpha}+\frac{1}{m}\overline{\alpha}+\sum_{j=m+1}^{2m-2}\frac{1}{m}\overline{\alpha}^2+\frac{\overline{\alpha}}{m}\overline{\alpha}\frac{t}{\tau}+\sum_{j=1}^{m-1}\alpha+\frac{1}{m}\frac{t}{\tau}$$
$$=\frac{1}{m}\left(\frac{1}{3}+\frac{4}{3}\frac{t}{\tau}\right)+\frac{m-1}{m}.$$

Integrating this equality from 0 to τ we obtain:

$$\int_0^\tau [W(\tau,t;\alpha_1,...,\alpha_{2m-1})]_1\,dt=\frac{1}{2m}\left(\frac{1}{3}+\frac{2}{3}\right)\tau+\frac{m-1}{2m}\tau=\frac{1}{2}\tau.$$

Analogously we get

$$\int_0^\tau [W(\tau,t;\alpha_1,...,\alpha_{2m-1})]_m\,dt=\frac{1}{2}\tau.$$

Finally we have:

$$\int_0^\tau [W(\tau, t; \alpha_1, ..., \alpha_{2m-1})]_i \, dt = \frac{1}{2}\tau, \quad i = 1, 2, ...m. \tag{3.20}$$

Let us compute the coefficients $\left[V_s^{(j)}(\tau, t)\right]_{k,i}$ $(s = 1, 2)$ including in the formula (3.5). Clearly, we have:

$$\left[V_s^{(j)}(\tau, t)\right]_{i,i} = \begin{cases} [U(\tau, \overline{\alpha} A_i) U(\tau, \alpha A_i)]_{i,i} = [U(\tau, A_i)]_{i,i} = 1/2, & j < l, \\ [U(\tau, \overline{\alpha} A_i) U(t, \alpha A_i)]_{i,i} = 1/2\overline{\alpha}^2 + \alpha^2 t^2/\left(2\tau^2\right) & \\ \qquad + \alpha\overline{\alpha} t/\tau, & j = l, \\ [U(\tau, \overline{\alpha} A_i)]_{i,i} = 1/2\overline{\alpha}^2, & l < j < 2m - l, \\ [U(t, \overline{\alpha} A_i)]_{i,i} = \overline{\alpha}^2 t^2/\left(2\tau^2\right), & j = 2m - l, \\ 0, & j > 2m - l, \end{cases}$$

where $l = i$, when $s = 1$ and $l = m - i + 1$, when $s = 2$; $\quad i = s, ..., m - 2 + s, \quad j = 1, .., 2m - 2;$

$$\left[V_s^{(j)}(\tau, t)\right]_{l,l} = \begin{cases} [U(\tau, A_l)]_{l,l} = 1/2, & j < m, \\ [U(t, A_l)]_{l,l} = t^2/\left(2\tau^2\right), & j = m, \\ 0, & j > m, \end{cases}$$

where $l = m$, when $s = 1$ and $l = 1$, when $s = 2$; $\quad j = 1, .., 2m - 2;$

$$\left[V_s^{(2m-1)}(t)\right]_{i,i} = \begin{cases} [U(t, \overline{\alpha} A_l)]_{l,l} = \overline{\alpha}^2 t^2/\left(2\tau^2\right), & i = l, \\ 0, & i \neq l, \end{cases}$$

where $l = 1$, when $s = 1$ and $l = m$, when $s = 2$;

$$\left[V_s^{(j)}(\tau, t)\right]_{k,i} = \begin{cases} [U(\tau, \overline{\alpha} A_k) U(\tau, A_i)]_{k,i} = \overline{\alpha}, & j < l, \\ [U(\tau, \overline{\alpha} A_k) U(\tau, \overline{\alpha} A_i) U(t, \alpha A_i)]_{k,i} & \\ \qquad = \overline{\alpha}^2 + \alpha\overline{\alpha} t/\tau, & j = l, \\ [U(\tau, \overline{\alpha} A_k) U(\tau, \overline{\alpha} A_i)]_{k,i} = \overline{\alpha}^2, & l < j < 2m - l, \\ [U(\tau, \overline{\alpha} A_k) U(t, \overline{\alpha} A_i)]_{k,i} = \overline{\alpha}^2 t/\tau, & j = 2m - l, \\ 0, & j > 2m - l, \end{cases}$$

where $\quad i, k = 1, .., m - 1, \quad k \neq i, \quad j = 1, ..., 2m - 2; \quad l = i$, when $s = 1$ and $l = m - i + 1$, when $s = 2$; $\quad$ also $k < i$ when $s = 1$ and $k > i$ when $s = 2$;

$$\left[V_s^{(j)}(\tau, t)\right]_{k,i} = \begin{cases} [U(\tau, A_k) U(\tau, \alpha A_i)]_{k,i} = \alpha, & j < l, \\ [U(\tau, A_k) U(t, \alpha A_i)]_{k,i} = \alpha t/\tau, & j = l, \\ 0, & j > l, \end{cases}$$

where $i,k=1,..,m-1$, $k\neq i$, $j=1,...,2m-2$; $l=i$, when $s=1$ and $l=m-i+1$, when $s=2$; also $k>i$ when $s=1$ and $k<i$ when $s=2$;

$$\left[V_s^{(j)}(\tau,t)\right]_{k,l}=\begin{cases}[U(\tau,\overline{\alpha}A_k)\,U(\tau,A_l)]_{k,l}=\overline{\alpha}, & j<m,\\ [U(\tau,\overline{\alpha}A_k)\,U(t,A_l)]_{k,l}=\overline{\alpha}t/\tau, & j=m,\\ 0, & j>m,\end{cases}$$

where $k=s,..,m-2+s$, $j=1,...,2m-2$; $l=m$, when $s=1$ and $l=1$, when $s=2$;

$$\left[V_1^{(2m-1)}(t)\right]_{k,i}=\left[V_2^{(2m-1)}(t)\right]_{k,i}=0,\quad k\neq i.$$

