A PRIORI ESTIMATES IN THE THEORY OF VARIATIONAL INEQUALITIES VIA STOCHASTIC ANALYSIS

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Abstract

This paper is devoted to the proof of new a priori estimates in the theory of variational inequalities (in particular for the obstacle problems) by the techniques of stochastic analysis. We essentially use the semimartingale inequalities for the Snell envelopes and the connection between optimal stopping problems and variational inequalities. Based on these a priori estimates we establish the stability of the solutions of the obstacle problem in the second order Sobolev space $W^{2,p}(D)$.

Key words and phrases: variational inequalities, multidimensional diffusion processes optimalstoping, shell envelopes, semimartingals.

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1. *Introduction*

Consider the probability space (Ω, F, P) and a $n-$ dimensional diffusion process $X_t = (X_t^1, \dots, X_t^n)_{t\geq 0}$ on this space.The process is the solution of the following stochastic differential equation

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dw_t, \quad X_0(w) = x, \tag{1.1}$$

where $w_t = (w_t^1, \dots, w_t^n)$ is the $n-$ dimensional standard Brownian motion, $\sigma(x)$ is a function C^2 from R^n into the space of $n \times n$ matrices and $b(x) = (b_1(x), \dots, b_n(x))$ is a function from R^n into R^n with components

$$b_i(x) = \sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j}, \quad i = 1, \dots, n,$$

where $a_{ij}(x)$ are the entries of $a(x) = \sigma(x)\,\sigma^*(x)/2$ (where $*$ denotes transposition). We suppose that the functions $a_{ij}(x)$, $b_i(x)$ are bounded

$$|a_{ij}(x)| \leq B, \quad |b_i(x)| \leq B, \quad i, j, = 1, \dots, n,$$

and that the following uniform ellipticity condition is satisfied:

$$\exists \alpha > 0, \quad \forall x \in R^n, \quad \forall y \in R^n$$
$$\alpha |y|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\, y_i\, y_j. \tag{1.2}$$

Let D be a bounded domain in R^n with its closer $\overline{D}$ and with a smooth (say, belonging to C^2) boundary ∂D. Denote $\sigma(D) = \inf(t \geq 0 : X_t \notin D)$ $-$ the first time the process X_t leaves domain D and let $g = g(x)$, $c = c(x)$ be functions defined on $\overline{D}$ such that

$$E_x \sup_{t \leq \sigma(D)} |g(X_t)| < \infty, \quad E_x \int_0^{\sigma(D)} |c(X_s)|\, ds < \infty, \quad \forall x \in \overline{D}.$$

Define now the following optimal stopping problem in the domain D

$$S(x) = \sup_{\tau \in M} E_x \left(g(X_{\tau \wedge \sigma(D)}) + \int_0^{\tau \wedge \sigma(D)} c(X_s)\, ds \right), \tag{1.3}$$

where P_x is the probability measure corresponding to the initial condition $X_0(w) = x$, M is the class of all stopping times τ with respect to the filtration $F^w = (F_t^w)_{t \geq 0}$.

The function $g = g(x)$ is called the payoff, $-c = -c(x)$ has a meaning of the instantaneous cost of observation. The function $S = S(x)$ is called the value function of the corresponding optimal stopping problem. The objective is to find the value function $S(x)$ and to determine the optimal stopping time τ^*, at which the supremum is achieved..

In [1] Bensoussan and Lions have developed the variational inequality approach for determining the value function $S(x)$ of the optimal stopping problem (1.3). Let us briefly discuss the corresponding basic results. Denote $H^1(D) \equiv W^{1,2}(D)$ $-$ the first order Sobolev space of functions $v = v(x)$ defined on $\overline{D}$, that is

$$v(x) \in L^2(D), \quad \frac{\partial v(x)}{\partial x_i} \in L^2(D), \quad i = 1, \ldots, n,$$

where $\dfrac{\partial v(x)}{\partial x_i}$ are the first order generalized derivatives of the function $v = v(x)$.

It is well-known that if we introduce the scalar product

$$(u, v)_{H^1(D)} = \int_D u(x)\, v(x)\, dx + \sum_{i=1}^{n} \int_D \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i} dx, \tag{1.4}$$

then this space becomes the Hilbert space.

