SPECTRAL ELEMENT DISCRETIZATIONS OF THE POISSON EQUATION WITH MIXED BOUNDARY CONDITIONS

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(Received: 05.03.01; revised: 19.04.01)

Abstract

The aim of this paper is to investigate the rate of convergence of the spectral element discretization of the Poisson equation in a square when it is provided with boundary conditions of mixed Dirichlet and Neumann types. We consider the two situations where the parts of the boundary corresponding to Dirichlet and Neumann conditions intersect with angles equal to $\pi/2$ or π.

Key words and phrases: Mixed boundary conditions, Spectral elements.
AMS subject classification: 65N35, 65N55.

1. *Introduction*

Let Ω denote the unit square $]-1,1[^2$. We consider the following Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_d, \\ \partial_n u = g & \text{on } \Gamma_n, \end{cases} \tag{1.1}$$

where Γ_d and Γ_n are two open parts of the boundary $\partial\Omega$ such that

$$\partial\Omega = \overline{\Gamma}_d \cup \overline{\Gamma}_n \qquad \text{and} \qquad \Gamma_d \cap \Gamma_n = \emptyset. \tag{1.2}$$

Such an equation models the repartition of temperature in a homogeneous medium Ω heated by an internal source f, when the temperature is enforced to be constant on Γ_d (here equal to zero for simplicity) and the heat flux is given equal to g on Γ_n.

We specifically consider the two following situations which seem to be generic when Ω is a square:

- either Γ_n is a whole edge of Ω,
- or Γ_n is the union of the parts of two neighbouring edges of Ω and contains their common vertex.

Equivalently the angle between Γ_d and Γ_n at each of the two points in $\overline{\Gamma}_d \cap \overline{\Gamma}_n$ is equal to $\frac{\pi}{2}$ in the first case, to π in the second case.

In the first step, we write the variational formulation of problem (1.1). It can be observed that the lack of regularity of the solution of this problem for smooth data has two origins: in the general case, the solution "contains", in a sense which is made precise later on, singular functions linked to the vertices of the square and also singular functions issued from the change of boundary conditions. We are interested in the characterization of the leading singular functions: relying on [12] and [6], [7], [8], we write the explicit form of these functions and state the corresponding regularity properties of the solution. Indeed this determines the convergence order of the best polynomial approximation, which is the key term in error estimates for any type of spectral discretization.

Next, we consider the spectral element discretization of problem (1.1). In the first situation, the simplest idea consists in using the pure spectral method, i.e. without domain decomposition. We prove an error estimate where the convergence order is explicit and check the optimality of this estimate. In the second situation, using spectral elements allows to improve the accuracy of the discretization, since piecewise polynomial functions on an appropriate partition of the domain fit better the singular functions than polynomials on the whole domain. So we introduce a conforming decomposition of the square Ω into four rectangles and we describe the corresponding spectral element discretization of problem (1.1). This discretization is conforming, however in order to take into consideration the fact that the ratio of the length of Γ_n to the length of Γ_d can be very small, we use different degrees of polynomials on the different subdomains. There also we exhibit the optimal order of convergence of the discretization.

An outline of the paper is as follows.

- In Section 2 we recall the main properties of the continuous problem.
- Section 3 is devoted to the description and analysis of the spectral discretization of problem (1.1) in the first situation.
- In Section 4 we present and analyze the spectral element discretization of problem (1.1) in the second situation.

Acknowledgment: The authors are very grateful to Monique Dauge for interesting and helpful discussions concerning the regularity of the problem.

2. *The continuous problem*

The generic point in Ω is denoted by $\mathbf{x} = (x, y)$. We first introduce the space $L^2(\Omega)$ of measurable real-valued functions v such that

$$\int_\Omega v^2(\mathbf{x})\,d\mathbf{x} < \infty.$$

For any nonnegative real number s, we also need the Hilbert Sobolev spaces $H^s(\Omega)$: when s is an integer m, $H^m(\Omega)$ is the space of functions in $L^2(\Omega)$ such that all their partial derivatives of order $\leq m$ belong to $L^2(\Omega)$, provided with the usual norm and semi-norm

$$\|v\|_{H^m(\Omega)} = \Big(\sum_{k=0}^{m}\sum_{\ell=0}^{k} \|\partial_x^\ell \partial_y^{k-\ell} v\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}, \;\; |v|_{H^m(\Omega)} = \Big(\sum_{\ell=0}^{m} \|\partial_x^\ell \partial_y^{m-\ell} v\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}};$$

when s is not an integer, $H^s(\Omega)$ is defined by Hilbertian interpolation between $H^{m+1}(\Omega)$ and $H^m(\Omega)$, with m equal to the integral part of s. As usual, $H_0^s(\Omega)$ stands for the closure in $H^s(\Omega)$ of the space $\mathcal{D}(\Omega)$ of infinitely differentiable functions with a compact support in Ω, and $H^{-s}(\Omega)$ denotes its dual space. On the global boundary $\partial\Omega$, $H^{s-\frac{1}{2}}(\partial\Omega)$ for all $s > \frac{1}{2}$ stands for the space of traces of functions in $H^s(\Omega)$. Similar spaces on a part of $\partial\Omega$ can be defined by restriction.

From now on, we assume that $\partial\Omega$ admits the decomposition (1.2) and that both Γ_d and Γ_n have positive measures in $\partial\Omega$. The variational space

$$X = \{v \in H^1(\Omega);\; v = 0 \text{ on } \Gamma_d\}, \tag{2.1}$$

is a closed subspace of $H^1(\Omega)$. Moreover the trace operator: $v \mapsto v_{|\Gamma_n}$ is linear and continuous from X onto $H_{00}^{\frac{1}{2}}(\Gamma_n)$ (we refer to [13] (Chap. 1, Th. 11.7) for the definition of $H_{00}^{\frac{1}{2}}(\Gamma_n)$).

Now, for any f in $L^2(\Omega)$ and g in the dual space $H_{00}^{\frac{1}{2}}(\Gamma_n)'$ of $H_{00}^{\frac{1}{2}}(\Gamma_n)$, we consider the following variational problem:

Find u in X such that

$$\forall v \in X, \quad a(u, v) = \int_\Omega f(\mathbf{x})v(\mathbf{x})\,d\mathbf{x} + \langle g, v\rangle, \tag{2.2}$$

where the bilinear form $a(\cdot,\cdot)$ is defined by

$$a(u, v) = \int_\Omega \mathbf{grad}\,u \cdot \mathbf{grad}\,v\,d\mathbf{x},$$

while $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $H_{00}^{\frac{1}{2}}(\Gamma_n)'$ and $H_{00}^{\frac{1}{2}}(\Gamma_n)$. Let $\mathcal{C}^\infty(\overline{\Omega})$ denote the space of restrictions to Ω of infinitely differentiable functions on $I\!R^2$. Then, it is readily checked from the density of $\mathcal{D}(\Omega)$ in $H_0^1(\Omega)$ and of $\mathcal{C}^\infty(\overline{\Omega}) \cap X$ in X, that this problem is equivalent to system (1.1).

The ellipticity of the form $a(\cdot,\cdot)$ on X is derived from a generalized Poincaré–Friedrichs inequality. Thus, the well-posedness of problem (2.2) is an easy consequence of Lax-Milgram lemma.

Proposition 2.1. *For any data f in $L^2(\Omega)$ and g in $H_{00}^{\frac{1}{2}}(\Gamma_n)'$, problem (2.2) has a unique solution u in X. Moreover, this solution satisfies, for a constant c_Ω only depending on the geometry of Ω,*

$$\|u\|_{H^1(\Omega)} \le c_\Omega \, (\|f\|_{L^2(\Omega)} + \|g\|_{H_{00}^{\frac{1}{2}}(\Gamma_n)'}). \tag{2.3}$$

In order to state the regularity results and since Γ_d and Γ_n are connected in the two situations we are interested in, we denote by $\mathbf{b}_1$ and $\mathbf{b}_2$ the two points in $\overline{\Gamma}_d \cap \overline{\Gamma}_n$. Let also $\mathcal{V}_1$ and $\mathcal{V}_2$ be open neighbourhoods of $\mathbf{b}_1$ and $\mathbf{b}_2$, respectively, such that $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ and also that the distance of each $\mathcal{V}_i$ to the vertices of Ω that do not coincide with $\mathbf{b}_1$ or $\mathbf{b}_2$ is positive. We first recall [12] (Chap. 5), [15] (Chap. I, § II & III) the regularity property outside $\mathcal{V}_1 \cup \mathcal{V}_2$.

Lemma 2.2. *The mapping: $(f,g) \mapsto u$, where u is the solution of problem (2.2), is continuous from $(L^2(\Omega) \cap H^{s-2}(\Omega)) \times H_\diamond^{s-\frac{3}{2}}(\Gamma_n)$ into $H^s(\Omega \setminus (\mathcal{V}_1 \cup \mathcal{V}_2))$ for all s, $1 < s < 3$, where $H_\diamond^{s-\frac{3}{2}}(\Gamma_n)$ for $s < 3$ stands for the space of functions such that their restrictions to any edge e contained in $\overline{\Gamma}_n$ belong to $H^{s-\frac{3}{2}}(e)$.*

This regularity property is optimal in the sense that the solution u of problem (2.2) does not necessarily belong to $H^3(\Omega)$ even for very smooth data f, as explained in the next lines. This requires some notation.

Let $\mathbf{a}_i$, $1 \le i \le 4$, denote the vertices of Ω, and let I_d, resp. I_n, be the set of indices i in $\{1,\dots,4\}$ such that $\mathbf{a}_i$ is inside Γ_d, resp. inside Γ_n. We introduce the system of polar coordinates (r_i, θ_i), where r_i is the distance to $\mathbf{a}_i$ and θ_i is the angular coordinate equal to 0 or $\frac{\pi}{2}$ on the two edges of $\partial\Omega$ that contain $\mathbf{a}_i$. We also introduce a regular function χ with compact support in $[0, \frac{1}{4}[$, equal to 1 in a neighbourhood of zero. Next, we consider the functions

$$\begin{aligned} S_i^d(r_i,\theta_i) &= \chi(r_i)\, r_i^2 \,((\log r_i)\, \sin(2\theta_i) + \theta_i \, \cos(2\theta_i) + \tfrac{\pi}{2} \, \sin^2\theta_i), \\ S_i^n(r_i,\theta_i) &= \chi(r_i)\, r_i^2 \,((\log r_i)\, \cos(2\theta_i) - \theta_i \, \sin(2\theta_i)). \end{aligned} \tag{2.4}$$

For simplicity, we use the same notations S_i^d and S_i^n for these functions in Cartesian coordinates (x,y) on Ω. It is readily checked that the functions S_i^d and S_i^n do not belong to $H^3(\Omega)$ while both ΔS_i^d and ΔS_i^n belong to

$H^1(\Omega)$ and the boundary value $\partial_n S_i^n$ belongs to $H^{\frac{3}{2}}(\Gamma_n)$. We refer to [1], [5], [6] and [12] (Chap. 5) for the following result.