For $2\leq i\leq m-1$ according to these formulas from the formula (3.19) we get:

$$2\left[W(\tau,t;\alpha_1,...,\alpha_{2m-1})\right]_{i,i}=\sum_{j=1}^{i-1}\left[V_1^{(j)}(\tau,t)\right]_{i,i}\alpha_j+\left[V_1^{(i)}(\tau,t)\right]_{i,i}\alpha_i$$

$$+\sum_{j=i+1}^{2m-i-1}\left[V_1^{(j)}(\tau,t)\right]_{i,i}\alpha_j+\left[V_1^{(2m-i)}(\tau,t)\right]_{i,i}\alpha_{2m-i}$$

$$+\sum_{j=1}^{m-i}\left[V_2^{(j)}(\tau,t)\right]_{i,i}\alpha_j+\left[V_2^{(m-i+1)}(\tau,t)\right]_{i,i}\alpha_{m-i+1}$$

$$+\sum_{j=m-i+2}^{m+i-2}\left[V_2^{(j)}(\tau,t)\right]_{i,i}\alpha_j+\left[V_2^{(m+i)}(\tau,t)\right]_{i,i}\alpha_{m+i}$$

$$=\sum_{j=1}^{i-1}\frac{1}{2}\alpha_j+\alpha_i\left(\frac{1}{2}\overline{\alpha}^2+\frac{1}{2}\alpha^2\frac{t^2}{\tau^2}+\alpha\overline{\alpha}\frac{t}{\tau}\right)+\sum_{j=i+1}^{2m-i-1}\frac{1}{2}\alpha_j\overline{\alpha}^2+\frac{1}{2}\alpha_{2m-i}\overline{\alpha}^2\frac{t^2}{\tau^2}$$

$$+\sum_{j=1}^{m-i}\frac{1}{2}\alpha_j+\alpha_{m-i+1}\left(\frac{1}{2}\overline{\alpha}^2+\frac{1}{2}\alpha^2\frac{t^2}{\tau^2}+\alpha\overline{\alpha}\frac{t}{\tau}\right)+\sum_{j=m-i+2}^{m+i-2}\frac{1}{2}\alpha_j\overline{\alpha}^2+\frac{1}{2}\alpha_{m+i}\overline{\alpha}^2\frac{t^2}{\tau^2}$$

$$=\sum_{j=1}^{i-1}\frac{1}{2}\alpha_j+\alpha_i\left(\frac{1}{2}\overline{\alpha}^2+\frac{1}{2}\alpha^2\frac{t^2}{\tau^2}+\alpha\overline{\alpha}\frac{t}{\tau}\right)+\sum_{j=i+1}^{m-1}\frac{1}{2}\alpha_j\overline{\alpha}^2$$

$$+\frac{1}{2}\alpha_m\overline{\alpha}^2+\sum_{j=m+1}^{2m-i-1}\frac{1}{2}\alpha_j\overline{\alpha}^2+\frac{1}{2}\alpha_{2m-i}\overline{\alpha}^2\frac{t^2}{\tau^2}$$

$$+\sum_{j=1}^{m-i}\frac{1}{2}\alpha_j+\alpha_{m-i+1}\left(\frac{1}{2}\overline{\alpha}^2+\frac{1}{2}\alpha^2\frac{t^2}{\tau^2}+\alpha\overline{\alpha}\frac{t}{\tau}\right)+\sum_{j=m-i+2}^{m-1}\overline{\alpha}^2\frac{1}{2}\alpha_j$$

$$+\overline{\alpha}^2\frac{1}{2}\alpha_m+\sum_{j=m+1}^{m+i-2}\overline{\alpha}^2\frac{1}{2}\alpha_j+\frac{1}{2}\overline{\alpha}^2\frac{t^2}{\tau^2}\alpha_{m+i}$$

$$=\sum_{j=1}^{i-1}\frac{\alpha}{2m}+\frac{\alpha}{m}\left(\frac{1}{2}\overline{\alpha}^2+\frac{1}{2}\alpha^2\frac{t^2}{\tau^2}+\alpha\overline{\alpha}\frac{t}{\tau}\right)+\sum_{j=i+1}^{m-1}\frac{1}{2}\frac{\alpha}{m}\overline{\alpha}^2$$

$$+\frac{1}{2m}\overline{\alpha}^2+\sum_{j=m+1}^{2m-i-1}\frac{\overline{\alpha}}{2m}\overline{\alpha}^2+\frac{\overline{\alpha}}{2m}\overline{\alpha}^2\frac{t^2}{\tau^2}$$

$$+\sum_{j=1}^{m-i}\frac{\alpha}{2m}+\frac{\alpha}{m}\left(\frac{1}{2}\overline{\alpha}^2+\frac{1}{2}\alpha^2\frac{t^2}{\tau^2}+\alpha\overline{\alpha}\frac{t}{\tau}\right)+\sum_{j=m-i+2}^{m-1}\frac{\alpha}{2m}\overline{\alpha}^2$$

$$+\frac{1}{2m}\overline{\alpha}^2+\sum_{j=m+1}^{m+i-2}\frac{\overline{\alpha}}{2m}\overline{\alpha}^2+\frac{\overline{\alpha}}{2m}\overline{\alpha}^2\frac{t^2}{\tau^2}$$

$$=\frac{(m-1)}{3m}+\frac{1}{m}\left(\frac{1}{3}\overline{\alpha}+\frac{2}{3}\alpha\frac{t}{\tau}\right).$$

If we integrate this equality from 0 to τ, we obtain:

$$\int_0^\tau\left[W\left(\tau,t;\alpha_1,...,\alpha_{2m-1}\right)\right]_{i,i}dt=\frac{1}{2}\tau\left(\frac{m-1}{3m}+\frac{1}{m}\left(\frac{1}{3}\overline{\alpha}+\frac{1}{3}\alpha\right)\right)=\frac{1}{6}\tau.$$

Let us consider the case when $i=m$:

$$2\left[W\left(\tau,t;\alpha_1,...,\alpha_{2m-1}\right)\right]_{m,m}=\sum_{j=1}^{m-1}\left[V_1^{(j)}\left(\tau,t\right)\right]_{m,m}\alpha_j+\left[V_1^{(m)}\left(\tau,t\right)\right]_{m,m}\alpha_m$$

$$+\left[V_2^{(1)}\left(\tau,t\right)\right]_{m,m}\alpha_1+\sum_{j=2}^{2m-2}\left[V_2^{(j)}\left(\tau,t\right)\right]_{m,m}\alpha_j+\left[V_2^{(2m-1)}\left(\tau,t\right)\right]_{m,m}\alpha_{2m-1}$$

$$=\sum_{j=1}^{m-1}\frac{1}{2}\alpha_j+\frac{1}{2}\frac{t^2}{\tau^2}\alpha_m+\left(\frac{1}{2}\overline{\alpha}^2+\frac{1}{2}\alpha^2\frac{t^2}{\tau^2}+\alpha\overline{\alpha}\frac{t}{\tau}\right)\alpha_1$$

$$+\sum_{j=2}^{2m-2}\overline{\alpha}^2\frac{1}{2}\alpha_j+\frac{1}{2}\overline{\alpha}^2\frac{t^2}{\tau^2}\alpha_{2m-1}=\sum_{j=1}^{m-1}\frac{1}{2}\alpha_j+\frac{1}{2}\frac{t^2}{\tau^2}\alpha_m$$

$$+\left(\frac{1}{2}\overline{\alpha}^2+\frac{1}{2}\alpha^2\frac{t^2}{\tau^2}+\alpha\overline{\alpha}\frac{t}{\tau}\right)\alpha_1+\sum_{j=2}^{m-1}\frac{1}{2}\overline{\alpha}^2\alpha_j$$