Consider a symmetric bilinear form on $H^1(D)$

$$a(u,v) = \sum_{i,j=1}^{n} \int_D a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_j} \, dx. \tag{1.5}$$

Let us suppose that $g(x) \in H^1(D)$ and $c(x) \in L^2(D)$ and define the closed convex subset K of the space $H^1(D)$ as

$$K = \{v : \ v \in H^1(D), \quad v \geq g \quad a.e., \quad v - g \in H_0^1(D)\}, \tag{1.6}$$

where $H_0^1(D)$ is the subspace of the space $H^1(D)$ consisting of those functions $v = v(x)$, which are zero on the boundary ∂D (in the sense of the space $H^1(D)$).

Consider now the following variational inequality:
Find $u \in K$ such that

$$a(u, v-u) \geq \int_D c(x)(v(x) - u(x)) \, dx, \quad \forall v \in K. \tag{1.7}$$

In [1] (Chapter 2, Theorem 5.1) Bensoussan and Lions state the result which says that the variational inequality (1.7) has a unique solution. Moreover, they further establish the fundamental connection between optimal stopping and variational inequalities. Namely, provided that $g(x) \in W^{2,p}(D)$, $c(x) \in L^p(D)$, $p > n$, it turns out (Chapter 2, Theorem 7.1), that $u(x) \in W^{2,p}(D)$ and

$$u(x) = S(x), \quad x \in \overline{D}. \tag{1.8}$$

In this case the value function $S(x)$ of the optimal stopping problem is the unique solution of the following obstacle problem: For the initial data $g(x) \in W^{2,p}(D)$, $c(x) \in L^p(D)$, $p > n$, find $u(x) \in W^{2,p}(D)$ such that

$$\begin{cases} Au(x) + c(x) \leq 0, \quad u(x) - g(x) \geq 0 & \\ & a.e. in \ D, \\ (Au(x) + c(x))(u(x) - g(x)) = 0 & \\ & where \ \ u(x) - g(x) \in H_0^1(D) \end{cases} \tag{1.9}$$

and Au is the elliptic operator

$$Au(x) = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u(x)}{\partial x_j} \right) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x) \frac{\partial u(x)}{\partial x_j}. \tag{1.10}$$

The initial motivation for us to study the (1.9)−(1.10) obstacle problem was the following question:

Suppose the payoff functions converge

$$g_n(x) \to g(x) \ \ in \ \ W^{2,p}(D),$$

what can be said about the convergence of the second order partial derivatives $\dfrac{\partial^2 S_n(x)}{\partial x_i \partial x_j}$ (where $S_n(x)$ are the value functions of the corresponding optimal stopping problems)? Searching the answer to the above mentioned question, we found out new a priori inequality between Green potentials, which has the following form

$$\begin{aligned} &|(u_1(x) - u_2(x)) - (g_1(x) - g_2(x))| \\ &+ \int\limits_D |A(u_1(y) - u_2(y)) + (c_1(y) - c_2(y))| \cdot G_D(x,y)\,dy \\ &\le \int\limits_D |A(g_1(y) - g_2(y)) + (c_1(y) - c_2(y))| \cdot G_D(x,y)\,dy \\ &\qquad \forall x \in \overline{D}, \end{aligned} \tag{1.11}$$

where $u_i(x)$, $i = 1, 2$ are the corresponding solutions of the obstacle problem for the initial data

$$g_i(x),\ c_i(x),\ i = 1,2,\ g_i(x) \in W^{2,p}(D),\ c_i(x) \in L^p(D),\ p > n,\ i = 1,2,$$

and $G_D(x, y)$ is a Green function for the operator A in the domain D.

Based on the estimate (1.11), we prove the following stability result for the solutions of the obstacle problem:

Fix p', $p' < p$ and let $\widetilde{D}$ be arbitrary domain strictly imbedded in domain D (i.e. $\widetilde{D} \subset\subset D$), then for arbitrary data $(g_m(x),\ c_m(x))$, $(g(x),\ c(x))$ such that

$$g_m(x) \to g(x) \quad in \ \ W^{2,p}(D), \quad c_m(x) \to c(x) \quad in \ L^p(D), \quad p > n,$$

we have

$$u_m(x) \to u(x) \quad in \ \ W^{2,p'}(\widetilde{D}),$$

where $u_m(x)$, $u(x)$ are the corresponding solutions of the obstacle problem (1.9)−(1.10).

2. *The proof of the a priori estimate for Green potentials*

We shall based ourselves several times hereafter on the following lemma proved in [1] (Chapter 6, lemma 1.2) by Bensoussan and Lions.