Lemma 2.3. *For any data* (f, g) *in* $H^{s-2}(\Omega) \times H_\diamond^{s-\frac{3}{2}}(\Gamma_n)$, $3 \le s < 5$, *there exist constants* λ_i^d *and* λ_i^n *and a function* u_r *in* $H^s(\Omega \setminus (\mathcal{V}_1 \cup \mathcal{V}_2))$ *such that the solution* u *of problem* (2.2) *admits the following expansion*

$$u = u_r + \sum_{i \in I_d} \lambda_i^d S_i^d + \sum_{i \in I_n} \lambda_i^n S_i^n \qquad \text{in } \Omega \setminus (\mathcal{V}_1 \cup \mathcal{V}_2), \tag{2.5}$$

where $H_\diamond^{s-\frac{3}{2}}(\Gamma_n)$ *stands for the space of functions* g *such that*
(i) for $s > 3$, *their restrictions to any edge* e *contained in* $\overline{\Gamma}_n$ *belong to* $H^{s-\frac{3}{2}}(e)$ *and, if* $\overline{\Gamma}_n$ *contains two edges* e *and* e' *with common endpoint* $\mathbf{a}$, *these restrictions satisfy* $g'_{|e}(\mathbf{a}) = -g'_{|e'}(\mathbf{a})$,
(ii) for $s = 3$, $H_\diamond^{\frac{3}{2}}(\Gamma_n)$ *is defined by Hilbertian interpolation between* $H_\diamond^{\frac{7}{4}}(\Gamma_n)$ *and* $H_\diamond^{\frac{5}{4}}(\Gamma_n)$.

The compatibility condition in part (i) of this statement comes from the fact that smooth solutions u are such that $\partial_x \partial_y u(\mathbf{a})$ is uniquely defined. It only appears in the second situation where Γ_n is made of two edges (and we work with less regular solutions in this case, as explained below).

Remark. *The coefficients* λ_i^d *and* λ_i^n *can be computed explicitly as a function of the data. For instance,* λ_i^d *is equal to* $\frac{1}{\pi} f(\mathbf{a}_i)$, *see* [6]. *This proves the optimality of the regularity result stated in Lemma* 2.2.

So it remains to investigate the regularity properties in $\mathcal{V}_1 \cup \mathcal{V}_2$. Here, we consider separately the two situations described in the introduction. The case where Γ_n is a whole edge of Ω is called Situation I. We refer to [5] and [8] (§8) for the next results.

Lemma 2.4. *In Situation I, the mapping:* $(f, g) \mapsto u$, *where* u *is the solution of problem* (2.2), *is continuous from* $(L^2(\Omega) \cap H^{s-2}(\Omega)) \times H_*^{s-\frac{3}{2}}(\Gamma_n)$ *into* $H^s(\mathcal{V}_1 \cup \mathcal{V}_2)$ *for all* s, $1 \le s < 4$, *where* $H_*^{s-\frac{3}{2}}(\Gamma_n)$ *stands for the space*

$$H_*^{s-\frac{3}{2}}(\Gamma_n) = \begin{cases} H_{00}^{\frac{1}{2}}(\Gamma_n)' & \text{for } s = 1, \\ H^{\frac{3}{2}-s}(\Gamma_n)' & \text{for } 1 < s < \frac{3}{2}, \\ L^2(\Gamma_n) & \text{for } s = \frac{3}{2}, \\ H^{s-\frac{3}{2}}(\Gamma_n) & \text{for } \frac{3}{2} < s < 2, \\ H_{00}^{\frac{1}{2}}(\Gamma_n) & \text{for } s = 2, \\ H^{s-\frac{3}{2}}(\Gamma_n) \cap H_{00}^{\frac{1}{2}}(\Gamma_n) & \text{for } s > 2. \end{cases} \tag{2.6}$$

Remark. *Note that the result of Lemma* 2.4 *is no longer valid in the case* $2 \le s < 4$ *when the function* g *does not vanish at the endpoints of*

Γ_n. *Indeed, a singular function, equal to* $\chi(r_i)\,(r_i\,(\log r_i)\,\sin\theta_i+\theta_i\,\cos\theta_i)$ *appears in a neighbourhood of each vertex* $\mathbf{a}_i$ *which is an endpoint of* Γ_n, *and its coefficient is equal to* $-\frac{2}{\pi}\,g(\mathbf{a}_i)$, *as proven in* [7] *and* [8].

We need a slightly more precise result. Let us assume that the endpoints of Γ_n coincide with the vertices $\mathbf{a}_i$ of Ω, $i=1$ and 2. We consider the singular functions

$$S_i^{dn}(r_i,\theta_i)=\chi(r_i)\,r_i^3\,((\log r_i)\,\sin(3\theta_i)+\theta_i\,\cos(3\theta_i)), \tag{2.7}$$

where θ_i is chosen to be zero on Γ_d and equal to $\frac{\pi}{2}$ on Γ_n. We again refer to [8] (§8) for the following result.

Lemma 2.5. *In Situation I, for any data* (f,g) *in* $H^{s-2}(\Omega)\times H_*^{s-\frac{3}{2}}(\Gamma_n)$, $4\le s<6$, *there exist constants* λ_i^{dn}, $i=1$ *and* 2, *and a function* u_r *in* $H^s(\mathcal{V}_1\cup\mathcal{V}_2)$ *such that the solution* u *of problem* (2.2) *admits the following expansion*

$$u=u_r+\sum_{i=1}^{2}\lambda_i^{dn}\,S_i^{dn}\qquad \text{in } \mathcal{V}_1\cup\mathcal{V}_2. \tag{2.8}$$

We now consider the case, called Situation II, where Γ_n is the union of a vertex, say $\mathbf{a}_1$ for simplicity, and of parts of the two edges that contain $\mathbf{a}_1$. More precisely, $\overline{\Gamma}_n$ is the union of the two segments $\mathbf{b}_1\mathbf{a}_1$ and $\mathbf{a}_1\mathbf{b}_2$. Here, we denote by (ρ_j,η_j) the system of polar coordinates such that ρ_j is the distance to $\mathbf{b}_j$ and η_j is the angular coordinate equal to 0 on Γ_d and to π on Γ_n. We introduce the functions

$$\Sigma_j(\rho_j,\eta_j)=\chi(\rho_j)\,\rho_j^{\frac{1}{2}}\,\sin(\frac{\eta_j}{2}),\qquad j=1,2. \tag{2.9}$$

The proof of the following result can be found in [12] (Chap. 5).

Lemma 2.6. *In Situation II, the mapping:* $(f,g)\mapsto u$, *where* u *is the solution of problem* (2.2), *is continuous from* $L^2(\Omega)\times H^{s-\frac{3}{2}}(\Gamma_n)$ *into* $H^s(\mathcal{V}_1\cup\mathcal{V}_2)$ *for all* s, $1\le s<\frac{3}{2}$. *Moreover, for any data* (f,g) *in* $(L^2(\Omega)\cap H^{s-2}(\Omega))\times H^{s-\frac{3}{2}}(\Gamma_n)$, $\frac{3}{2}\le s<\frac{5}{2}$, *there exist constants* μ_j, $j=1$ *and* 2, *and a function* u_r *in* $H^s(\mathcal{V}_1\cup\mathcal{V}_2)$ *such that the solution* u *of problem* (2.2) *admits the following expansion*

$$u=u_r+\sum_{j=1}^{2}\mu_j\,\Sigma_j\qquad \text{in } \mathcal{V}_1\cup\mathcal{V}_2. \tag{2.10}$$

Remark. *Note that all expansions* (2.5), (2.8) *and* (2.10) *are continuous, in the sense that the* H^s*-norm of the function* u_r *and the absolute values of the constants are bounded as a function of the norms of* f *and* g.

Remark. *The singularities* S_i^d, S_i^n, S_i^{dn} *and* Σ_j, *introduced previously, are the local leading ones. More precisely, for appropriate singular but smoother functions* S_i^{d*}, S_i^{n*}, S_i^{dn*} *and* Σ_j^*, *expansions* (2.5), (2.8) *and* (2.10) *can be increased, respectively, as follows*

$$u = u_r^* + \sum_{i \in I_d} \lambda_i^d \, S_i^d + \sum_{i \in I_d} \lambda_i^{d*} \, S_i^{d*} + \sum_{i \in I_n} \lambda_i^n \, S_i^n + \sum_{i \in I_n} \lambda_i^{n*} \, S_i^{n*} \;\; in \; \Omega \setminus (\mathcal{V}_1 \cup \mathcal{V}_2), \tag{2.11}$$

$$u = u_r^* + \sum_{i=1}^{2} \lambda_i^{dn} \, S_i^{dn} + \sum_{i=1}^{2} \lambda_i^{dn*} \, S_i^{dn*} \;\; in \; \mathcal{V}_1 \cup \mathcal{V}_2, \tag{2.12}$$

$$u = u_r^* + \sum_{j=1}^{2} \mu_j \, \Sigma_j + \sum_{j=1}^{2} \mu_j^* \, \Sigma_j^* \quad in \; \mathcal{V}_1 \cup \mathcal{V}_2, \tag{2.13}$$

and for regular data f *and* g_n, *the function* u_r^* *now belongs to the spaces* $H^s(\Omega \setminus (\mathcal{V}_1 \cup \mathcal{V}_2))$ *or to* $H^s(\mathcal{V}_1 \cup \mathcal{V}_2)$ *for higher values of* s.

3. *Spectral discretization in the first situation*

For simplicity and as illustrated in the following figure, we now assume that Γ_n coincides with the edge $]-1,1[\times\{1\}$ of Ω.

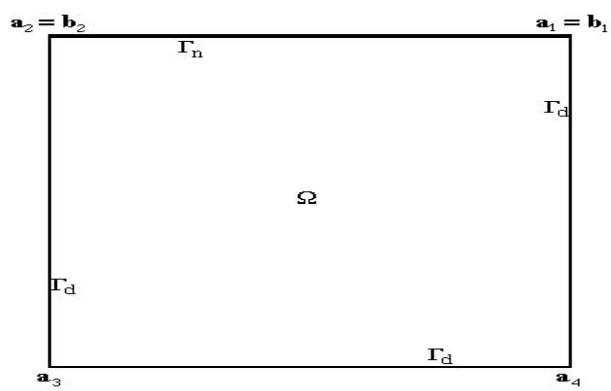

Figure 1

The discretization parameter is a positive integer N. For any nonnegative integer n, let $\mathbb{P}_n(\Omega)$ denote the space of restrictions to Ω of polynomials with two variables x and y and degree $\leq n$ with respect to each variable. The discrete space is

$$X_N = \mathbb{P}_N(\Omega) \cap X.$$

It is an easy matter to check that the dimension of X_N is $N(N-1)$.

We recall the main properties of the Gauss–Lobatto formula on the interval $]-1,1[$: with $\xi_0 = -1$ and $\xi_N = 1$, there exists a unique set of $N-1$ nodes ξ_j, $1 \le j \le N-1$, and $N+1$ weights σ_j, $0 \le j \le N$, such that the following equality holds for all polynomials Φ of degree $\le 2N-1$:

$$\int_{-1}^{1} \Phi(\zeta)\, d\zeta = \sum_{j=0}^{N} \Phi(\xi_j)\, \sigma_j.$$

If $(L_n)_{n\ge 0}$ denotes the family of Legendre polynomials (where each L_n has degree n and satisfies: $L_n(1) = 1$), the ξ_j, $1 \le j \le N-1$, are the zeros of L'_N and the weights σ_j, $0 \le j \le N$, are positive, given by

$$\sigma_j = \frac{2}{N(N+1)\, L_N^2(\xi_j)}.$$

Thus, for any functions u and v continuous on $\overline{\Omega}$, we are in a position to define by tensorization a discrete product

$$((u,v))_N = \sum_{i=0}^{N}\sum_{j=0}^{N} u(\xi_i,\xi_j) v(\xi_i,\xi_j)\, \sigma_i \sigma_j.$$

A similar one can be defined on all functions u and v continuous on $\overline{\Gamma}_n$:

$$((u,v))_{N,\Gamma_n} = \sum_{i=0}^{N} u_{|\Gamma_n}(\xi_i) v_{|\Gamma_n}(\xi_i)\, \sigma_i.$$

Let also $\mathcal{I}_N$ stand for the Lagrange interpolation operator at the nodes (ξ_i,ξ_j), $0 \le i,j \le N$, with values in $\mathbb{P}_N(\Omega)$ and $\mathcal{I}_N^{\Gamma_n}$ be the Lagrange interpolation operator at the nodes ξ_i of $\overline{\Gamma}_n$, $0 \le i \le N$, with values in the space $\mathbb{P}_N(\Gamma_n)$ of polynomials with degree $\le N$ with respect to the tangential coordinate on Γ_n.