$$+\frac{1}{2}\overline{\alpha}^2\alpha_m+\sum_{j=m+1}^{2m-2}\frac{1}{2}\overline{\alpha}^2\alpha_j+\frac{1}{2}\overline{\alpha}^2\frac{t^2}{\tau^2}\alpha_{2m-1}$$

$$=\sum_{j=1}^{m-1}\frac{1}{2m}\alpha+\frac{1}{2m}\frac{t^2}{\tau^2}+\frac{1}{m}\alpha\left(\frac{1}{2}\overline{\alpha}^2+\frac{1}{2}\alpha^2\frac{t^2}{\tau^2}+\alpha\overline{\alpha}\frac{t}{\tau}\right)$$

$$+\sum_{j=2}^{m-1}\frac{1}{2m}\alpha\overline{\alpha}^2+\frac{1}{2m}\overline{\alpha}^2+\sum_{j=m+1}^{2m-2}\frac{1}{2}\overline{\alpha}^3+\frac{1}{2m}\overline{\alpha}^3\frac{t^2}{\tau^2}$$

$$=\frac{m-1}{2m}\alpha+\frac{1}{2m}\frac{t^2}{\tau^2}+\frac{1}{m}\alpha\left(\frac{1}{2}\overline{\alpha}^2+\frac{1}{2}\alpha^2\frac{t^2}{\tau^2}+\alpha\overline{\alpha}\frac{t}{\tau}\right)$$

$$+\frac{m-2}{2m}\alpha\overline{\alpha}^2+\frac{1}{2m}\overline{\alpha}^2+\frac{m-2}{2m}\overline{\alpha}^3+\frac{1}{2m}\overline{\alpha}^3\frac{t^2}{\tau^2}$$

$$=\frac{m-1}{2m}\left(\alpha+\overline{\alpha}^2\right)+\frac{1}{2m}\frac{t^2}{\tau^2}+\frac{1}{m}\left(\frac{1}{2}\alpha\overline{\alpha}^2+\frac{1}{2}\alpha^3\frac{t^2}{\tau^2}+\alpha^2\overline{\alpha}\frac{t}{\tau}+\frac{1}{2}\overline{\alpha}^3\frac{t^2}{\tau^2}\right)$$

$$=\frac{m-1}{3m}+\frac{1}{2m}\frac{t^2}{\tau^2}+\frac{1}{m}\left(\frac{1}{6}\overline{\alpha}+\frac{1}{3}\alpha\frac{t}{\tau}\right).$$

If we integrate this equality from 0 to τ, we obtain:

$$\int_0^\tau\left[W\left(\tau,t;\alpha_1,...,\alpha_{2m-1}\right)\right]_{m,m}dt=\frac{1}{2}\tau\left(\frac{m-1}{3m}+\frac{1}{6m}+\frac{1}{m}\left(\frac{1}{6}\overline{\alpha}+\frac{1}{6}\alpha\right)\right)=\frac{1}{6}\tau.$$

Analogously we obtain that,

$$\int_0^\tau\left[W\left(\tau,t;\alpha_1,...,\alpha_{2m-1}\right)\right]_{1,1}dt=\frac{1}{6}\tau.$$

Finally we have:

$$\int_0^\tau\left[W\left(\tau,t;\alpha_1,...,\alpha_{2m-1}\right)\right]_{i,i}dt=\frac{1}{6}\tau,\quad i=1,2,\cdots,m.$$

Now let us compute the coefficients $2\left[W\left(\tau,t,\alpha_1,...,\alpha_{2m-1}\right)\right]_{k,i}$ in case, when $k<i$, where $k=1,2,...,m-1$ and $i=2,3,...,m$:

$$2\left[W\left(\tau,t;\alpha_1,...,\alpha_{2m-1}\right)\right]_{k,i}=\sum_{j=1}^{i-1}\left[V_1^{(j)}\left(\tau,t\right)\right]_{k,i}\alpha_j+\left[V_1^{(i)}\left(\tau,t\right)\right]_{k,i}\alpha_i$$

$$+\sum_{j=i+1}^{2m-i-1}\left[V_1^{(j)}(\tau,t)\right]_{k,i}\alpha_j+\left[V_1^{(2m-i)}(\tau,t)\right]_{k,i}\alpha_{2m-i}$$

$$+\sum_{j=1}^{m-i}\left[V_2^{(j)}(\tau,t)\right]_{k,i}\alpha_j+\left[V_2^{(m-i+1)}(\tau,t)\right]_{k,i}\alpha_{m-i+1}$$

$$+\sum_{j=m-i+2}^{m+i-2}\left[V_2^{(j)}(\tau,t)\right]_{k,i}\alpha_j+\left[V_2^{(m+i)}(\tau,t)\right]_{k,i}\alpha_{m+i}$$

$$=\sum_{j=1}^{i-1}\overline{\alpha}\alpha_j+\left(\overline{\alpha}^2+\alpha\overline{\alpha}\frac{t}{\tau}\right)\alpha_i+\sum_{j=i+1}^{2m-i-1}\overline{\alpha}^2\alpha_j$$

$$+\overline{\alpha}^2\frac{t}{\tau}\alpha_{2m-i}+\sum_{j=1}^{m-i}\alpha\alpha_j+\alpha\frac{t}{\tau}\alpha_{m-i+1}$$

$$=\sum_{j=1}^{i-1}\overline{\alpha}\alpha_j+\left(\overline{\alpha}^2+\alpha\overline{\alpha}\frac{t}{\tau}\right)\alpha_i+\sum_{j=i+1}^{m-1}\overline{\alpha}^2\alpha_j+\overline{\alpha}^2\alpha_m$$

$$+\sum_{j=m+1}^{2m-i-1}\overline{\alpha}^2\alpha_j+\overline{\alpha}^2\frac{t}{\tau}\alpha_{2m-i}+\sum_{j=1}^{m-i}\alpha\alpha_j+\alpha\frac{t}{\tau}\alpha_{m-i+1}$$

$$=\sum_{j=1}^{i-1}\overline{\alpha}\frac{\alpha}{m}+\frac{\alpha}{m}\left(\overline{\alpha}^2+\alpha\overline{\alpha}\frac{t}{\tau}\right)+\sum_{j=i+1}^{m-1}\frac{\alpha}{m}\overline{\alpha}^2+\frac{1}{m}\overline{\alpha}^2$$

$$+\sum_{j=m+1}^{2m-i-1}\frac{\overline{\alpha}}{m}\overline{\alpha}^2+\frac{\overline{\alpha}}{m}\overline{\alpha}^2\frac{t}{\tau}+\sum_{j=1}^{m-i}\alpha\frac{\alpha}{m}+\frac{\alpha}{m}\alpha\frac{t}{\tau}$$

$$=\frac{(i-1)}{m}\alpha\overline{\alpha}+\frac{1}{m}\alpha\left(\overline{\alpha}^2+\alpha\overline{\alpha}\frac{t}{\tau}\right)+\frac{(m-i-1)}{m}\alpha\overline{\alpha}^2+\frac{1}{m}\overline{\alpha}^2$$

$$+\frac{(m-i-1)}{m}\overline{\alpha}^3+\frac{1}{m}\overline{\alpha}^3\frac{t}{\tau}+\frac{(m-i)}{m}\alpha^2+\frac{1}{m}\alpha^2\frac{t}{\tau}$$

$$=\frac{(m-1)}{3m}+\frac{1}{3m}\left(\overline{\alpha}+2\alpha\frac{t}{\tau}\right).$$