Lemma 2.1. *Suppose $f(x) \in L^{p/2}(D)$, where $p > n$. Then the following estimate holds*

$$\sup_{x \in \overline{D}} E_x \left(\int_0^{\sigma(D)} |f(X_s)| ds \right) \leq c \cdot \|f(x)\|_{L^{p/2}(D)}, \tag{2.1}$$

where constant c is independent of $f(x)$.

Remark 2.1. *Let us take $f(x) \equiv 1$ in the estimate above, then we get the well-known fact, that*

$$\sup_{x \in \overline{D}} E_x \sigma(D) < \infty. \tag{2.2}$$

Denote now $X_t^{\sigma(D)} \equiv X_{t \wedge \sigma(D)}$, $t \geq 0$ and note the following obvious relationships (where $\chi_D(x)$ is the characteristic function of the domain D).

$$\begin{gathered} I_{(s<\sigma(D))} = \chi(X_s^{\sigma(D)}), \\ \int_0^{t \wedge \sigma(D)} c(X_s) ds = \int_0^t I_{(s<\sigma(D))} \cdot c(X_s) ds = \int_0^t \widetilde{c}(X_s^{\sigma(D)}) ds, \\ \int_0^{t \wedge \sigma(D)} \sigma(X_s) dw_s = \int_0^t I_{(s<\sigma(D))} \cdot \sigma(X_s) dw_s = \int_0^t \widetilde{\sigma}(X_s^{\sigma(D)}) dw_s, \end{gathered} \tag{2.3}$$

where

$$\widetilde{c}(x) = c(x) \cdot \chi_D(x), \quad \widetilde{\sigma}(x) = \sigma(x) \cdot \chi_D(x).$$

From these relationships and the stochastic differential equation (1.1), we have

$$\begin{aligned} X_t^{\sigma(D)} &= X_0 + \int_0^{t \wedge \sigma(D)} b(X_s) ds + \int_0^{t \wedge \sigma(D)} \sigma(X_s) dw_s \\ &= X_0 + \int_0^t \widetilde{b}(X_s^{\sigma(D)}) ds + \int_0^t \widetilde{\sigma}(X_s^{\sigma(D)}) dw_s. \end{aligned} \tag{2.4}$$

The Markov property of the stopped process $X_t^{\sigma(D)}$, $t \geq 0$ can be glimpsed from this equation. In fact, due to general results on Markov processes in [2] (chapter 10, Theorem 10.3) the triple $(X_t^{\sigma(D)}, F_t^w, P_x)$, $t \geq 0$, $x \in \overline{D}$ defines a standard Markov process.

From now on throughout the work we shall assume that

$$g(x) \in W^{2,p}(D), \quad c(x) \in L^p(D), \quad \textit{where} \;\; p > n. \tag{2.5}$$

Then

$$g(x) \in C(\overline{D}), \quad \frac{\partial g(x)}{\partial x_i} \in C(\overline{D}), i = 1, \quad \ldots, n \tag{2.6}$$

according to well-known Sobolev lemma.

Rewriting optimal stopping problem (1.3) in terms of the standard Markov process $(X_t^{\sigma(D)}, F_t^w, P_x)$, $t \geq 0$

$$S(x) = \sup_{\tau \in M} E_x \left(g(X_\tau^{\sigma(D)}) + \int_0^\tau \widetilde{c}(X_s^{\sigma(D)}) ds \right), \quad x \in \overline{D}, \tag{2.7}$$

we have

$$\left| g(X_\tau^{\sigma(D)}) \right| \leq \sup_{x \in \overline{D}} |g(x)| < \infty,$$

$$\int_0^\infty \left| \widetilde{c}(X_s^{\sigma(D)}) \right| ds = \int_0^{\sigma(D)} |c(X_s)| \, ds$$

and we get from lemma 2.1, that

$$\sup_{x \in \overline{D}} E_x \int_0^{\sigma(D)} |c(X_s)| ds \leq c \cdot \|c(x)\|_{L^{p/2}(D)} < \infty,$$

therefore the optimal stopping problem is well defined and $S(x)$ is a bounded function of x

$$\sup_{x \in \overline{D}} |S(x)| < \infty.$$

Introduce the notation

$$f(x) \equiv E_x \left(\int_0^\infty \widetilde{c}(X_s^{\sigma(D)}) ds \right) = E_x \int_0^{\sigma(D)} c(X_s) ds. \tag{2.8}$$