Now, for any f continuous on $\overline{\Omega}$ and g continuous on $\overline{\Gamma}_n$, we consider the following variational problem:

Find u_N in X_N such that

$$\forall v_N \in X_N, \quad a_N(u_N, v_N) = ((f, v_N))_N + ((g, v_N))_{N,\Gamma_n}, \tag{3.1}$$

where the bilinear form $a_N(\cdot,\cdot)$ is defined by

$$a_N(u_N, v_N) = ((\partial_x u_N, \partial_x v_N))_N + ((\partial_y u_N, \partial_y v_N))_N.$$

We first check the well-posedness of this problem.

Lemma 3.1. *There exist positive constants* γ *and* α *independent of* N *such that the form* $a_N(\cdot,\cdot)$ *satisfies the following properties of continuity*

$$\forall u_N \in X_N, \forall v_N \in X_N, \quad a_N(u_N, v_N) \le \gamma \, \|u_N\|_{H^1(\Omega)} \|v_N\|_{H^1(\Omega)}, \tag{3.2}$$

and of ellipticity

$$\forall u_N \in X_N, \quad a_N(u_N, u_N) \ge \alpha \, \|u_N\|^2_{H^1(\Omega)}. \tag{3.3}$$

Proof. By using a Cauchy–Schwarz inequality in $a_N(u_N, v_N)$ and the equivalence of the semi-norm $|\cdot|_{H^1(\Omega)}$ and norm $\|\cdot\|_{H^1(\Omega)}$ on X (see Proposition 2.1), we are reduced to prove that, for appropriate constants α' and γ',

$$\forall u_N \in X_N, \quad \alpha' \, |u_N|^2_{H^1(\Omega)} \le a_N(u_N, u_N) \le \gamma' \, |u_N|^2_{H^1(\Omega)}. \tag{3.4}$$

Note that the term $(\partial_x u_N)^2$ has degree $\le 2N-2$ with respect to x, so that the exactness property of the quadrature formula implies that the sum on the i in the first term of $a_N(u_N, u_N)$ can be replaced by the integral with respect to x. Using a similar argument for the second term yields that

$$a_N(u_N, u_N) = \int_{-1}^{1} \sum_{j=0}^{N} (\partial_x u_N)^2(x, \xi_j) \, \sigma_j \, dx + \int_{-1}^{1} \sum_{i=0}^{N} (\partial_y u_N)^2(\xi_i, y) \, \sigma_i \, dy.$$

We recall [3] (Rem. 13.3), [4] (Lemma 2.2) that, with obvious definition for $\mathbb{P}_N(-1,1)$,

$$\forall \varphi_N \in \mathbb{P}_N(-1,1), \;\; \|\varphi_N\|^2_{L^2(-1,1)} \le \sum_{j=0}^{N} \varphi_N^2(\xi_j) \, \sigma_j \le 3 \, \|\varphi_N\|^2_{L^2(-1,1)}. \tag{3.5}$$

Inserting this into the previous line gives (3.4) with $\alpha' = 1$ and $\gamma' = 3$.

Proposition 3.2. *For any data* f *continuous on* $\overline{\Omega}$ *and* g *continuous on* $\overline{\Gamma}_n$*, problem* (3.1) *has a unique solution* u_N *in* X_N*. Moreover, this solution satisfies, for a constant* c_Ω *only depending on* Ω*,*

$$\|u_N\|_{H^1(\Omega)} \le c_\Omega \, (\|\mathcal{I}_N f\|_{L^2(\Omega)} + \|\mathcal{I}_N^{\Gamma_n} g\|_{L^2(\Gamma_n)}). \tag{3.6}$$

Proof. The existence and uniqueness of the solution is an obvious consequence of Lemma 3.1 combined with Lax-Milgram lemma. In order to prove (3.6), we take v_N equal to u_N in (3.1) and use (3.3), which leads to

$$\alpha \, \|u_N\|^2_{H^1(\Omega)} \le a_N(u_N, u_N) = ((f, u_N))_N + ((g, u_N))_{N,\Gamma_n}.$$

It follows from the definition of the scalar products that, in the previous line, f and g can be replaced by $\mathcal{I}_N f$ and $\mathcal{I}_N^{\Gamma_n} g$, respectively. By using Cauchy–Schwarz inequality, we derive

$$\alpha \|u_N\|^2_{H^1(\Omega)} \leq ((\mathcal{I}_N f, \mathcal{I}_N f))_N^{\frac{1}{2}} ((u_N, u_N))_N^{\frac{1}{2}} + ((\mathcal{I}_N^{\Gamma_n} g, \mathcal{I}_N^{\Gamma_n} g))_{N,\Gamma_n}^{\frac{1}{2}} ((u_N, u_N))_{N,\Gamma_n}^{\frac{1}{2}}.$$

Using once more (3.5) in each direction gives

$$\alpha \|u_N\|^2_{H^1(\Omega)} \leq 9\, \|\mathcal{I}_N f\|_{L^2(\Omega)} \|u_N\|_{L^2(\Omega)} + 3\, \|\mathcal{I}_N^{\Gamma_n} g\|_{L^2(\Gamma_n)} \|u_N\|_{L^2(\Gamma_n)}.$$

So the desired inequality follows thanks to the trace theorem.

We are now interested in the a priori error estimate between the solution u of problem (2.2) and the solution u_N of problem (3.1). Let v_N be any approximation of u in the space $X_{N-1} = \mathbb{P}_{N-1}(\Omega) \cap X$ of polynomials of degree $\leq N-1$ with respect to x and y which satisfy the boundary conditions in X. We deduce from (3.3) and the discrete problem (3.1) that

$$\begin{aligned}\alpha \|u_N - v_N\|^2_{H^1(\Omega)} &\leq a_N(u_N - v_N, u_N - v_N)\\ &= -a_N(v_N, u_N - v_N) + ((f, u_N - v_N))_N + ((g, u_N - v_N))_{N,\Gamma_n}.\end{aligned}$$

It follows from the exactness property of the quadrature formula that $a_N(v_N, u_N - v_N)$ coincides with $a(v_N, u_N - v_N)$. So using problem (2.2) yields

$$\begin{aligned}\alpha \|u_N - v_N\|^2_{H^1(\Omega)} &\leq a(u - v_N, u_N - v_N)\\ &\quad - \int_\Omega f(\mathbf{x})(u_N - v_N)(\mathbf{x})\, d\mathbf{x} + ((f, u_N - v_N))_N\\ &\quad - \langle g, u_N - v_N\rangle + ((g, u_N - v_N))_{N,\Gamma_n}.\end{aligned}$$

Combining this with a triangle inequality leads to the following version of Strang's second lemma, for a constant c independent of N,

$$\begin{aligned}\|u - u_N\|_{H^1(\Omega)} \leq c\Big(&\inf_{v_N \in X_{N-1}} \|u - v_N\|_{H^1(\Omega)}\\ &+ \sup_{w_N \in X_N} \frac{\int_\Omega f(\mathbf{x}) w_N(\mathbf{x})\, d\mathbf{x} - ((f, w_N))_N}{\|w_N\|_{H^1(\Omega)}}\\ &+ \sup_{w_N \in X_N} \frac{\langle g, w_N\rangle - ((g, w_N))_{N,\Gamma_n}}{\|w_N\|_{H^1(\Omega)}}\Big). \qquad (3.7)\end{aligned}$$

The last two terms in the right-hand side of (3.7) are issued from the use of numerical integration, and evaluating them relies on standard arguments. If Π_{N-1} denotes the orthogonal projection operator from $L^2(\Omega)$ onto $\mathbb{P}_{N-1}(\Omega)$, we have for any w_N in X_N

$$\int_\Omega f(\mathbf{x}) w_N(\mathbf{x})\,d\mathbf{x} - ((f, w_N))_N$$

$$= \int_\Omega (f - \Pi_{N-1} f)(\mathbf{x}) w_N(\mathbf{x})\,d\mathbf{x} - ((\mathcal{I}_N f - \Pi_{N-1} f, w_N))_N,$$

whence, due to (3.5),

$$\sup_{w_N \in X_N} \frac{\int_\Omega f(\mathbf{x}) w_N(\mathbf{x})\,d\mathbf{x} - ((f, w_N))_N}{\|w_N\|_{H^1(\Omega)}}$$

$$\leq 10\,\|f - \Pi_{N-1} f\|_{L^2(\Omega)} + 9\,\|f - \mathcal{I}_N f\|_{L^2(\Omega)}.$$

The approximation properties of the operators Π_{N-1} and $\mathcal{I}_N$ are well known [3] (Thms 7.1 & 14.2), [4] (Prop. 2.4 and 2.9), they lead to the following estimate: if the function f belongs to $H^\sigma(\Omega)$, $\sigma > 1$, then

$$\sup_{w_N \in X_N} \frac{\int_\Omega f(\mathbf{x}) w_N(\mathbf{x})\,d\mathbf{x} - ((f, w_N))_N}{\|w_N\|_{H^1(\Omega)}} \leq c\,N^{-\sigma}\,\|f\|_{H^\sigma(\Omega)}. \tag{3.8}$$

Similarly, if $\Pi_{N-1}^{\Gamma_n}$ denotes the orthogonal projection operator from $L^2(\Gamma_n)$ onto $\mathbb{P}_{N-1}(\Gamma_n)$, we have

$$\sup_{w_N \in X_N} \frac{\langle g, w_N\rangle - ((g, w_N))_{N,\Gamma_n}}{\|w_N\|_{H^1(\Omega)}} \leq 4\,\|g - \Pi_{N-1}^{\Gamma_n} g\|_{L^2(\Gamma_n)} + 3\,\|g - \mathcal{I}_N^{\Gamma_n} g\|_{L^2(\Gamma_n)},$$

which gives the second estimate [3] (Thms 6.1 & 13.4), [4] (Prop. 2.1 & 2.7): if the function g belongs to $H^\tau(\Gamma_n)$, $\tau > \frac{1}{2}$,

$$\sup_{w_N \in X_N} \frac{\langle g, w_N\rangle - ((g, w_N))_{N,\Gamma_n}}{\|w_N\|_{H^1(\Omega)}} \leq c\,N^{-\tau}\,\|g\|_{H^\tau(\Gamma_n)}. \tag{3.9}$$

So it remains to study the approximation error $\inf_{v_N \in X_{N-1}} \|u - v_N\|_{H^1(\Omega)}$. The standard estimate is proven in [3] (§7) and in [4] (Prop. 2.5 & 2.6): for any function v in $H^s(\Omega)$, $s \geq 1$,

$$\inf_{v_N \in X_N} \|v - v_N\|_{H^1(\Omega)} \leq c\,N^{1-s}\,\|v\|_{H^s(\Omega)}. \tag{3.10}$$

However we intend to slightly improve it, by taking into account the explicit knowledge of the singular functions introduced in Section 2. We just recall the main steps for deriving an optimal bound.