Integrating this equality from 0 to τ we obtain:

$$\int_0^{\tau}\left[W(\tau,t;\alpha_1,...,\alpha_{2m-1})\right]_{k,i}dt=\frac{1}{2}\tau\left(\frac{(m-1)}{3m}+\frac{1}{3m}(\overline{\alpha}+\alpha)\right)=\frac{1}{6}\tau,$$

where $k < i,\ k = 1, ..., m-1$ and $i = 2, 3, ..., m$. Analogously we can obtain same equality, when $k > i,\ k = 2, 3, ..., m$ and $i = 1, ..., m-1$. Let us compute the coefficients $2\left[V\left(\tau, t, \alpha_1, ..., \alpha_{2m-1}\right)\right]_{k,m},\ k = 2, ..., m-1$:

$$2\left[W\left(\tau, t; \alpha_1, ..., \alpha_{2m-1}\right)\right]_{k,m} = \sum_{j=1}^{m-1}\left[V_1^{(j)}\left(\tau, t\right)\right]_{k,m}\alpha_j + \left[V_1^{(i)}\left(\tau, t\right)\right]_{k,m}\alpha_i$$

$$+\sum_{j=i+1}^{2m-i-1}\left[V_1^{(j)}\left(\tau, t\right)\right]_{k,m}\alpha_j + \left[V_1^{(2m-i)}\left(\tau, t\right)\right]_{k,m}\alpha_{2m-i}$$

$$+\sum_{j=1}^{m-i}\left[V_2^{(j)}\left(\tau, t\right)\right]_{k,m}\alpha_j + \left[V_2^{(m-i+1)}\left(\tau, t\right)\right]_{k,m}\alpha_{m-i+1}$$

$$+\sum_{j=m-i+2}^{m+i-2}\left[V_2^{(j)}\left(\tau, t\right)\right]_{k,m}\alpha_j + \left[V_2^{(m+i)}\left(\tau, t\right)\right]_{k,m}\alpha_{m+i}$$

$$= \sum_{j=1}^{m-1}\overline{\alpha}\alpha_j + \overline{\alpha}\frac{t}{\tau}\alpha_m + \alpha\frac{t}{\tau}\alpha_1 = (m-1)\frac{\alpha}{m}\overline{\alpha} + \frac{1}{m}\overline{\alpha}\frac{t}{\tau} + \frac{\alpha}{m}\alpha\frac{t}{\tau}$$

$$= \frac{m-1}{3m} + \frac{2}{3m}\frac{t}{\tau}.$$

Let us integrate this equality from 0 to τ. We obtain:

$$\int_0^\tau\left[W\left(\tau, t; \alpha_1, ..., \alpha_{2m-1}\right)\right]_{k,m}dt = \frac{1}{2}\tau\left(\frac{m-1}{3m} + \frac{1}{3m}\right) = \frac{1}{6}\tau.$$

Analogously we get

$$\int_0^\tau\left[W\left(\tau, t; \alpha_1, ..., \alpha_{2m-1}\right)\right]_{m,k}dt = \frac{1}{6}\tau,$$

$$\int_0^\tau\left[W\left(\tau, t; \alpha_1, ..., \alpha_{2m-1}\right)\right]_{1,k}dt = \frac{1}{6}\tau,$$

$$\int_0^\tau\left[W\left(\tau, t; \alpha_1, ..., \alpha_{2m-1}\right)\right]_{k,1}dt = \frac{1}{6}\tau,$$

where $k = 2, ..., m-1$. Finally we have:

$$\int_0^\tau\left[W\left(\tau, t; \alpha_1, ..., \alpha_{2m-1}\right)\right]_{k,i}dt = \frac{1}{6}\tau, \quad k, i = 1, 2, ..., m. \tag{3.21}$$

Let us integrate the equality (3.16) from 0 to τ and take into account equalities (3.20) and (3.21), we get:

$$\int_0^\tau W(\tau, t; \alpha_1, ..., \alpha_{2m-1})\, dt = \tau I - \frac{1}{2}\tau^2 \sum_{i=1}^m A_i + \frac{1}{6}\tau^3 \sum_{k,i=1}^m A_k A_i$$

$$+ \int_0^\tau \widetilde{R}_3(\tau, t)\, dt = \tau I - \frac{1}{2}\tau^2 \sum_{i=1}^m A_i + \frac{1}{6}\tau^3 \left(\sum_{k,i=1}^m A_k A_i\right)^2 + \int_0^\tau \widetilde{R}_3(\tau, t)\, dt$$

$$= \tau I - \frac{1}{2}\tau^2 A + \frac{1}{6}\tau^3 A^2 + \int_0^\tau \widetilde{R}_3(\tau, t)\, dt. \qquad (3.22)$$

According to the formula (3.1) we have:

$$\int_0^\tau U(t, A)\, dt = \int_0^\tau \left(I - tA + \frac{1}{2}t^2 A^2 + R_3(t, A)\right) dt$$

$$= \tau I - \frac{1}{2}\tau^2 A + \frac{1}{6}\tau^3 A^2 + \int_0^\tau R_3(t, A)\, dt. \qquad (3.23)$$

The following estimation is true:

$$\left\| \int_0^\tau R_3(t, A)\, \varphi dt \right\| \leq c e^{\omega_0 \tau} \tau^4 \|\varphi\|_{A^3,} \quad \varphi \in D\left(A^3\right). \qquad (3.24)$$

From the equalities (3.22) and (3.23) we obtain:

$$\int_0^\tau [U(t, A) - W(\tau, t; \alpha_1, ..., \alpha_{2m-1})]\, \varphi dt = \int_0^\tau \widetilde{R}_3(\tau, t)\, \varphi dt - \int_0^\tau R_3(t, A)\, \varphi dt.$$

Hence according to the inequalities (3.17) and (3.24) we obtain the sought estimation. ■

Lemma 3.4. *Let the conditions a) and b) of the theorem be satisfied, then the following estimation holds:*

$$\left\| \int_0^\tau [U(t, A) - 2W(\tau, t; \sigma_1, ..., \sigma_{2m-1})]\, t\varphi ds \right\| \leq c e^{\omega_0 \tau} \tau^4 \|\varphi\|_{A^2}, \quad \varphi \in D\left(A^2\right),$$

where c, ω_0 are positive constants.