The strong Markov property gives us

$$E_x \left(\int_\tau^\infty \widetilde{c}(X_s^{\sigma(D)}) ds | F_\tau^w \right) = f(X_\tau^{\sigma(D)}), \quad i.e.$$

$$\int_0^\tau \widetilde{c}(X_s^{\sigma(D)}) ds = E_x \left(\int_0^\infty \widetilde{c}(X_s^{\sigma(D)}) ds | F_\tau^w \right) - f(X_\tau^{\sigma(D)}),$$

therefore we get

$$S(x) = \sup_{\tau \in M} E_x \left(g(X_\tau^{\sigma(D)}) - f(X_\tau^{\sigma(D)}) \right) + f(x);$$

hence

$$S(x) - f(x) = \sup_{\tau \in M} E_x \left(g(X_\tau^{\sigma(D)} - f(X_\tau^{\sigma(D)}) \right). \tag{2.9}$$

It follows from relationship (2.9) and the general theory of optimal stopping of standard Markov processes developed in [6] (chapter 3) that the stochastic process $S(X_t^{\sigma(D)}) - f(X_t^{\sigma(D)})$ is the minimal supermartingale (on the time interval $[0, \infty]$) bounding the process $g(X_t^{\sigma(D)}) - f(X_t^{\sigma(D)})$ from above. Our next objective is to obtain the semimartingale decomposition of these processes.

Lemma 2.2. *Suppose $v(x) \in W^{2,p}(D)$, $p > n$. Then the Ito formula is fulfilled for the process $v(X_t^{\sigma(D)})$*

$$v(X_{t\wedge\sigma(D)}) = v(x) + \int_0^{t\wedge\sigma(D)} grad\, v(X_s) \cdot \sigma(X_s)\, dw_s + \int_0^{t\wedge\sigma(D)} Av(X_s) ds,$$
$$t \geq 0, \quad P_x \; - \; a.s. \tag{2.10}$$

Proof. Let us take the sequence $v_n(x) \in C^2(\overline{D})$ such that

$$\|v_n(x) - v(x)\|^{W_{2,p}(D)} \to 0 \quad as \quad n \to \infty.$$

The Ito formula for the processes $v_n(X_{t\wedge\sigma(D)})$ has the following form

$$v_n(X_{t\wedge\sigma(D)}) = v_n(x) + \int_0^{t\wedge\sigma(D)} Av_n(X_s) ds + \int_0^{t\wedge\sigma(D)} grad\, v_n(X_s) \cdot \sigma(X_s)\, dw_s,$$
$$t \geq 0, \quad P_x \; - \; a.s. \tag{2.11}$$

Consider at first the expressions

$$E_x \int_0^{\sigma(D)} |Av(X_s)| ds, \quad E_x \int_0^{\sigma(D)} |grad\, v(X_s) \cdot \sigma(X_s)|^2 ds.$$

We have by virtue of lemma 2.1

$$E_x \int\limits_0^{\sigma(D)} |Av(X_s)|ds \le c \left\| Av(x) \right\|_{L^{p/2}(D)} \le c_1 B \left\| v(x) \right\|_{W^{2,p/2}(D)} < \infty,$$

$$E_x \int\limits_0^{\sigma(D)} |grad\, v(X_s) \cdot \sigma(X_s)|^2 ds$$

$$= E_x \int\limits_0^{\sigma(D)} 2 \cdot (a(X_s) \cdot grad\, v(X_s), grad\, v(X_s))ds \le 2\,B\,E_x \int\limits_0^{\sigma(D)} |grad\, v(X_s)|^2 ds$$

$$\le 2\,B\,c \left\| |grad\, v(x)|^2 \right\|_{L^{p/2}(D)} = 2\,B\,c \left\| |grad\, v(x)| \right\|^2_{L^p(D)}$$

$$\le 2\,B\,c_1 \left\| v(x) \right\|^2_{W^{2,p}(D)} < \infty,$$

therefore the process

$$\int\limits_0^{t\wedge\sigma(D)} grad\, v(X_s) \cdot \sigma(X_s) dw_s$$

is the square integrable martingale and the process $\int\limits_0^{t\wedge\sigma(D)} Av(X_s)ds$ is of integrable variation. Similarly to the above mentioned estimates we have