Step 1: Let π_N denote the orthogonal projection operator from $L^2(-1,1)$ onto $I\!P_N(-1,1)$. We recall [3] (§3) that the Legendre polynomials L_n satisfy the differential equation

$$((1-\zeta^2)\,L_n')' + n(n+1)\,L_n = 0,$$

hence they are the eigenfunctions of the Sturm–Liouville operator A defined by

$$A\varphi = -((1-\zeta^2)\,\varphi')'. \tag{3.11}$$

This operator is self-adjoint and positive definite, so that we can introduce for all nonnegative real numbers s the domain of A^s:

$$D(A^s) = \{\varphi \in L^2(-1,1);\ A^s\varphi \in L^2(-1,1)\},$$

which is provided with the graph norm. Then, the following estimate is proven in [2] (Lemma 5), [3] (Rem. 6.3), for all functions φ in $D(A^{\frac{s}{2}})$, $s \geq 0$,

$$\|\varphi - \pi_N\varphi\|_{L^2(-1,1)} \leq N^{-s}\,\|\varphi\|_{D(A^{\frac{s}{2}})}. \tag{3.12}$$

Let us also recall from [2] (Lemma 7) the following less standard property, valid for all functions φ in $D(A^{\frac{s}{2}})$, $s > 1$,

$$\|\pi_N\varphi' - (\pi_N\varphi)'\|_{L^2(-1,1)} \leq N^{2-s}\,\|\varphi\|_{D(A^{\frac{s}{2}})}. \tag{3.13}$$

Remark. *It is readily checked that* $H^s(-1,1)$ *is imbedded in* $D(A^{\frac{s}{2}})$, *so that estimate* (3.12) *is most often used with the norm* $\|\cdot\|_{D(A^{\frac{s}{2}})}$ *replaced by* $\|\cdot\|_{H^s(-1,1)}$. *However this imbedding is strict: indeed it is proven in* [9] *and* [10] *that, when* s *is an integer* m,

$$D(A^{\frac{m}{2}}) = \{\varphi \in L^2(-1,1);\ (1-\zeta^2)^{\frac{m}{2}}\,d^m\varphi \in L^2(-1,1)\}. \tag{3.14}$$

Step 2: On the square Ω, we define the following spaces by tensorization, for any nonnegative real number s,

$$V^s(\Omega) = L^2(-1,1; D(A_y^{\frac{s}{2}})) \cap D(A_x^{\frac{s}{2}}; L^2(-1,1)), \tag{3.15}$$

where A_x and A_y denote the operator A applied with respect to the x and y variables, respectively. Indeed, the orthogonal projection operator Π_N from $L^2(\Omega)$ onto $I\!P_N(\Omega)$ coincides with the tensorized product $\pi_N^x \circ \pi_N^y$, where π_N^x and π_N^y stand for the one-dimensional operator π_N applied with respect to the x and y directions. Noting that π_N^y commutes with both

π_N^x and the derivative with respect to x, and starting from the triangle inequality

$$\|\partial_x(v-\Pi_N v)\|_{L^2(\Omega)} \le \|\partial_x v - \pi_N^y(\partial_x v)\|_{L^2(\Omega)} + \|\pi_N^y(\partial_x v - \pi_N^x(\partial_x v))\|_{L^2(\Omega)} + \|\pi_N^y(\pi_N^x(\partial_x v) - \partial_x(\pi_N^x v))\|_{L^2(\Omega)}$$

together with its analogue for $\partial_y(v-\Pi_N v)$, we derive the following result from (3.12) and (3.13)(see [2] (Prop. 10) for more details): for any function v in $V^s(\Omega)$, $s>1$, such that $\mathbf{grad}\, v$ belongs to $V^t(\Omega)^2$, $t \ge 0$,

$$\|v-\Pi_N v\|_{H^1(\Omega)} \le c\,(N^{2-s}\,\|v\|_{V^s(\Omega)} + N^{-t}\,\|\mathbf{grad}\, v\|_{V^t(\Omega)^2}). \tag{3.16}$$

However, the operator Π_N does not preserve the zero conditions on Γ_d, hence it does not map X into X_N.

Step 3: Let Γ_ℓ, $1 \le \ell \le 4$, denote the edges of Ω. We recall from [14] the following result.

Lemma 3.3. *For $1 \le \ell \le 4$, there exists a lifting operator R_N^ℓ from the space $\mathbb{P}_N(\Gamma_\ell)$ into $\mathbb{P}_N(\Omega)$ such that, for any φ_N in $\mathbb{P}_N(\Gamma_\ell)$, $R_N^\ell\varphi_N$ is equal to φ_N on Γ_ℓ, vanishes on the opposite edge to Γ_ℓ, and satisfies*

$$\|R_N^\ell\varphi_N\|_{H^1(\Omega)} \le c\,\|\varphi_N\|_{H^{\frac12}(\Gamma_\ell)}. \tag{3.17}$$

Moreover, there exists a lifting operator $\tilde{R}_N^\ell$ from the space of polynomials in $\mathbb{P}_N(\Gamma_\ell)$ vanishing at one or two endpoints $\mathbf{a}_i$ of Γ_ℓ into $\mathbb{P}_N(\Omega)$ such that, for any φ_N in $\mathbb{P}_N(\Gamma_\ell)$, $\tilde{R}_N^\ell\varphi_N$ is equal to φ_N on Γ_ℓ, vanishes on the opposite edge to Γ_ℓ and on the other edge containing the $\mathbf{a}_i$, and satisfies

$$\|\tilde{R}_N^\ell\varphi_N\|_{H^1(\Omega)} \le c\,\|\varphi_N\|_{\tilde{H}^{\frac12}(\Gamma_\ell)}, \tag{3.18}$$

for an appropriate norm $\|\cdot\|_{\tilde{H}^{\frac12}(\Gamma_\ell)}$ (see [14]).

We do not write explicitly the norm $\|\cdot\|_{\tilde{H}^{\frac12}(\Gamma_\ell)}$ for simplicity, however note the following property: for any function v in $H^1(\Omega)$ vanishing on the other edge containing $\mathbf{a}_i$,

$$\|v\|_{\tilde{H}^{\frac12}(\Gamma_\ell)} \le c\,\|v\|_{H^1(\Omega)}. \tag{3.19}$$

Assume that the three edges in Γ_d are the Γ_ℓ, $1 \le \ell \le 3$, and that Γ_2 stands for the opposite edge to Γ_n. The final idea is to set

$$w_N = \Pi_N v - R_N^2[(\Pi_N v)_{|\Gamma_2}], \quad v_N = w_N - \tilde{R}_N^1[w_{N\,|\Gamma_1}] - \tilde{R}_N^3[w_{N\,|\Gamma_3}].$$

Indeed, w_N vanishes on the two vertices in $\overline{\Gamma}_1 \cap \overline{\Gamma}_2$ and $\overline{\Gamma}_3 \cap \overline{\Gamma}_2$, so that v_N belongs to X_N. Moreover we derive from (3.17) and (3.18) that

$$\|v-w_N\|_{H^1(\Omega)} \le \|v-\Pi_N v\|_{H^1(\Omega)} + c\,\|\Pi_N v\|_{H^{\frac12}(\Gamma_2)},$$

$$\|v - v_N\|_{H^1(\Omega)} \le \|v - w_N\|_{H^1(\Omega)} + c\,\|w_N\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_1)} + c\,\|w_N\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_3)}.$$

By noting that v vanishes on Γ_1, Γ_2 and Γ_3, this yields

$$\|v - w_N\|_{H^1(\Omega)} \le \|v - \Pi_N v\|_{H^1(\Omega)} + c\,\|v - \Pi_N v\|_{H^{\frac{1}{2}}(\Gamma_2)},$$

$$\|v - v_N\|_{H^1(\Omega)} \le \|v - w_N\|_{H^1(\Omega)} + c\,\|v - w_N\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_1)} + c\,\|v - w_N\|_{\tilde{H}^{\frac{1}{2}}(\Gamma_3)}.$$

Thanks to the trace theorem and (3.19), this leads to the final estimate: for any function v in $V^s(\Omega) \cap X$, $s > 1$, such that $\mathbf{grad}\, v$ belongs to $V^t(\Omega)^2$, $t \ge 0$,

$$\inf_{v_N \in X_N} \|v - v_N\|_{H^1(\Omega)} \le c\,(N^{2-s}\,\|v\|_{V^s(\Omega)} + N^{-t}\,\|\mathbf{grad}\, v\|_{V^t(\Omega)^2}). \quad (3.20)$$

And, of course, estimates (3.10) and (3.20) hold with X_N replaced by X_{N-1} (but with a slightly modified constant).

The interest of working with the spaces $V^s(\Omega)$ instead of $H^s(\Omega)$ comes from the following lemma. We refer to [2] (Prop. 14 & Rem. 15) for its proof.

Lemma 3.4. *Let S be any function of type*

$$S(r_i, \theta_i) = \chi(r_i)\, r_i^\lambda\, (\log r_i)^p\, \varphi(\theta_i),$$

where i belongs to $\{1, \cdots, 4\}$, $\lambda > 0$, p is either zero or 1, and φ belongs to $\mathcal{C}^\infty([0, \frac{\pi}{2}])$. Then the function S belongs to $H^s(\Omega)$ for all $s < \lambda + 1$ and to $V^s(\Omega)$ for all $s < 2(\lambda + 1)$. Moreover, there exists a constant c such that, for all real number η, $0 < \eta < 1$,

$$\|S\|_{V^{2(\lambda+1)-\eta}(\Omega)} \le c\,\eta^{-\frac{1}{2}}. \quad (3.21)$$

We are now in a position to state the main result of this section.

Theorem 3.5. *Assume that the function f belongs to $H^\sigma(\Omega)$, $\sigma > 1$, and that the function g belongs to $H^\tau_*(\Gamma_n)$, $\tau > \frac{1}{2}$. Then, the following error estimate holds between the solution u of problem (2.2) and the solution u_N of problem (3.1):*

$$\|u - u_N\|_{H^1(\Omega)} \le c\,\sup\{N^{-\sigma}, N^{-\tau}, N^{-4}\,(\log N)^{\frac{1}{2}}\}\,(\|f\|_{H^\sigma(\Omega)} + \|g\|_{H^\tau(\Gamma_n)}). \quad (3.22)$$

Proof. Thanks to (3.7) to (3.9), it suffices to estimate the approximation error term $\inf_{v_N \in X_{N-1}} \|u - v_N\|_{H^1(\Omega)}$. We first assume that the data f and g are smooth, namely that they belong to $H^\sigma(\Omega)$, $\sigma \ge 3$, and to $H^\tau_*(\Gamma_n)$, $\tau \geq \frac{7}{2}$. By combining Lemmas 2.3 and 2.5, we observe that u admits the continuous expansion

$$u = u_r + \sum_{i=3}^{4} \lambda_i^d \, S_i^d + \sum_{i=1}^{2} \lambda_i^{dn} \, S_i^{dn}$$

with u_r in $H^s(\Omega)$ for all $s < 5$. So the idea is to choose an approximation v_N of u of type

$$v_N = v_{rN} + \sum_{i=3}^{4} \lambda_i^d \, S_{iN}^d + \sum_{i=1}^{2} \lambda_i^{dn} \, S_{iN}^{dn}.$$