Proof. If we decompose all semigroups in the $W(\tau, t; \sigma_1, ..., \sigma_{2m-1})$ from right to left according to the formula (3.1) so that each residual member is of the second degree, we get the following formula:

$$2W(\tau, t; \sigma_1, ..., \sigma_{2m-1}) = \sum_{i=1}^{2m-1} 2\sigma_i I$$

$$-\tau \sum_{i=1}^{m} 2\left[W(\tau, t; \sigma_1, ..., \sigma_{2m-1})\right]_i A_i + \widetilde{R}_2(\tau, t), \tag{3.25}$$

where the following estimation is true for $\widetilde{R}_2(\tau, t)$:

$$\left\| \int_0^\tau t\widetilde{R}_2(\tau, t)\varphi dt \right\| \le ce^{\omega_0 \tau}\tau^4 \|\varphi\|_{A^2,} \quad \varphi \in D\left(A^2\right). \tag{3.26}$$

In case of two addends ($m = 2$) the residual member is explicitly written in [25].

Let us compute the following coefficients $2\left[V(\tau, t, \sigma_1, ..., \sigma_{2m-1})\right]_i$, $i = 2, ..., m-1$:

$$2\left[W(\tau, t; \sigma_1, ..., \sigma_{2m-1})\right]_i = \sum_{j=1}^{i-1}\left[V_1^{(j)}(\tau, t)\right]_i \sigma_j + \left[V_1^{(i)}(\tau, t)\right]_i \sigma_i$$

$$+ \sum_{j=i+1}^{2m-i-1}\left[V_1^{(j)}(\tau, t)\right]_i \sigma_j + \left[V_1^{(2m-i)}(\tau, t)\right]_i \sigma_{2m-i}$$

$$+ \sum_{j=1}^{m-i}\left[V_2^{(j)}(\tau, t)\right]_i \sigma_j + \left[V_2^{(m-i+1)}(\tau, t)\right]_i \sigma_{m-i+1}$$

$$+ \sum_{j=m-i+2}^{m+i-2}\left[V_2^{(j)}(\tau, t)\right]_i \sigma_j + \left[V_2^{(m+i)}(\tau, t)\right]_i \sigma_{m+i}$$

$$= \sum_{j=1}^{i-1}\sigma_j + \left(\overline{\alpha} + \alpha\frac{t}{\tau}\right)\sigma_i + \sum_{j=i+1}^{2m-i-1}\overline{\alpha}\sigma_j + \overline{\alpha}\frac{t}{\tau}\sigma_{2m-i}$$

$$+ \sum_{j=1}^{m-i}\sigma_j + \left(\overline{\alpha} + \alpha\frac{t}{\tau}\right)\sigma_{m-i+1} + \sum_{j=m-i+2}^{m+i-2}\overline{\alpha}\sigma_j + \overline{\alpha}\frac{t}{\tau}\sigma_{m+i}$$

$$= \sum_{j=1}^{i-1}\sigma_j + \left(\overline{\alpha} + \alpha\frac{t}{\tau}\right)\sigma_i + \sum_{j=i+1}^{m-1}\overline{\alpha}\sigma_j + \sigma_m\overline{\alpha}\sigma_j + \sum_{j=m+1}^{2m-i-1}\overline{\alpha}\sigma_j + \overline{\alpha}\frac{t}{\tau}\sigma_{2m-i}$$

$$+\sum_{j=1}^{m-i}\sigma_j+\left(\overline{\alpha}+\alpha\frac{t}{\tau}\right)\sigma_{m-i+1}+\sum_{j=m-i+2}^{m-1}\overline{\alpha}\sigma_j+\overline{\alpha}\sigma_m+\sum_{j=m+1}^{m+i-2}\overline{\alpha}\sigma_j+\overline{\alpha}\frac{t}{\tau}\sigma_{m+i}$$

$$=\sum_{j=1}^{i-1}\widetilde{\sigma}_1+\left(\overline{\alpha}+\alpha\frac{t}{\tau}\right)\widetilde{\sigma}_1+\sum_{j=i+1}^{m-1}\overline{\alpha}\widetilde{\sigma}_1+\overline{\alpha}\widetilde{\sigma}_2+\sum_{j=m+1}^{2m-i-1}\overline{\alpha}\widetilde{\sigma}_3+\overline{\alpha}\frac{t}{\tau}\widetilde{\sigma}_3$$

$$+\sum_{j=1}^{m-i}\widetilde{\sigma}_1+\left(\overline{\alpha}+\alpha\frac{t}{\tau}\right)\widetilde{\sigma}_1+\sum_{j=m-i+2}^{m-1}\overline{\alpha}\widetilde{\sigma}_1+\overline{\alpha}\widetilde{\sigma}_2+\sum_{j=m+1}^{m+i-2}\overline{\alpha}\widetilde{\sigma}_3+\overline{\alpha}\frac{t}{\tau}\widetilde{\sigma}_3$$

$$=(i-1)\,\widetilde{\sigma}_1+\left(\overline{\alpha}+\alpha\frac{t}{\tau}\right)\widetilde{\sigma}_1+(m-i-1)\,\overline{\alpha}\widetilde{\sigma}_1+\overline{\alpha}\widetilde{\sigma}_2+(m-i-1)\,\overline{\alpha}\widetilde{\sigma}_3+\overline{\alpha}\frac{t}{\tau}\widetilde{\sigma}_3$$

$$+(m-i)\,\widetilde{\sigma}_1+\left(\overline{\alpha}+\frac{t}{\tau}\alpha\right)\widetilde{\sigma}_1+(i-2)\,\overline{\alpha}\widetilde{\sigma}_1+\overline{\alpha}\widetilde{\sigma}_2+(i-2)\,\overline{\alpha}\widetilde{\sigma}_3+\overline{\alpha}\frac{t}{\tau}\widetilde{\sigma}_3$$

$$=(m-1)\,\widetilde{\sigma}_1+2\left(\overline{\alpha}+\alpha\frac{t}{\tau}\right)\widetilde{\sigma}_1+(m-3)\,\overline{\alpha}\widetilde{\sigma}_1+2\overline{\alpha}\widetilde{\sigma}_2+(m-3)\,\overline{\alpha}\widetilde{\sigma}_3+2\overline{\alpha}\frac{t}{\tau}\widetilde{\sigma}_3$$

$$=(m-1)\,\widetilde{\sigma}_1+2\left(\overline{\alpha}+\alpha\frac{t}{\tau}\right)\widetilde{\sigma}_1+(m-3)\,\overline{\alpha}\widetilde{\sigma}_1+(m-3)\,\overline{\alpha}\widetilde{\sigma}_3+2\overline{\alpha}\frac{t}{\tau}\widetilde{\sigma}_3$$

$$+2\overline{\alpha}\left((m-1)\,\widetilde{\sigma}_1+\widetilde{\sigma}_2+(m-1)\,\widetilde{\sigma}_3\right)-2\,(m-1)\,\overline{\alpha}\widetilde{\sigma}_1-2\overline{\alpha}\,(m-1)\,\widetilde{\sigma}_3$$

$$=\left[(m-1)+2\left(\overline{\alpha}+\alpha\frac{t}{\tau}\right)+(m-3)\,\overline{\alpha}-2\,(m-1)\,\overline{\alpha}\right]\widetilde{\sigma}_1$$

$$+\left[(m-3)\,\overline{\alpha}+2\overline{\alpha}\frac{t}{\tau}-2\overline{\alpha}\,(m-1)\right]\widetilde{\sigma}_3+\overline{\alpha}$$

$$=\left[(m-1)\,(1-\overline{\alpha})+2\alpha\frac{t}{\tau}\right]\widetilde{\sigma}_1+\left[2\overline{\alpha}\frac{t}{\tau}-(m+1)\,\overline{\alpha}\right]\widetilde{\sigma}_3+\overline{\alpha}$$

$$=\left[(m-1)+2\frac{t}{\tau}\right]\alpha\widetilde{\sigma}_1+\left[2\frac{t}{\tau}-(m+1)\right]\overline{\alpha}\widetilde{\sigma}_3+\overline{\alpha}.$$