$$E_x \int\limits_0^{t\wedge\sigma(D)} |Av_n(X_s) - Av(X_s)|ds \le c \left\| Av_n(x) - Av(x) \right\|_{L^{p/2}(D)}$$

$$\le c_1 B \left\| v_n(x) - v(x) \right\|_{W^{2,p/2}(D)},$$

$$E_x \left| \int\limits_0^{t\wedge\sigma(D)} (grad\, v_n(X_s) - grad\, v(X_s)) \cdot \sigma(X_s)\, dw_s \right|^2$$

$$= E_x \int\limits_0^{t\wedge\sigma(D)} |(grad\, v_n(X_s) - grad\, v(X_s)) \cdot \sigma(X_s)|^2\, ds$$

$$\le 2Bc \left\| |grad(v_n(x) - v(x))|^2 \right\|_{L^{p/2}(D)}$$

$$= 2Bc \left\| |grad(v_n(x) - v(x))| \right\|^2_{L^p(D)}$$

$$\le 2Bc_1 \left\| v_n(x) - v(x) \right\|^2_{W^{2,p}(D)}.$$

We can pass to the limit as $n \to \infty$ in the equality (2.11) after noting that (by wellknown Sobolev lemma)

$$\sup_{x\in\overline{D}} |v_n(x) - v(x)|_{n\to\infty} \to 0,$$

thus we obtain semimartingale decomposition (2.10).

Remark 2.2. *It is obvious from the proof of lemma 2.2 that the martingale part in decomposition (2.10) is square integrable and the other part is of integrable variation.*

Lemma 2.3. *function $f(x)$ defined by $f(x) = E_x \int\limits_0^{\sigma(D)} c(X_s)ds$ belongs to the space $W^{2,p}(D)$ and we have*

$$f(X_{t\wedge\sigma(D)}) = f(x) + \int\limits_0^{t\wedge\sigma(D)} grad\, f(X_s)\cdot\sigma(X_s)\, dw_s + \int\limits_0^{t\wedge\sigma(D)} (-c(X_s))ds, \quad t \geq 0, \quad P_x \; - \; a.s. \tag{2.12}$$

Proof. For $c(x) \in L^p(D)$, $p > n$, consider the following problem: find $v(x) \in W^{2,p}(D)$, such that

$$\begin{aligned} Av(x) &= -c(x), \quad x \in D, \\ v(x) &= 0, \quad x \in \partial D. \end{aligned}$$

It is proved in [3] (chapter 9, Theorem 9.15) that this problem has a unique solution $v(x) \in W^{2,p}(D)$. Now applying lemma 2.2 to this function, we have

$$v(X_{t\wedge\sigma(D)}) = v(x) + \int\limits_0^{t\wedge\sigma(D)} grad\, v(X_s)\cdot\sigma(X_s)\, dw_s + \int\limits_0^{t\wedge\sigma(D)} Av(X_s)ds, \quad t \geq 0, \quad P_x \; - \; a.s.$$

Taking the limit $t \to \infty$, we get

$$0 = v(x) + \int\limits_0^{\sigma(D)} grad\, v(X_s)\cdot\sigma(X_s)\, dw_s + \int\limits_0^{\sigma(D)} Av(X_s)ds,$$

but

$$\int\limits_0^{\sigma(D)} A\, v(X_s)ds = -\int\limits_0^{\sigma(D)} c(X_s)ds,$$

therefore taking the expectation in the above equality we can find

$$v(x) = E_x \int\limits_0^{\sigma(D)} c(X_s)ds = f(x).$$

Hence $f(x) = v(x)$ and the assertion of lemma 2.3 is verified.

Now we are ready to prove the basic result of this article.

Theorem 2.1. *Let* $(g_i(x),\ c_i(x))$, $i = 1, 2$ *be two pairs of initial data for the obstacle problem (1.9)–(1.10), such that*

$$g_i(x) \in W^{2,p}(D), \quad i = 1,2, \quad c_i(x) \in L^p(D), \quad i = 1,2, \quad p > n, \tag{2.13}$$

then the following a priori inequality between Green potentials holds

$$\begin{aligned} \int_D |A(u_1(y) - u_2(y)) + (c_1(y) - c_2(y))| \cdot G_D(x,y)dy \\ +|(u_1(x) - u_2(x)) - (g_1(y) - g_2(y))| \\ \le \int_D |A(g_1(y) - g_2(y)) + (c_1(y) - c_2(y))| \cdot G_D(x,y)dy, \\ \forall x \in \overline{D}, \end{aligned} \tag{2.14}$$

where $u_i(x)$, $i = 1,2$ *are the corresponding solutions of the obstacle problem (1.9)–(1.10) and* $G_D(x,y)$ *is a Green function for the operator* A *in the domain* D.