Indeed, we derive from Lemma 3.4 that, for all $\eta > 0$, the function S_i^d belongs to $V^{6-\eta}(\Omega)$ and also that its gradient belongs to $V^{4-\eta}(\Omega)^2$ (since both $\partial_x S_i^d$ and $\partial_y S_i^d$ are of the same type as the function S in Lemma 3.4, with $\lambda = 1$). Then, applying (3.20) yields that there exists a polynomial S_{iN}^d in X_{N-1} such that, for any $\eta > 0$,

$$\|S_i^d - S_{iN}^d\|_{H^1(\Omega)} \leq c\, N^{\eta-4} \, (\|S_i^d\|_{V^{6-\eta}(\Omega)} + \|\mathbf{grad}\, S_i^d\|_{V^{4-\eta}(\Omega)^2}).$$

Using (3.21) gives

$$\|S_i^d - S_{iN}^d\|_{H^1(\Omega)} \leq c\, N^{\eta-4} \, \eta^{-\frac{1}{2}},$$

so that taking η equal to $(\log\, N)^{-1}$ leads to

$$\|S_i^d - S_{iN}^d\|_{H^1(\Omega)} \leq c\, N^{-4} \, (\log\, N)^{\frac{1}{2}}. \tag{3.23}$$

By a simpler argument (the S_i^{dn} are smoother than the S_i^d), still relying on (3.20) and Lemma 3.4 , we also derive the existence of a S_{iN}^{dn} in X_{N-1} such that

$$\|S_i^{dn} - S_{iN}^{dn}\|_{H^1(\Omega)} \leq c\, N^{-4}. \tag{3.24}$$

Finally, the function u_r belongs to $H^s(\Omega)$, $s < 5$, so that using (3.10) gives, for any $\eta > 0$,

$$\inf_{v_{rN} \in X_{N-1}} \|u_r - v_{rN}\|_{H^1(\Omega)} \leq c\, N^{\eta-4} \, \|u_r\|_{H^{5-\eta}(\Omega)}.$$

Moreover it follows from expansion (2.11) and (3.21) that, there also, $\|u\|_{H^{5-\eta}(\Omega)}$ behaves like $c\,\eta^{-\frac{1}{2}}$. Taking once more η equal to $(\log N)^{-1}$ and using the continuity of expansions (2.5) and (2.8) gives the desired result.

When one of the data f and g is not smooth, the solution u or its regular part u_r belongs to $H^s(\Omega)$, $s \le \min\{\sigma+2, \tau+\frac{3}{2}\}$. So the approximation error on u behaves like $c\sup\{N^{-\sigma-1}, N^{-\tau-\frac{1}{2}}\}$, which is smaller than $c\sup\{N^{-\sigma}, N^{-\tau}\}$. There also we obtain the desired estimate.

We conclude this section by deriving an estimate of the error in $L^2(\Omega)$ thanks to the Aubin–Nitsche duality argument.

Corollary 3.6. *If the assumptions of Theorem* 3.5 *are satisfied, the following error estimate holds between the solution u of problem* (2.2) *and the solution u_N of problem* (3.1)*:*

$$\|u-u_N\|_{L^2(\Omega)} \le c\sup\{N^{-\sigma}, N^{-\tau},\, N^{-5}\,(\log N)^{\frac{1}{2}}\}\,(\|f\|_{H^\sigma(\Omega)} + \|g\|_{H^\tau(\Omega)}). \tag{3.25}$$

Proof. As standard, proving the estimate relies on the formula

$$\|u-u_N\|_{L^2(\Omega)} = \sup_{h\in L^2(\Omega)} \frac{\int_\Omega (u-u_N)(\mathbf{x})h(\mathbf{x})\,d\mathbf{x}}{\|h\|_{L^2(\Omega)}}. \tag{3.26}$$

Indeed, for any h in $L^2(\Omega)$, we consider the following problem with mixed boundary conditions

$$\begin{cases} -\Delta w = h & \text{in } \Omega,\\ w = 0 & \text{on } \Gamma_d,\\ \partial_n w = 0 & \text{on } \Gamma_n. \end{cases} \tag{3.27}$$

It follows from Lemmas 2.2 and 2.4 that its solution w belongs to $H^2(\Omega)$ and satisfies

$$\|w\|_{H^2(\Omega)} \le c\,\|h\|_{L^2(\Omega)}. \tag{3.28}$$

By integrating by parts, we obtain

$$\int_\Omega (u-u_N)(\mathbf{x})h(\mathbf{x})\,d\mathbf{x} = a(u-u_N, w).$$

So using problems (2.2) and (3.1) yields for any w_N in X_{N-1}

$$\int_\Omega (u-u_N)(\mathbf{x})h(\mathbf{x})\,d\mathbf{x} = a(u-u_N, w-w_N)$$
$$+\int_\Omega f(\mathbf{x})w_N(\mathbf{x})\,d\mathbf{x} - ((f,w_N))_N + \langle g, w_N\rangle - ((g,w_N))_{N,\Gamma_n}. \tag{3.29}$$

So choosing w_N as an accurate approximation of w (see (3.10)), and applying once more (3.8) and (3.9) lead to the desired estimate.

For smooth data f and g and up to the $(\log N)^{\frac{1}{2}}$ which is negligible, the error behaves like $c\,N^{-4}$ in $H^1(\Omega)$, like $c\,N^{-5}$ in $L^2(\Omega)$. It is exactly of the same order as for a standard Dirichlet problem (see [2] (§4.2)), which is not surprising since the leading singularities are those of the "Dirichlet" corners $\mathbf{a}_3$ and $\mathbf{a}_4$.

Remark. *Let us consider for a while the case where* Γ_n *is the union of two opposite edges of* Ω. *Thus, all the vertices of* Ω *belong to* $\overline{\Gamma}_d \cap \overline{\Gamma}_n$, *and the singularities there are weaker than for "Dirichlet" corners. So, if the assumptions of Theorem* 3.5 *hold, estimates* (3.22) *and* (3.25) *can be replaced by*

$$\|u-u_N\|_{H^1(\Omega)} \le c \sup\{N^{-\sigma}, N^{-\tau}, N^{-6}(\log N)^{\frac{1}{2}}\} (\|f\|_{H^\sigma(\Omega)} + \|g\|_{H^\tau(\Gamma_n)}), \tag{3.30}$$

$$\|u-u_N\|_{L^2(\Omega)} \le c \sup\{N^{-\sigma}, N^{-\tau}, N^{-7}(\log N)^{\frac{1}{2}}\} (\|f\|_{H^\sigma(\Omega)} + \|g\|_{H^\tau(\Gamma_n)}). \tag{3.31}$$

The convergence order is still higher here.

4. *Spectral element discretization in the second situation*

We now consider Situation II, in the following special geometry: Γ_n is the union of the two edges $\{1\}\times]1-\varepsilon,1]$ and $]1-\varepsilon,1]\times\{1\}$, with $0<\varepsilon\le 1$, so that it contains the corner $\mathbf{a}_1$ with coordinates $(1,1)$. As illustrated in Figure 2, we also denote by $\mathbf{b}_1$ and $\mathbf{b}_2$ the endpoints of Γ_n with coordinates $(1,1-\varepsilon)$ and $(1-\varepsilon,1)$, respectively, by Γ_{n1} the segment $\mathbf{b}_1\mathbf{a}_1$ and by Γ_{n2} the segment $\mathbf{a}_1\mathbf{b}_2$. The idea is to take into account the case where ε can be small.

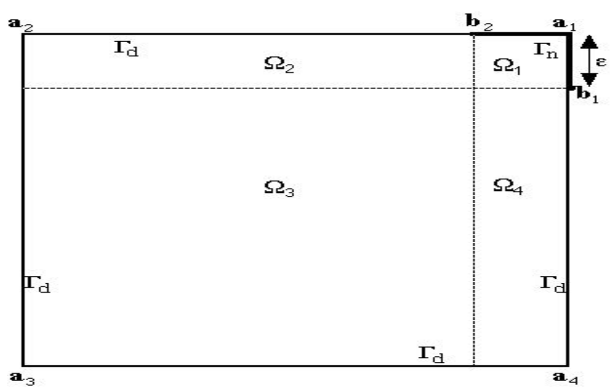

Figure 2

In view of the discretization, we introduce a partition of Ω into four rectangles Ω_k, $1\le k\le 4$, defined as follows:

$$\begin{aligned}\Omega_1 &=]1-\varepsilon,1[^2, \quad \Omega_2 =]-1,1-\varepsilon[\times]1-\varepsilon,1[,\\ \Omega_3 &=]-1,1-\varepsilon[^2, \quad \Omega_4 =]1-\varepsilon,1[\times]-1,1-\varepsilon[.\end{aligned}$$

It can be noted that each $\overline{\Omega}_k$, $1\le k\le 4$, contains the vertex $\mathbf{a}_k$ of Ω.

The discretization parameter δ here is a pair of positive integers (N, M). Indeed, in order to handle possibly small values of ε, we introduce the four local spaces: with the notation of Section 3, Z_δ^1 is the space $\mathbb{P}_N(\Omega_1)$ and Z_δ^3 is the space $\mathbb{P}_M(\Omega_3)$, while Z_δ^2 is the space $\mathbb{P}_{M,N}(\Omega_2)$ of restrictions to Ω_2 of polynomials with degree $\leq M$ with respect to x and $\leq N$ with respect to y and Z_δ^4 is the space $\mathbb{P}_{N,M}(\Omega_4)$ of restrictions to Ω_4 of polynomials with degree $\leq N$ with respect to x and $\leq M$ with respect to y. With these choices, it can be observed that the spaces of traces of functions in Z_δ^k and $Z_\delta^{k'}$, $1 \leq k < k' \leq 4$, coincide on the interface $\partial\Omega_k \cap \partial\Omega_{k'}$. So we define the discrete space

$$X_\delta = \{v_\delta \in X;\ v_{\delta\,|\Omega_k} \in Z_\delta^k,\ 1 \leq k \leq 4\}.$$

Note that, since X_δ is imbedded in X, the functions in X_δ are continuous through the interfaces and satisfy the right boundary conditions.

By translation and homothety of the Gauss–Lobatto nodes and weights introduced in Section 3, we introduce the $N+1$ nodes ξ_j^N and weights σ_j^N, $0 \leq j \leq N$, on $[1-\varepsilon, 1]$ such that the quadrature formula is exact on $\mathbb{P}_{2N-1}(1-\varepsilon, 1)$, the $M+1$ nodes ξ_j^M and weights σ_j^M, $0 \leq j \leq M$, on $[-1, 1-\varepsilon]$ such that the quadrature formula is exact on $\mathbb{P}_{2M-1}(-1, 1-\varepsilon)$. This leads to define the four "local" discrete products

$$((u,v))_\delta^1 = \sum_{i=0}^{N}\sum_{j=0}^{N} u_{|\Omega_1}(\xi_i^N, \xi_j^N) v_{|\Omega_1}(\xi_i^N, \xi_j^N)\, \sigma_i^N \sigma_j^N,$$

$$((u,v))_\delta^2 = \sum_{i=0}^{M}\sum_{j=0}^{N} u_{|\Omega_2}(\xi_i^M, \xi_j^N) v_{|\Omega_2}(\xi_i^M, \xi_j^N)\, \sigma_i^M \sigma_j^N,$$

$$((u,v))_\delta^3 = \sum_{i=0}^{M}\sum_{j=0}^{M} u_{|\Omega_3}(\xi_i^M, \xi_j^M) v_{|\Omega_3}(\xi_i^M, \xi_j^M)\, \sigma_i^M \sigma_j^M,$$

$$((u,v))_\delta^4 = \sum_{i=0}^{N}\sum_{j=0}^{M} u_{|\Omega_4}(\xi_i^N, \xi_j^M) v_{|\Omega_4}(\xi_i^N, \xi_j^M)\, \sigma_i^N \sigma_j^M,$$

and finally the global product

$$((u,v))_\delta = \sum_{k=1}^{4} ((u,v))_\delta^k.$$

Similarly, on Γ_n, we define the local products

$$((u,v))_{\delta,\Gamma_n}^1 \quad = \quad \sum_{j=0}^{N} u_{|\Gamma_{n1}}(\xi_j^N) v_{|\Gamma_{n1}}(\xi_j^N)\, \sigma_j^N,$$

$$((u,v))^2_{\delta,\Gamma_n} \;=\; \sum_{i=0}^{N} u_{|\Gamma_{n2}}(\xi_i^N) v_{|\Gamma_{n2}}(\xi_i^N)\,\sigma_i^N,$$

and the global one

$$((u,v))_{\delta,\Gamma_n} = \sum_{\ell=1}^{2} ((u,v))^\ell_{\delta,\Gamma_n}.$$

We also denote by $\mathcal{I}^k_\delta$ the Lagrange interpolation operator associated with the nodes of the discrete product $((\cdot,\cdot))^k_\delta$ and by $\mathcal{I}_\delta$ the global interpolation operator (i.e. such that its restriction to each Ω_k, $1 \le k \le 4$, coincides with $\mathcal{I}^k_\delta$). Similar definitions are used for the operators $\mathcal{I}^{\Gamma_n,\ell}_\delta$ and $\mathcal{I}^{\Gamma_n}_\delta$.