Let us multiply this equality on t and integrate it from 0 to τ, we get:

$$\int_0^\tau 2\left[W\left(\tau,t;\sigma_1,...,\sigma_{2m-1}\right)\right]_i tdt=\left[\frac{(m-1)}{2}+\frac{2}{3}\right]\alpha\widetilde{\sigma}_1$$

$$+\left[\frac{2}{3}-\frac{(m+1)}{2}\right]\overline{\alpha}\widetilde{\sigma}_3+\frac{1}{2}\overline{\alpha}=\frac{3m+1}{6}\alpha\widetilde{\sigma}_1-\frac{3m-1}{6}\overline{\alpha}\widetilde{\sigma}_3+\frac{1}{2}\overline{\alpha}.$$

Let us insert into this equality values of parameters $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_3$. We get:

$$\int_0^\tau 2\left[W\left(\tau,t;\sigma_1,...,\sigma_{2m-1}\right)\right]_i tdt=\frac{1}{3}\tau^2,\quad i=2,...,m-1.$$

Now let us consider the case when $i = 1$:

$$2\left[W\left(\tau,t;\sigma_1,...,\sigma_{2m-1}\right)\right]_1 = \left[V_1^{(1)}\left(\tau,t\right)\right]_1\sigma_1 + \sum_{j=2}^{2m-2}\left[V_1^{(j)}\left(\tau,t\right)\right]_1\sigma_j$$

$$+\left[V_1^{(2m-1)}\left(\tau,t\right)\right]_1\sigma_{2m-1} + \sum_{j=1}^{m-1}\left[V_2^{(j)}\left(\tau,t\right)\right]_i\sigma_j + \left[V_2^{(m)}\left(\tau,t\right)\right]_1\sigma_m$$

$$= \left(\overline{\alpha} + \alpha\frac{t}{\tau}\right)\sigma_1 + \sum_{j=2}^{2m-2}\overline{\alpha}\sigma_j + \overline{\alpha}\frac{t}{\tau}\sigma_{2m-1} + \sum_{j=1}^{m-1}\sigma_j + \frac{t}{\tau}\sigma_m$$

$$= \left(\overline{\alpha} + \alpha\frac{t}{\tau}\right)\sigma_1 + \sum_{j=2}^{m-1}\overline{\alpha}\sigma_j + \overline{\alpha}\sigma_m + \sum_{j=m+1}^{2m-2}\overline{\alpha}\sigma_j + \overline{\alpha}\frac{t}{\tau}\sigma_{2m-1} + \sum_{j=1}^{m-1}\sigma_j + \frac{t}{\tau}\sigma_m$$

$$= \left(\overline{\alpha} + \alpha\frac{t}{\tau}\right)\widetilde{\sigma}_1 + \sum_{j=2}^{m-1}\overline{\alpha}\widetilde{\sigma}_1 + \overline{\alpha}\widetilde{\sigma}_2 + \sum_{j=m+1}^{2m-2}\overline{\alpha}\widetilde{\sigma}_3 + \overline{\alpha}\frac{t}{\tau}\widetilde{\sigma}_3 + \sum_{j=1}^{m-1}\widetilde{\sigma}_1 + \frac{t}{\tau}\widetilde{\sigma}_2$$

$$= \left(\overline{\alpha} + \alpha\frac{t}{\tau}\right)\widetilde{\sigma}_1 + (m-2)\,\overline{\alpha}\widetilde{\sigma}_1 + \overline{\alpha}\widetilde{\sigma}_2$$

$$+(m-2)\,\overline{\alpha}\widetilde{\sigma}_3 + \overline{\alpha}\frac{t}{\tau}\widetilde{\sigma}_3 + (m-1)\,\widetilde{\sigma}_1 + \frac{t}{\tau}\widetilde{\sigma}_2$$

$$= \left(\overline{\alpha} + \alpha\frac{t}{\tau}\right)\widetilde{\sigma}_1 + (m-2)\,\overline{\alpha}\widetilde{\sigma}_1 + (m-2)\,\overline{\alpha}\widetilde{\sigma}_3 + \overline{\alpha}\frac{t}{\tau}\widetilde{\sigma}_3$$

$$+(m-1)\,\widetilde{\sigma}_1 + \left(\overline{\alpha} + \frac{t}{\tau}\right)\left(\widetilde{\sigma}_2 + (m-1)\,\widetilde{\sigma}_1 + (m-1)\,\widetilde{\sigma}_3\right)$$

$$-(m-1)\left(\overline{\alpha} + \frac{t}{\tau}\right)\widetilde{\sigma}_1 - (m-1)\left(\overline{\alpha} + \frac{t}{\tau}\right)\widetilde{\sigma}_3$$

$$= \left(\overline{\alpha} + \alpha\frac{t}{\tau} + (m-2)\,\overline{\alpha} + (m-1) - (m-1)\left(\overline{\alpha} + \frac{t}{\tau}\right)\right)\widetilde{\sigma}_1$$

$$+\left((m-2)\,\overline{\alpha} + \overline{\alpha}\frac{t}{\tau} - (m-1)\left(\overline{\alpha} + \frac{t}{\tau}\right)\right)\widetilde{\sigma}_3 + \frac{1}{2}\left(\overline{\alpha} + \frac{t}{\tau}\right)$$

$$= \left(\alpha\frac{t}{\tau} + (m-1)\left(1 - \frac{t}{\tau}\right)\right)\widetilde{\sigma}_1 + \left(\overline{\alpha}\left(\frac{t}{\tau} - 1\right) - (m-1)\frac{t}{\tau}\right)\widetilde{\sigma}_3 + \frac{1}{2}\left(\overline{\alpha} + \frac{t}{\tau}\right).$$