Proof. The main tools in proving the above mentioned inequality are the general results on the semimartingale distance between the Snell envelopes developed in Shashiashvili [5], especially theorem 2 therein.

It was already mentioned, that the stochastic process $u_i(X_t^{\sigma(D)}) - f_i(X_t^{\sigma(D)})$, $i = 1,2$ is the minimal supermartingale (on the time interval $[0,\infty]$) bounding the process $g_i(X_t^{\sigma(D)}) - f_i(X_t^{\sigma(D)})$ above, where $f_i(x)$, $i = 1,2$ was introduced in (2.8) as follows

$$f_i(x) = E_x \int_0^{\sigma(D)} c_i(X_s)ds, \quad i = 1,2.$$

Consider the semimartingale decomposition of the processes

$$u_i(X_t^{\sigma(D)}) - f_i(X_t^{\sigma(D)}), \quad g_i(X_t^{\sigma(D)}) - f_i(X_t^{\sigma(D)}), \quad i = 1,2$$

taking into account lemmas 2.2 and 2.3

We have

$$u_i(X_{t\wedge\sigma(D)}) - f_i(X_{t\wedge\sigma(D)}) = u_i(x) - f_i(x) + \int_0^{t\wedge\sigma(D)} grad(u_i - f_i)(X_s)\cdot\sigma(X_s)dw_s$$
$$+ \int_0^{t\wedge\sigma(D)} (Au_i(X_s) + c_i(X_s))ds,$$
$$g_i(X_{t\wedge\sigma(D)}) - f_i(X_{t\wedge\sigma(D)}) = g_i(x) - f_i(x) + \int_0^{t\wedge\sigma(D)} grad(g_i - f_i)(X_s)\cdot\sigma(X_s)dw_s$$
$$+ \int_0^{t\wedge\sigma(D)} (Ag_i(X_s) + c_i(X_s))ds, \quad i = 1,2, \quad 0 \le t \le \infty, \quad P_x \; - \; a.s. \tag{2.15}$$

Now we apply theorem 2 from [5]. Taking the mathematical expectation from it with respect to the measure P_x, $x \in \overline{D}$ (for $t = 0$ and infinite time interval $[0,\infty]$), we have

$$\begin{aligned} &|u_1(x) - u_2(x) - (g_1(x) - g_2(x))| \\ &+E_x \int_0^{\sigma(D)} |A(u_1 - u_2)(X_s) + (c_1(X_s) - c_2(X_s))|\, ds \\ &\le E_x \int_0^{\sigma(D)} |A(g_1 - g_2)(X_s) + (c_1(X_s) - c_2(X_s))|\, ds, \quad x \in \overline{D}. \end{aligned} \tag{2.16}$$

To finish the proof of the inequality (2.14) we should only note, that as it is well-known (see, for example, section 5, chapter 13 in [2]) for any nonnegative measurable function $\varphi(x)$,

$$E_x \int_0^{\sigma(D)} \varphi(X_s)ds = \int_D \varphi(y)\cdot G_D(x,y)dy, \quad x \in \overline{D}. \tag{2.17}$$

Corollary 2.1. *The following estimate takes place*

$$\begin{aligned} &\int_D |A(u_1 - u_2)(y) + (c_1(y) - c_2(y))|\, E_y\sigma(D)\, dy \\ &\le \int_D |A(g_1 - g_2)(y) + (c_1(y) - c_2(y))|\, E_y\sigma(D)\, dy. \end{aligned} \tag{2.18}$$

Proof. The Green function $G_D(x,y)$ is symmetric, i.e. $G_D(x,y) = G_D(y,x)$, as the operator Au is self-adjoint. Therefore (take $\varphi(y) \equiv 1$ in (2.17))

$$\int\limits_D G_D(x,y)dx = \int\limits_D G_D(y,x)dx = E_y\sigma(D).$$

Now to get the estimate (2.18) it only remains to integrate inequality (2.14) with respect to x and to change the order of integration.