The discrete problem now reads, for any f continuous on $\overline{\Omega}$ and g continuous on $\overline{\Gamma}_n$,

Find u_δ in X_δ such that

$$\forall v_\delta \in X_\delta, \quad a_\delta(u_\delta, v_\delta) = ((f, v_\delta))_\delta + ((g, v_\delta))_{\delta,\Gamma_n}, \tag{4.1}$$

where the bilinear form $a_\delta(\cdot,\cdot)$ is defined by

$$a_\delta(u_\delta, v_\delta) = ((\partial_x u_\delta, \partial_x v_\delta))_\delta + ((\partial_y u_\delta, \partial_y v_\delta))_\delta.$$

Remark. *Writing the discrete problem is a little more complex than in Section* 3*, because of the use of domain decomposition. However, let us for a while consider a discretization of problem* (2.2) *in this situation and without domain decomposition.*
1) If a conforming discretization is used, relying on the basic conforming space $X_N = \mathbb{P}_N(\Omega) \cap X$, it is readily checked that functions in this space have a zero trace on the whole boundary $\partial\Omega$. Thus there is no convergence of the best approximation of u in X_N for all functions u in X that do not vanish on Γ_n.
2) In the case of a nonconforming discretization, relying for instance on the space of polynomials in $\mathbb{P}_N(\Omega)$ that vanish at all nodes ξ_j (introduced in Section 3*) which belong to $\overline{\Gamma}_d$, the distance of each function Σ_j defined in* (2.9) *to each best approximation would behave like $c\,N^{-\frac{1}{2}}\,(\log N)^{\frac{1}{2}}$. We intend to prove that the convergence order of problem* (4.1) *is higher.*

The properties of the form $a_\delta(\cdot,\cdot)$ are derived from the equivalence of the semi-norm $|\cdot|_{H^1(\Omega)}$ and norm $\|\cdot\|_{H^1(\Omega)}$ on X together with exactly the same arguments as in the proof of Lemma 3.1 applied on each subdomain Ω_k. So we skip the proof of the following lemma.

Lemma 4.1. *There exist positive constants γ and α independent of δ such that the form $a_\delta(\cdot,\cdot)$ satisfies the following properties of continuity*

$$\forall u_\delta \in X_\delta, \forall v_\delta \in X_\delta, \quad a_\delta(u_\delta, v_\delta) \le \gamma\,\|u_\delta\|_{H^1(\Omega)} \|v_\delta\|_{H^1(\Omega)}, \tag{4.2}$$

and of ellipticity

$$\forall u_\delta \in X_\delta, \quad a_\delta(u_\delta, u_\delta) \geq \alpha \, \|u_\delta\|^2_{H^1(\Omega)}. \tag{4.3}$$

This leads to the well-posedness result.

Proposition 4.2. *For any data f continuous on $\overline{\Omega}$ and g continuous on $\overline{\Gamma}_n$, problem (4.1) has a unique solution u_δ in X_δ. Moreover, this solution satisfies, for a constant c_Ω only depending on the geometry of Ω,*

$$\|u_\delta\|_{H^1(\Omega)} \leq c_\Omega \, (\|\mathcal{I}_\delta f\|_{L^2(\Omega)} + \|\mathcal{I}_\delta^{\Gamma_n} g\|_{L^2(\Gamma_n)}). \tag{4.4}$$

Let now δ_- be the pair of integers $(M-1, N-1)$. The space $X_{\delta-}$ is obviously defined as the space X_δ when replacing N by $N-1$ and M by $M-1$. Then, the same arguments as in Section 3 leads to the analogue of (3.7), for a constant c independent of δ,

$$\begin{aligned} \|u - u_\delta\|_{H^1(\Omega)} \leq c \Big(& \inf_{v_\delta \in X_{\delta-}} \|u - v_\delta\|_{H^1(\Omega)} \\ & + \sup_{w_\delta \in X_\delta} \frac{\int_\Omega f(\mathbf{x}) w_\delta(\mathbf{x}) \, d\mathbf{x} - ((f, w_\delta))_\delta}{\|w_\delta\|_{H^1(\Omega)}} \\ & + \sup_{w_\delta \in X_\delta} \frac{\langle g, w_\delta \rangle - ((g, w_\delta))_{\delta, \Gamma_n}}{\|w_\delta\|_{H^1(\Omega)}} \Big). \end{aligned} \tag{4.5}$$

Let now Z_δ stand for the space

$$Z_\delta = \{ v_\delta \in L^2(\Omega); \; v_{\delta \, |\Omega_k} \in Z_\delta^k, \, 1 \leq k \leq 4 \},$$

and let $Z_{\delta-}$ be its analogue with N replaced by $N-1$ and M replaced by $M-1$. If $\Pi_{\delta-}$ denotes the orthogonal projection operator from $L^2(\Omega)$ onto $Z_{\delta-}$, we derive from the same arguments as in Section 3 that

$$\sup_{w_\delta \in X_\delta} \frac{\int_\Omega f(\mathbf{x}) w_\delta(\mathbf{x}) \, d\mathbf{x} - ((f, w_\delta))_\delta}{\|w_\delta\|_{H^1(\Omega)}} \leq 10 \, \|f - \Pi_{\delta-} f\|_{L^2(\Omega)} + 9 \, \|f - \mathcal{I}_\delta f\|_{L^2(\Omega)}. \tag{4.6}$$

Note that each $\|f - \Pi_{\delta-} f\|_{L^2(\Omega)}$ and $\|f - \mathcal{I}_\delta f\|_{L^2(\Omega)}$ can be evaluated separately on each subdomain Ω_k. However, in order to take into account the large aspect ratio of the anisotropic domains Ω_2 and Ω_4, we need some special arguments.

We introduce the continuous piecewise affine mapping F that sends the domain Ω onto the square $\hat{\Omega} =]-1, 3-2\varepsilon[^2$ and each subdomain Ω_k, $1 \leq k \leq 4$, onto a square $\hat{\Omega}_k$ with length of edge equal to $2 - \varepsilon$ (see Figure

3). More precisely, if φ stands for the mapping $\zeta \mapsto 1-\varepsilon+\frac{2-\varepsilon}{\varepsilon}(\zeta-1+\varepsilon)$,
- the function $F_{|\Omega_1}$: $(x,y) \mapsto (\hat{x}=\varphi(x), \hat{y}=\varphi(y))$ maps Ω_1 onto $\hat{\Omega}_1 =]1-\varepsilon, 3-2\varepsilon[^2$,
- the function $F_{|\Omega_2}$: $(x,y) \mapsto (\hat{x}=x, \hat{y}=\varphi(y))$ maps Ω_2 onto $\hat{\Omega}_2 =]-1, 1-\varepsilon[\times]1-\varepsilon, 3-2\varepsilon[$,
- the function $F_{|\Omega_3}$ is simply the identity, so that $\hat{\Omega}_3 = \Omega_3$,
- the function $F_{|\Omega_4}$: $(x,y) \mapsto (\hat{x}=\varphi(x), \hat{y}=y)$ maps Ω_4 onto $\hat{\Omega}_4 =]1-\varepsilon, 3-2\varepsilon[\times]-1, 1-\varepsilon[$.

We use the standard notation $\hat{w} = w \circ F^{-1}$ for all functions w defined on Ω.

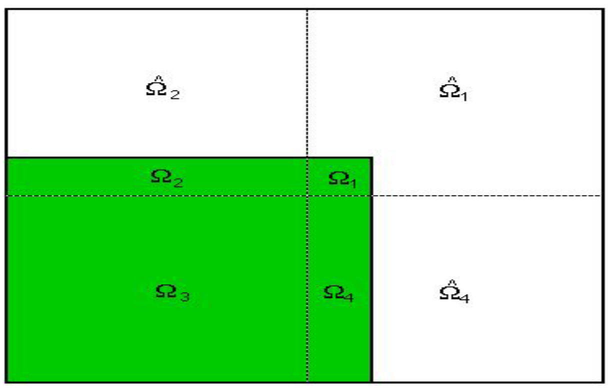

Figure 3

We agree to denote $\hat{\Lambda}_k^{\hat{x}}$ and $\hat{\Lambda}_k^{\hat{y}}$ the intervals such that the domain $\hat{\Omega}_k$ is equal to the product $\hat{\Lambda}_k^{\hat{x}} \times \hat{\Lambda}_k^{\hat{y}}$. We need the "anisotropic" semi-norms

$$\begin{aligned} |\hat{w}|_{H^s_{\hat{x}}(\hat{\Omega}_k)} &= \Big(\int_{\hat{\Lambda}_k^{\hat{y}}} |v(\cdot,\hat{y})|^2_{H^s(\hat{\Lambda}_k^{\hat{x}})}\, d\hat{y} \Big)^{\frac{1}{2}}, \\ |\hat{w}|_{H^s_{\hat{y}}(\hat{\Omega}_k)} &= \Big(\int_{\hat{\Lambda}_k^{\hat{x}}} |v(\hat{x},\cdot)|^2_{H^s(\hat{\Lambda}_k^{\hat{y}})}\, d\hat{x} \Big)^{\frac{1}{2}}. \end{aligned}$$

Lemma 4.3. *Assume that the function f belongs to $H^\sigma(\Omega)$, $\sigma > 1$. Then, the following estimate holds*

$$\sup_{w_\delta \in X_\delta} \frac{\int_\Omega f(\mathbf{x}) w_\delta(\mathbf{x})\, d\mathbf{x} - ((f, w_\delta))_\delta}{\|w_\delta\|_{H^1(\Omega)}} \le c\,(\varepsilon^\sigma N^{-\sigma} + M^{-\sigma})\, \|f\|_{H^\sigma(\Omega)}. \quad (4.7)$$

Proof. Relying on (4.6), we first estimate the quantity $\|f - \Pi_{\delta-} f\|_{L^2(\Omega)}$. We agree to denote by $\pi_n^{\hat{x}}$ and $\pi_n^{\hat{y}}$ the operator π_n introduced in Section 3 (see (3.12)) when translated to $\hat{\Lambda}_k^{\hat{x}}$ and $\hat{\Lambda}_k^{\hat{y}}$, respectively (we skip the k in this notation for simplicity). Then it can be checked that, for $1 \le k \le 4$, the function $\hat{f}_\delta^k = ((\Pi_{\delta-} f) \circ F^{-1})_{|\hat{\Omega}_k}$ coincides with $\pi_{m_k}^{\hat{x}} \circ \pi_{n_k}^{\hat{y}} \hat{f}_{|\hat{\Omega}_k}$, with

$$(m_1, n_1) = (N-1, N-1), \qquad (m_2, n_2) = (M-1, N-1),$$
$$(m_3, n_3) = (M-1, M-1), \qquad (m_4, n_4) = (N-1, M-1). \tag{4.8}$$

From (3.12) and (3.14) combined with Bramble–Hilbert inequality, we derive

$$\|\hat{f} - \hat{f}_\delta^k\|_{L^2(\hat{\Omega}_k)} \le c\, m_k^{-\sigma}\, |\hat{f}|_{H_{\hat{x}}^\sigma(\hat{\Omega}_k)} + n_k^{-\sigma}\, |\hat{f}|_{H_{\hat{y}}^\sigma(\hat{\Omega}_k)}.$$

When going back to Ω_k, this yields the desired estimate. A similar argument holds for the term $\|f - \mathcal{I}_\delta f\|_{L^2(\Omega)}$ since, there also, each $\mathcal{I}_\delta^k$ is the tensorized product of two one-dimensional interpolation operators.