Let us multiply this equality on t and integrate it from 0 to τ, we get:

$$\int_0^\tau 2\left[W\left(\tau,t;\sigma_1,...,\sigma_{2m-1}\right)\right]_1 tdt = \left(\frac{1}{3}\alpha + (m-1)\left(\frac{1}{2} - \frac{1}{3}\right)\right)\widetilde{\sigma}_1$$

$$+\left(\overline{\alpha}\left(\frac{1}{3}-\frac{1}{2}\right)-\frac{m-1}{3}\right)\widetilde{\sigma}_3+\frac{1}{2}\left(\frac{1}{2}\overline{\alpha}+\frac{1}{3}\right)$$
$$=\frac{m-1+2\alpha}{6}\widetilde{\sigma}_1-\frac{2m-2+\overline{\alpha}}{6}\widetilde{\sigma}_3+\frac{3\overline{\alpha}+2}{12}.$$

By inserting into this equality the values of the parameters $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_3$, we get:

$$\int_0^\tau 2\left[W\left(\tau,t;\sigma_1,...,\sigma_{2m-1}\right)\right]_1 tdt=\frac{1}{3}\tau^2.$$

Analogously for $i=m$ we have:

$$\int_0^\tau 2\left[W\left(\tau,t;\sigma_1,...,\sigma_{2m-1}\right)\right]_m tdt=\frac{1}{3}\tau^2.$$

Finally we have:

$$\int_0^\tau 2\left[W\left(\tau,t;\sigma_1,...,\sigma_{2m-1}\right)\right]_i tdt=\frac{1}{3}\tau^2,\quad i=1,2,..,m. \tag{3.27}$$

Let us multiply the equality (3.25) on t, integrate it from 0 to τ and take into account the formula (3.27), then we obtain:

$$\int_0^\tau 2W\left(\tau,t;\sigma_1,...,\sigma_{2m-1}\right)tdt=\frac{1}{2}\tau^2 I-\frac{1}{3}\tau^3\sum_{i=1}^m A_i+\int_0^\tau t\widetilde{R}_2\left(\tau,t\right)dt$$
$$=\frac{1}{2}\tau^2 I-\frac{1}{3}\tau^3 A+\int_0^\tau t\widetilde{R}_2\left(\tau,t\right)dt. \tag{3.28}$$

according to the formula (3.1) we have:

$$\int_0^\tau U\left(t,A\right)tdt=\int_0^\tau\left(tI-t^2A+tR_2\left(t,A\right)\right)dt$$
$$=\frac{1}{2}\tau^2 I-\frac{1}{3}\tau^3 A+\int_0^\tau tR_2\left(t,A\right)dt. \tag{3.29}$$

Clearly the following estimation:

$$\left\|\int_0^\tau tR_2\left(t,A\right)\varphi dt\right\|\le ce^{\omega_0\tau}\tau^4\left\|\varphi\right\|_{A^2,}\quad\varphi\in D\left(A^2\right). \tag{3.30}$$

From the equalities (3.28) and (3.29) we get:

$$\int_0^\tau \left[U(t,A) - 2W(\tau,t,\sigma_1,...,\sigma_{2m-1})\right] t\varphi dt$$

$$= \int_0^\tau t\widetilde{R}_2(\tau,t)\,\varphi dt - \int_0^\tau tR_2(t,A)\,\varphi dt.$$

Hence according to the inequalities (3.26) and (3.30) we obtain the sought estimation. ■

Lemma 3.5. *Let the conditions a) and b) of the theorem be satisfied, then the following estimation holds:*

$$\left\| \int_0^\tau \left(U(s,A) - \frac{1}{2}\left[V_1^{(2m-1)}(s) + V_2^{(2m-1)}(s)\right]\right) \frac{s^2}{2}\varphi ds \right\|$$

$$\leq ce^{\omega_0 \tau}\tau^4 \|\varphi\|_A, \quad \varphi \in D(A),$$

where c, ω_0 are positive constants.

Proof. According to the formulas (3.1),(3.2) and the condition b) of the theorem we obtain the following estimation:

$$\left\| \int_0^\tau \left[U(s,A) - \frac{1}{2}\left[V_1^{(2m-1)}(s) + V_2^{(2m-1)}(s)\right]\right] \frac{s^2}{2}\varphi ds \right\|$$

$$= \left\| \int_0^\tau \left[U(s,A) - \frac{1}{2}\left(U(s,\overline{\alpha}A_1) + U(s,\overline{\alpha}A_m)\right)\right] \frac{s^2}{2}\varphi ds \right\|$$

$$= \left\| \int_0^\tau \left[I + R_1(s,A) - \frac{1}{2}\left(I + R_1(s,\overline{\alpha}A_1) + I + R_1(s,\overline{\alpha}A_m)\right)\right] \frac{s^2}{2}\varphi ds \right\|$$

$$\leq ce^{\omega_0 \tau}\tau^4 \|\varphi\|_A, \quad \varphi \in D(A). \quad ■$$

4. *Proof of the theorem*

Let us return to the proof of the theorem. According to the property of a semigroup, the solution of the problem (2.1) at the point $t = t_k$ can be written as follows:

$$u(t_k) = U(t_k, A)\varphi + \int\limits_0^{t_k} U(t_k - s, A) f(s) ds$$

$$= U^k(\tau, A)\varphi + \sum_{i=1}^{k} U^{k-i}(\tau, A) F_i^{(1)}, \qquad (4.1)$$

where

$$F_i^{(1)} = \int\limits_{t_{i-1}}^{t_i} U(t_i - s, A) f(s) ds$$

$$= \int\limits_{t_{i-1}}^{t_i} U(t_i - s, A) \left[f(t_i) - (t_i - s) f'(t_i) + \frac{(t_i - s)^2}{2} f''(t_i) + R_3^- (f, t_i, s) \right] ds$$

$$= \int\limits_{t_{i-1}}^{t_i} U(t_i - s, A) f(t_i) ds - \int\limits_{t_{i-1}}^{t_i} U(t_i - s, A)(t_i - s) f'(t_i) ds$$

$$+ \int\limits_{t_{i-1}}^{t_i} U(t_i - s, A) \frac{(t_i - s)^2}{2} f''(t_i) ds + \int\limits_{t_{i-1}}^{t_i} U(t_i - s, A) R_3^- (f, t_i, s)\, ds, \quad (4.2)$$

where

$$R_3^- (f, t_i, s) = - \int\limits_s^{t_i} \int\limits_{\xi_1}^{t_i} \int\limits_{\xi_2}^{t_i} f'''(\xi) d\xi d\xi_2 d\xi_1.$$

From the (3.12) and (4.1) we have:

$$u(t_k) - u_k(t_k) = \left[U^k(\tau, A) - V^k(\tau) \right] \varphi$$

$$+ \sum_{i=0}^{k} \left[U^{k-i}(\tau, A) F_i^{(1)} - V^{k-i}(\tau) F_i^{(2)} \right] = \left[U^k(\tau, A) - V^k(\tau) \right] \varphi$$

$$+ \sum_{i=1}^{k} \left[\left(U^{k-i}(\tau, A) - V^{k-i}(\tau) \right) F_i^{(1)} + V^{k-i}(\tau) \left(F_i^{(1)} - F_i^{(2)} \right) \right]. \qquad (4.3)$$