3. *The stability result for the solutions of the obstacle problem*

Fix p', $n < p' < p$ and consider an arbitrary domain $\widetilde{D}$ strictly imbedded in the domain D (that is the closer of $\widetilde{D}$). Let $\delta = 1/2 \cdot dist(\widetilde{D}, \partial D)$ and denote by $\widetilde{D}_\delta$ the $\delta-$ neighborhood of $\widetilde{D}$. In this section we shall be based on the inner estimates for the solutions of the second order elliptic partial differential equations

$$Au = \varphi \quad on \quad \widetilde{D}_\delta. \tag{3.1}$$

In particular it is proved in [4] (chapter 9, theorem 9.11) that

$$\|u(x)\|_{W^{2,p}(\widetilde{D})} \leq c\left(\|u(x)\|_{L^p(\widetilde{D}_\delta)} + \|\varphi(x)\|_{L^p(\widetilde{D}_\delta)}\right) \tag{3.2}$$

where the constant c does not depend on $\varphi(x)$, but is dependent on n, p, α, B, $\widetilde{D}$, D and δ and the modulus of continuity of the functions $a_{ij}(x)$ in the domain D.

We shall need also Hëlder inequality in the following form

$$\|\psi(x)\|_{L^{p'}(\widetilde{D}_\delta)} \leq \|\psi(x)\|^{1-\lambda}_{L^p(\widetilde{D}_\delta)} \cdot \|\psi(x)\|^{\lambda}_{L^1(\widetilde{D}_\delta)}, \tag{3.3}$$

where

$$\lambda = \frac{p-p'}{p'(p-1)}, \quad 1-\lambda = \frac{p(p'-1)}{p'(p-1)}.$$

Theorem 2. Suppose the assumptions of theorem 1 are satisfied. Then the following inner estimate holds for the solutions of the obstacle problem (9)−(10)

$$\begin{aligned}
&\|u_1(x) - u_2(x)\|_{W^{2,p'}(\widetilde{D})} \leq c \cdot \left(\|g_1(x) - g_2(x)\|_{W^{2,p}(D)} + \|c_1(x) - c_2(x)\|_{L^p(D)}\right) \\
&+c \cdot \left(\|g_1(x)\|_{W^{2,p}(D)} + \|g_2(x)\|_{W^{2,p}(D)} + \|c_1(x)\|_{L^p(D)} + \|c_2(x)\|_{L^p(D)}\right)^{1-\lambda} \\
&\cdot \left(\|g_1(x) - g_2(x)\|_{W^{2,p}(D)} + \|c_1(x) - c_2(x)\|_{L^p(D)}\right)^{\lambda},
\end{aligned} \tag{3.4}$$

where the constant c does not depend on the initial data $(g_i(x), c_i(x))$, $i = 1, 2$, but depends on $n, p, p', \alpha, B, \widetilde{D}, D$ and δ (and the modulus of continuity of $a_{ij}(x)$ in the domain D).

Proof. From the estimate (31) we get

$$\|u_1(x) - u_2(x)\|_{W^{2,p'}(\widetilde{D})}$$
$$\leq c \cdot \Big(\|u_1(x) - u_2(x)\|_{L^{p'}(\widetilde{D}_\delta)} + \|A(u_1 - u_2)(x)\|_{L^{p'}(\widetilde{D}_\delta)} \Big) . \tag{3.5}$$

The first term in the right side of this inequality can be estimated in the following manner

$$\|u_1(x) - u_2(x)\|_{L^{p'}(\widetilde{D}_\delta)} \leq \|u_1(x) - u_2(x)\|_{L^{p'}(D)}$$
$$\leq c_1 \cdot \Big(\|u_1(x) - u_2(x) - (g_1(x) - g_2(x))\|_{L^\infty(D)} + \|g_1(x) - g_2(x)\|_{L^p(D)} \Big) .$$

but from the inequalities (12) and (27) we have

$$\|u_1(x) - u_2(x) - (g_1(x) - g_2(x))\|_{L^\infty(D)}$$
$$\leq c_2 \cdot \|A(g_1 - g_2)(x) + (c_1(x) - c_2(x))\|_{L^p(D)} ,$$

therefore

$$\|u_1(x) - u_2(x)\|_{L^{p'}(\widetilde{D}_\delta)} \leq c_3 \cdot \Big(\|g_1(x) - g_2(x)\|_{W^{2,p}(D)} + \|c_1(x) - c_2(x)\|_{L^p(D)} \Big) . \tag{3.6}$$