We skip the proof of the next lemma which is much simpler, since Γ_n is made of two segments with equal lengths and is mapped by F onto two edges of the square $\hat{\Omega}_1$.

Lemma 4.4. *Assume that the function g belongs to $H^\tau(\Gamma_n)$, $\tau > \frac{1}{2}$. Then, the following estimate holds*

$$\sup_{w_\delta \in X_\delta} \frac{\langle g, w_\delta \rangle - ((g, w_\delta))_{\delta, \Gamma_n}}{\|w_\delta\|_{H^1(\Omega)}} \le c\, \varepsilon^\tau\, N^{-\tau}\, \|g\|_{H^\tau(\Gamma_n)}. \tag{4.9}$$

Remark. *Local regularity of the functions f and g on the subdomains Ω_k or parts $\Gamma_{n\ell}$ of the boundary Γ_n can also be taken into account in the previous estimates* (4.7) *and* (4.9)*. However this would lead to a rather technical statement and we have no applications for that.*

It remains to estimate the approximation error, i.e. the distance of u to $X_{\delta-}$. The anisotropy of the domains Ω_2 and Ω_4 makes the proof rather technical. We first prove the analogue of estimate (3.10).

Lemma 4.5. *The following estimate holds for all functions v in X such that each $v_{|\Omega_k}$, $1 \le k \le 4$, belongs to $H^s(\Omega_k)$, $s \ge 1$,*

$$\inf_{v_\delta \in X_\delta} \|v - v_\delta\|_{H^1(\Omega)} \le c\, (\varepsilon^{1-s}\, N^{1-s} + M^{1-s}) \sum_{k=1}^{4} \|v\|_{H^s(\Omega_k)}. \tag{4.10}$$

Proof. We recall [3] (Thm 6.3), [4] (Prop. 2.3) that, for each interval $\hat{\Lambda}$ and each positive integer n, there exists an operator π_n^1 from $H^1(\hat{\Lambda})$ onto $\mathbb{P}_n(\hat{\Lambda})$ such that, for any function $\hat{\varphi}$ in $H^t(\hat{\Lambda})$, $t \ge 1$, $\pi_n^1 \hat{\varphi}$ coincides with $\hat{\varphi}$ at the two endpoints of $\hat{\Lambda}$ and satisfies

$$|\hat{\varphi} - \pi_n^1 \hat{\varphi}|_{H^1(\hat{\Lambda})} + n\, \|\hat{\varphi} - \pi_n^1 \hat{\varphi}\|_{L^2(\hat{\Lambda})} \le c\, n^{1-t}\, |\hat{\varphi}|_{H^t(\hat{\Lambda})}. \tag{4.11}$$

We denote by $\pi_n^{1\hat{x}}$ this operator applied in the $\hat{x}$ direction on $\hat{\Lambda}_k^{\hat{x}}$ and $\pi_n^{1\hat{y}}$ this operator applied in the $\hat{y}$ direction on $\hat{\Lambda}_k^{\hat{y}}$. We assume that $s \ge 2$, since the general result can then be derived thanks to an interpolation argument. The idea is now to associate with any function v in X such that each $v_{|\Omega_k}$ belongs to $H^2(\Omega_k)$, the function $\hat{v}_\delta$ defined by

$$\hat{v}_{\delta\,|\hat{\Omega}_1} = \pi_N^{1\hat{x}} \circ \pi_N^{1\hat{y}} \hat{v}_{|\hat{\Omega}_1}, \quad \hat{v}_{\delta\,|\hat{\Omega}_2} = \pi_M^{1\hat{x}} \circ \pi_N^{1\hat{y}} \hat{v}_{|\hat{\Omega}_2},$$

$$\hat{v}_{\delta\,|\hat{\Omega}_3} = \pi_M^{1\hat{x}} \circ \pi_M^{1\hat{y}} \hat{v}_{|\hat{\Omega}_3}, \quad v_{\delta\,|\hat{\Omega}_4} = \pi_N^{1\hat{x}} \circ \pi_M^{1\hat{y}} \hat{v}_{|\hat{\Omega}_4},$$

next the function $v_\delta = \hat{v}_\delta \circ F$. Indeed, on each edge Γ of an $\hat{\Omega}_k$, $\hat{v}_{\delta\,|\Gamma}$ coincides with a $\hat{\pi}_n^\zeta \hat{v}_{|\Gamma}$, with n equal to N or M and ζ equal to $\hat{x}$ or $\hat{y}$, according as Γ is parallel to the x or y axis. So, since v belongs to X, $\hat{v}_\delta$ vanishes on $\partial\hat{\Omega} \setminus \partial\hat{\Omega}_1$ and is continuous on $\hat{\Omega}$. This means that v_δ belongs to X_δ. Next, on $\hat{\Omega}_2$ for instance, we use the triangle inequality

$$\|\partial_{\hat{x}}(\hat{v} - \pi_M^{1\hat{x}} \circ \pi_N^{1\hat{y}} \hat{v})\|_{L^2(\hat{\Omega}_2)} \leq \|\partial_{\hat{x}}(\hat{v} - \pi_M^{1\hat{x}} \hat{v})\|_{L^2(\hat{\Omega}_2)} + \|\partial_x \pi_M^{1\hat{x}}(\hat{v} - \pi_N^{1\hat{y}} \hat{v})\|_{L^2(\hat{\Omega}_2)}.$$

By applying (4.11) both with $t = 1$ and $t = s$, we derive

$$\|\partial_{\hat{x}}(\hat{v} - \pi_M^{1\hat{x}} \circ \pi_N^{1\hat{y}} \hat{v})\|_{L^2(\hat{\Omega}_2)} \leq c\, M^{1-s}\, |\hat{v}|_{H^s_{\hat{x}}(\hat{\Omega}_2)} + N^{1-s}\, |\partial_{\hat{x}} \hat{v}|_{H^{s-1}_{\hat{y}}(\hat{\Omega}_2)}.$$

Using this estimate, its analogue for $\partial_{\hat{y}} \hat{v}$ and their analogues on the other subdomains, we obtain the desired estimate by going back to each subdomain Ω_k.

We now try to improve this estimate, as in (3.20). We observe that, on each interval $]a, b[$, the analogue of the operator A defined in (3.11) writes

$$A\varphi = -((\zeta - a)(b - \zeta)\, \varphi')'. \tag{4.12}$$

This leads to introducing the spaces $V^s(\Omega_k)$ as in (3.15). Also denoting by $A_{\hat{x}}$ and $A_{\hat{y}}$ the analogues of A on $\hat{\Lambda}_k^{\hat{x}}$ and $\hat{\Lambda}_k^{\hat{y}}$, respectively, we introduce the weaker anisotropic semi-norms

$$|\hat{w}|_{V^s_{\hat{x}}(\hat{\Omega}_k)} = \Big(\int_{\hat{\Omega}_k} (A_{\hat{x}}^{\frac{s}{2}} \hat{w})^2 \, d\hat{x}\, d\hat{y}\Big)^{\frac{1}{2}},$$

$$|\hat{w}|_{V^s_{\hat{y}}(\hat{\Omega}_k)} = \Big(\int_{\hat{\Omega}_k} (A_{\hat{y}}^{\frac{s}{2}} \hat{w})^2 \, d\hat{x}\, d\hat{y}\Big)^{\frac{1}{2}}.$$

Similar norms are used on the domain Ω_k, they are denoted by $|\cdot|_{V^s_x(\Omega_k)}$ and $|\cdot|_{V^s_y(\Omega_k)}$. From now on, we assume that

$$M \geq N, \tag{4.13}$$

which seems rather coherent in view of estimates (4.7) and (4.10).

Lemma 4.6. *If assumption* (4.13) *is satisfied, the following estimate holds for all functions* v *in* X *such that each* $v_{|\Omega_k}$ *belongs to* $V^{s+1}(\Omega_k)$ *and* $(\mathbf{grad}\, v)_{|\Omega_k}$ *belongs to* $V^{s-1}(\Omega_k)^2$, $1 \leq k \leq 4$, $s \geq 1$,

$$\inf_{v_\delta \in X_{\delta-}} \|v - v_\delta\|_{H^1(\Omega)}$$

$$\leq c\,(N^{1-s} + M^{1-s}) \sum_{k=1}^{4} (\varepsilon_k^{-1}\, \|v\|_{V^{s+1}(\Omega_k)} + \|\mathbf{grad}\, v\|_{V^{s-1}(\Omega_k)^2}), \qquad (4.14)$$

where ε_k *coincides with* ε *for* $k = 1$, 2 *and* 4 *and is equal to* 1 *for* $k = 3$.