From the equalities (4.2) and (3.13) we have:

$$F_i^{(1)} - F_i^{(2)} = \int\limits_{t_{i-1}}^{t_i} \left[U(t_i - s, A) - W(\tau, t_i - s, \alpha_1, ..., \alpha_{2m-1}) \right] f(t_i) ds$$

$$-\int\limits_{t_{i-1}}^{t_i}\left[U(t_i-s,A)-2W(\tau,t_i-s,\sigma_1,...,\sigma_{2m-1})\right](t_i-s)f'(t_i)ds$$

$$+\int\limits_{t_{i-1}}^{t_i}\left(U(t_i-s,A)-\frac{1}{2}\left[V_1^{(2m-1)}(t_i-s)+V_2^{(2m-1)}(t_i-s)\right]\right)$$

$$\times\frac{(t_i-s)^2}{2}f''(t_i)ds+\int\limits_{t_{i-1}}^{t_i}U(t_i-s,A)R_3^-(f,t_i,s)\,ds$$

$$=\int\limits_0^{\tau}\left[U(s,A)-W(\tau,s,\alpha_1,...,\alpha_{2m-1})\right]f(t_i)ds$$

$$-\int\limits_0^{\tau}\left[U(s,A)-2W(\tau,s,\sigma_1,...,\sigma_{2m-1})\right]sf'(t_i)ds$$

$$+\int\limits_0^{\tau}\left[U(s,A)-\frac{1}{2}\left[V_1^{(2m-1)}(s)+V_2^{(2m-1)}(s)\right]\right]\frac{s^2}{2}f''(t_k)ds$$

$$+\int\limits_0^{\tau}U(s,A)R_3^-(f,t_i,t_i-s)\,ds. \qquad (4.4)$$

Hence according to the Lemma 3.3, Lemma 3.4 and Lemma 3.5 we obtain the following estimation:

$$\left\|F_k^{(1)}-F_k^{(2)}\right\|=ce^{\omega_0\tau}\tau^4\left(\|f(t_k)\|_{A^3}\right.$$

$$\left.+\left\|f'(t_k)\right\|_{A^2}+\left\|f''(t_k)\right\|_A+\sup_{t\in[0,t_k]}\left\|f'''(t)\right\|\right). \qquad (4.5)$$

From the Lemma 3.1 the following estimation holds:

$$\left\|\sum_{i=1}^{k}\left(U^{k-i}(\tau,A)-V^{k-i}(\tau)\right)F_i^{(1)}\right\|$$

$$=\left\|\sum_{i=1}^{k}\left(U^{k-i}(\tau,A)-V^{k-i}(\tau)\right)\int\limits_{t_{i-1}}^{t_i}U(t_i-s,A)f(s)ds\right\|$$

$$= \left\| \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \left(U^{k-i}(\tau, A) - V^{k-i}(\tau) \right) U(t_i - s, A) f(s) ds \right\|$$

$$\leq c e^{\omega_0 \tau} \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} t_{k-i} \tau^3 \left\| U(t_i - s, A) f(s) ds \right\|_{A^4}$$

$$\leq c e^{\omega_0 t_k} t_k^2 \tau^3 \sup_{s,t \in [0,t_k]} \left\| U(s, A) f(t) \right\|_{A^4}. \tag{4.6}$$

From the equality (4.3) according to the inequalities (4.5), (4.6) and the Lemma 3.1 we obtain the following estimation:

$$\| u_k(t_k) - u(t_k) \| \leq \left\| \left[U^k(\tau, A) - V^k(\tau) \right] \varphi \right\|$$

$$+ \sum_{i=1}^{k} \left[\left\| \left[U^{k-i}(\tau, A) - V^{k-i}(\tau) \right] F_i^{(1)} \right\| + \left\| V^{k-i}(\tau) \right\| \left\| \left(F_i^{(1)} - F_i^{(2)} \right) \right\| \right]$$

$$\leq c e^{\omega_0 t_k} t_k \tau^3 \left(\sup_{s \in [0,t_k]} \| U(s, A) \varphi \|_{A^4} + t_k \sup_{s,t \in [0,t_k]} \| U(s, A) f(t) \|_{A^4} \right)$$

$$+ \sum_{i=1}^{k} e^{\omega (k-i) \tau} c e^{\omega_0 \tau} \tau^4 \left(\sup_{t \in [0,t_i]} \| f(t) \|_{A^3} \right.$$

$$\left. + \sup_{t \in [0,t_i]} \left\| f'(t) \right\|_{A^2} + \sup_{t \in [0,t_i]} \left\| f''(t) \right\|_A + \sup_{t \in [0,t_i]} \left\| f'''(t) \right\| \right)$$

$$\leq c e^{\omega_0 t_k} t_k \tau^3 \left(\sup_{s \in [0,t_k]} \| U(s, A) \varphi \|_{A^4} + t_k \sup_{s,t \in [0,t_k]} \| U(s, A) f(t) \|_{A^4} \right.$$

$$+ \sup_{t \in [0,t_k]} \| f(t) \|_{A^3} + \sup_{t \in [0,t_k]} \left\| f'(t) \right\|_{A^2} + \sup_{t \in [0,t_k]} \left\| f''(t) \right\|_A$$

$$\left. + \sup_{t \in [0,t_k]} \left\| f'''(t) \right\| \right), \quad \varphi \in D\left(A^4\right). \quad \blacksquare$$

5. *Conclusion*

In the case when operators $A_1, A_2, ..., A_m$ are matrices, it is obvious that conditions of the **Theorem** are automatically satisfied. Also conditions of the **Theorem** are satisfied, if $A_1, A_2, ..., A_m$ and A are self-adjoint, positive definite operators. The requirement αA operator $\left(\alpha = 1/\sqrt{3}\left(\cos 30^0 + i \sin 30^0\right)\right)$ must generate a strongly continuous semigroup puts the condition for the spectrum of A. Namely, the spectrum of A must be placed within sector with the angle less than 120 degrees, because in case of turning of spectrum by ± 30 degrees (this is caused by multiplying of A on α parameter) the spectrum area will stay in the positive (right) half-plane.

Third degree precision is reached by introducing a complex parameter. Because of this, each equation of the given decomposed system is changed by a pair of real equations, unlike lower degree precision schemas. To solve the specific problem, (for example) the matrix factorization may be used, where the coefficients are the matrices of the second order, unlike lower degree precision schemas, where the common factorization may be used.

It must be noted that the sum of the absolute values of coefficients of the addends of transition operator $V(\tau)$ equals to one, unlike the high degree precision decomposition formulas considered in [20]. because of this, the considered scheme is automatically stable for any bounded operators $A_1, A_2, ..., A_m$.

Remark: *The decomposition scheme considered in the present article and the theorem on error estimation of this scheme without proof are published in [8].*

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