We start now estimating the second term in the right side of the inequality (34). From Hëlder inequality (32) we have

$$\|A(u_1 - u_2)(x)\|_{L^{p'}(\widetilde{D}_\delta)} \leq \|A(u_1 - u_2)(x)\|^{1-\lambda}_{L^p(\widetilde{D}_\delta)} \cdot \|A(u_1 - u_2)(x)\|^{\lambda}_{L^1(\widetilde{D}_\delta)} . \tag{3.7}$$

The Levy-Stampacchia inequality ([1], chapter 2, corollary 5.4) gives us

$$|A(u_1 - u_2)(x)| \leq |Au_1(x)| + |Au_2(x)| \leq |Ag_1(x)| + |c_1(x)| + |Ag_2(x)| + |c_2(x)|,$$

hence

$$\|A(u_1 - u_2)(x)\|_{L^p(D)} \leq c_4 \Big(\|g_1(x)\|_{W^{2,p}(D)} + \|g_2(x)\|_{W^{2,p}(D)}$$
$$+ \|c_1(x)\|_{L^p(D)} + \|c_2(x)\|_{L^p(D)} \Big) .$$

It remains to estimate the expression $\|A(u_1-u_2)(x)\|_{L^1(\widetilde{D}_\delta)}$ and that is the key step in our proof. The starting point is the inequality (29), from which it follows, that

$$\begin{aligned}&\int\limits_D |A(g_1-g_2)(y)+(c_1(y)-c_2(y))|\, E_y\sigma(D)\\ &\geq \int\limits_{\widetilde{D}_\delta} |A(u_1-u_2)(y)+(c_1(y)-c_2(y))|\, E_y\sigma(D)\\ &\geq \inf_{y\in\widetilde{D}_\delta} E_y\sigma(D)\cdot\int\limits_{\widetilde{D}_\delta} |A(u_1-u_2)(y)+(c_1(y)-c_2(y))|\, dy,\end{aligned}$$

therefore

$$\begin{aligned}&\int\limits_{\widetilde{D}_\delta} |A(u_1-u_2)(y)+(c_1(y)-c_2(y))|\, dy\\ &\leq \frac{\sup\limits_{y\in D} E_y\sigma(D)}{\inf\limits_{y\in\widetilde{D}_\delta} E_y\sigma(D)}\int\limits_D |A(g_1-g_2)(y)+(c_1(y)-c_2(y))|\, dy.\end{aligned}$$

It is easy to get from here, that

$$\|A(u_1-u_2)(x)\|_{L^1(\widetilde{D}_\delta)}\leq c_5\left(\|g_1(x)-g_2(x)\|_{W^{2,p}(D)}+\|c_1(x)-c_2(x)\|_{L^p(D)}\right).$$

Thus we come to the following inequality

$$\begin{aligned}\|A(u_1-u_2)(x)\|_{L^{p'}(\widetilde{D}_\delta)}\leq c_6\Big(&\|g_1(x)\|_{W^{2,p}(D)}+\|g_2(x)\|_{W^{2,p}(D)}+\|c_1(x)\|_{L^p(D)}\\ &+\|c_2(x)\|_{L^p(D)}\Big)^{1-\lambda}\cdot\left(\|g_1(x)-g_2(x)\|_{W^{2,p}(D)}+\|c_1(x)-c_2(x)\|_{L^p(D)}\right)^{\lambda}.\end{aligned}\tag{3.8}$$

At last we obtain the desired estimate (33) after summing up the inequalities (35) and (37).

Corollary 2. The straightforward consequence of the estimate (33) is the following stability result:

For arbitrary data $(g_m(x),c_m(x))$ and $(g(x),c(x))$ such that

$$g_m(x)\to g(x)\ \ in\ \ W^{2,p}(D),\quad c_m(x)\to c(x)\quad in\ \ L^p(D),\quad p>n,$$

we have

$$u_m(x)\to u(x)\quad in\ \ W^{2,p'}(\widetilde{D})$$

where $u_m(x)$ and $u(x)$ are the corresponding solutions of the obstacle problem (9)−(10) (and $\widetilde{D}$ is any domain strictly imbedded in the domain D).

Remark 3. We have to note at the end of our paper that it has been very helpful for us to be acquainted with the theory of variational inequalities by the remarkable manual [6] of Kinderlehrer and Stampacchia, especially its chapter 4 on regularity.

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