Proof. In a first step, we define a function $\hat{w}_\delta^0$ exactly as in the proof of Lemma 4.3, i.e. such that each $\hat{w}^0_{\delta\,|\Omega_k}$, $1 \leq k \leq 4$, is equal to $\pi^{\hat{x}}_{m_k} \circ \pi^{\hat{y}}_{n_k} \hat{v}_{|\hat{\Omega}_k}$ for the same choice of the pairs (m_k, n_k) as in (4.8). Next, we apply several times the following procedure: starting from a $\hat{w}_\delta^n$, we construct a $\hat{w}_\delta^{n+1}$ by modifying the values of $\hat{w}_\delta^n$ on one $\hat{\Omega}_k$ and using the analogue $\hat{R}^{k,\ell}_{\delta-}$ of the lifting operators introduced in Lemma 3.3 (we do not make precise which of them, for simplicity), which maps the polynomial traces on one edge $\hat{\Gamma}_{k,\ell}$ of $\hat{\Omega}_k$ into polynomials on $\hat{\Omega}_k$. We lift successively:
- the values of $\hat{w}_\delta^n$, $0 \leq n \leq 5$, on the six edges of the $\hat{\Omega}_k$ which are images by F of edges contained in Γ_d, exactly as in Section 3,
- the jump $\hat{w}^6_{\delta\,|\hat{\Omega}_1} - \hat{w}^6_{\delta\,|\hat{\Omega}_2}$ on $\partial\hat{\Omega}_1 \cap \partial\hat{\Omega}_2$ into $\hat{\Omega}_2$,
- the jump $\hat{w}^7_{\delta\,|\hat{\Omega}_4} - \hat{w}^7_{\delta\,|\hat{\Omega}_3}$ on $\partial\hat{\Omega}_4 \cap \partial\hat{\Omega}_3$ into $\hat{\Omega}_3$,
- the jump $\hat{w}^8_{\delta\,|\hat{\Omega}_2} - \hat{w}^8_{\delta\,|\hat{\Omega}_3}$ on $\partial\hat{\Omega}_2 \cap \partial\hat{\Omega}_3$ into $\hat{\Omega}_3$,
- the jump $\hat{w}^9_{\delta\,|\hat{\Omega}_1} - \hat{w}^9_{\delta\,|\hat{\Omega}_4}$ on $\partial\hat{\Omega}_1 \cap \partial\hat{\Omega}_4$ into $\hat{\Omega}_4$

(note that the last four liftings require assumption (4.13)). Finally, we take $\hat{v}_\delta$ equal to $\hat{w}_\delta^{10}$, next v_δ equal to $\hat{v}_\delta \circ F$. It is readily checked that v_δ belongs to $X_{\delta-}$ and moreover it follows from (3.17) and (3.18) that

$$\|\hat{v} - \hat{v}_\delta\|_{H^1(\hat{\Omega})} \leq c \sum_{k=1}^{4} \|\hat{v} - \hat{w}_\delta^0\|_{H^1(\hat{\Omega}_k)}.$$

To estimate the last quantity, relying on the triangle inequality, on $\hat{\Omega}_2$ for instance,

$$\|\partial_{\hat{y}}(\hat{v} - \hat{w}_\delta^0)\|_{L^2(\hat{\Omega}_2)}$$

$$\leq \|\partial_{\hat{y}} v - \pi^{\hat{x}}_{M-1}(\partial_{\hat{y}} v)\|_{L^2(\hat{\Omega}_2)} + \|\pi^{\hat{x}}_{M-1}(\partial_{\hat{y}} v - \pi^{\hat{y}}_{N-1}(\partial_{\hat{y}} v))\|_{L^2(\hat{\Omega}_2)}$$

$$+ \|\pi^{\hat{x}}_{M-1}(\pi^{\hat{y}}_{N-1}(\partial_{\hat{y}} v) - \partial_{\hat{y}}(\pi^{\hat{y}}_{N-1} v))\|_{L^2(\hat{\Omega}_2)},$$

we derive from (3.12) and (3.13) that

$$\|\partial_{\hat y}(\hat v - \hat w_\delta^0)\|_{L^2(\hat\Omega_2)}$$

$$\le c\left(M^{1-s}\,|\partial_{\hat y} v|_{V_{\hat x}^{s-1}(\hat\Omega_2)} + N^{1-s}\,|\partial_{\hat y} v|_{V_{\hat y}^{s-1}(\hat\Omega_2)} + N^{1-s}\,|v|_{V_{\hat y}^{s+1}(\hat\Omega_2)}\right). \quad (4.15)$$

Next, by going back to Ω_2, we have

$$\|\partial_y(v - w_\delta^0)\|_{L^2(\Omega_2)}$$

$$\le c\left(M^{1-s}\,|\partial_y v|_{V_x^{s-1}(\Omega_2)} + N^{1-s}\,|\partial_y v|_{V_y^{s-1}(\Omega_2)} + \varepsilon^{-1}\,N^{1-s}\,|v|_{V_y^{s+1}(\Omega_2)}\right). \quad (4.16)$$

Relying on analogous estimates for $\partial_{\hat x}$ and also on the other domains $\hat\Omega_k$, we obtain the desired result.

Furthermore, estimate (4.14) can be improved by the following arguments. We again start from (4.16). On the interval $\Lambda_\varepsilon =]1-\varepsilon, 1[$, by using the analogue A_ε of A on Λ_ε as defined in (4.12), we observe from (3.14) that, for any positive integer m,

$$D(A_\varepsilon^{\frac{m}{2}}) = \{\varphi \in L^2(\Lambda_\varepsilon);\ (1-\zeta)^{\frac{m}{2}}(\zeta - 1 + \varepsilon)^{\frac{m}{2}}\, d^m\varphi \in L^2(\Lambda_\varepsilon)\}.$$

We define the function δ on Λ_ε as the distance to $\partial\Lambda_\varepsilon$, namely

$$\delta(\zeta) = \inf\,\{1-\zeta, \zeta - 1 + \varepsilon\}.$$

The idea is then to introduce a new scale of spaces $\widetilde V^s(\Lambda_\varepsilon)$: when s is a positive integer m,

$$\widetilde V^m(\Lambda_\varepsilon) = \{\varphi \in L^2(\Lambda_\varepsilon);\ \delta(\zeta)^{\frac{m}{2}}\, d^m\varphi \in L^2(\Lambda_\varepsilon)\}, \quad (4.17)$$

provided with the natural norm and semi-norm, and, otherwise, $\widetilde V^s(\Lambda_\varepsilon)$ is defined by Hilbertian interpolation between $\widetilde V^m(\Lambda_\varepsilon)$ and $L^2(\Lambda_\varepsilon)$, where m is the smallest integer $> s$. Thus, it can be checked by an interpolation argument that, for any $s > 0$, $\widetilde V^s(\Lambda_\varepsilon)$ is imbedded in $D(A_\varepsilon^{\frac{s}{2}})$ and, moreover, that, if N is $\ge s$,

$$\forall \varphi \in \widetilde V^s(\Lambda_\varepsilon), \qquad \inf_{\varphi_N \in \mathbb{P}_{N-1}(\Lambda_\varepsilon)} \|A_\varepsilon^{\frac{s}{2}}(\varphi - \varphi_N)\|_{L^2(\Lambda_\varepsilon)} \le c\,\varepsilon^{\frac{s}{2}}\,|\varphi|_{\widetilde V^s(\Lambda_\varepsilon)} \quad (4.18)$$

(this comes from the fact that, for any ζ in Λ_ε, $1-\zeta$ and $\zeta - 1 + \varepsilon$ are smaller than ε). Next, on each Ω_k, we define the space $\widetilde V^s(\Omega_k)$ by replacing in the definition (3.15) of $V^s(\Omega_k)$, $D(A_\varepsilon^{\frac{s}{2}})$ by $\widetilde V^s(\Lambda_\varepsilon)$ for all intervals Λ_ε involved in the definition of Ω_k (twice for Ω_1, once for Ω_2 and Ω_4).

Then combining (4.16) and its analogues together with (4.18) gives the next result.

Corollary 4.7. *If assumption* (4.13) *is satisfied, the following estimate holds for all functions v in X such that each $v_{|\Omega_k}$ belongs to $\widetilde{V}^{s+1}(\Omega_k)$ and $(\mathbf{grad}\, v)_{|\Omega_k}$ belongs to $\widetilde{V}^{s-1}(\Omega_k)^2$, $1 \le k \le 4$, $s \ge 1$,*

$$\inf_{v_\delta \in X_{\delta-}} \|v - v_\delta\|_{H^1(\Omega)}$$

$$\le c\,(\varepsilon^{\frac{s-1}{2}}\, N^{1-s} + M^{1-s}) \sum_{k=1}^{4} (\|v\|_{\widetilde{V}^{s+1}(\Omega_k)} + \|\mathbf{grad}\, v\|_{\widetilde{V}^{s-1}(\Omega_k)^2}). \quad (4.19)$$

Note to conclude that replacing $V^s(\Omega_k)$ by $\widetilde{V}^s(\Omega_k)$ does not modify, in a neighbourhood of $\mathbf{b}_1$ or $\mathbf{b}_2$, the part of the weight which is the distance to the edge containing $\mathbf{b}_1$ or $\mathbf{b}_2$, so that the following analogue of Lemma 3.4 holds.

Lemma 4.8. *The functions Σ_j, $j = 1$ and 2, introduced in* (2.9)*, belong to $\widetilde{V}^{s+1}(\Omega_k)$ and their gradients belong to $\widetilde{V}^{s-1}(\Omega_k)^2$, $1 \le k \le 4$, for all $s < 2$.*

We are now in a position to evaluate the error estimate for problem (4.1).

Theorem 4.9. *Assume that the function f belongs to $H^\sigma(\Omega)$, $\sigma > 1$, and that the function g belongs to $H^\tau(\Gamma_n)$, $\tau > \frac{1}{2}$. If assumption* (4.13) *is satisfied, the following error estimate holds between the solution u of problem* (2.2) *and the solution u_δ of problem* (4.1)*:*

$$\|u - u_\delta\|_{H^1(\Omega)}$$

$$\le c\, \sup\{\varepsilon^\sigma\, N^{-\sigma} + M^{-\sigma},\, \varepsilon^\tau\, N^{-\tau},\, (\varepsilon^{\frac{1}{2}}\, N^{-1} + M^{-1})\,(\log(MN\,\varepsilon^{-\frac{1}{2}}))^{\frac{1}{2}}\}$$

$$(\|f\|_{H^\sigma(\Omega)} + \|g\|_{H^\tau(\Gamma_n)}). \quad (4.20)$$

Proof. Here also, it remains to evaluate the approximation error. We use the expansion (2.10) where now the function u_r belongs to $H^2(\Omega)$ and choose the function v_δ of the form $v_{r\delta} + \sum_{j=1}^{2} \mu_j\, \Sigma_{j\delta}$. Indeed, it follows from Corollary 4.7 and Lemma 4.8, together with the fact that the norm of the Σ_j, resp. of $\mathbf{grad}\, \Sigma_j$, in $\widetilde{V}^{3-\eta}(\Omega)$, resp. in $\widetilde{V}^{1-\eta}(\Omega)^2$, behaves like $\eta^{-\frac{1}{2}}$, that

$$\inf_{\Sigma_{j\delta} \in X_{\delta-}} \|\Sigma_j - \Sigma_{j\delta}\|_{H^1(\Omega)} \le c\,\eta^{-\frac{1}{2}}\,(\varepsilon^{\frac{1-\eta}{2}}\, N^{\eta-1} + M^{\eta-1}).$$

So, taking η equal to $(\log(MN\,\varepsilon^{-\frac{1}{2}}))^{-1}$ gives

$$\inf_{\Sigma_{j\delta}\in X_{\delta-}} \|\Sigma_j - \Sigma_{j\delta}\|_{H^1(\Omega)} \leq c\,(\varepsilon^{\frac{1}{2}}\,N^{-1} + M^{-1})\,(\log(MN\,\varepsilon^{-\frac{1}{2}}))^{\frac{1}{2}}. \quad (4.21)$$

Similarly, it follows from Lemma 4.5 that

$$\inf_{u_{r\delta}\in X_{\delta-}} \|u_r - u_{r\delta}\|_{H^1(\Omega)} \leq c\,(\varepsilon\,N^{-1} + M^{-1})\,\|u_r\|_{H^2(\Omega)}.$$

Combining the last two estimates leads to the desired result.

Remark. *It follows from Lemmas* 2.2 *and* 2.3 *that the solution u of problem* (2.2) *is more regular in Ω_3 than in the other subdomains, so that using polynomials of much lower degree on Ω_3 will not diminish the convergence order. However, this would lead to different grids on the edges $\partial\Omega_2 \cap \partial\Omega_3$ and $\partial\Omega_3 \cap \partial\Omega_4$, so that we would rather avoid it.*

We conclude with the improved error estimate in $L^2(\Omega)$ which is derived via the Aubin–Nitsche duality argument. We only give an abridged proof, since it is very similar to that of Corollary 3.6.

Corollary 4.10. *If the assumptions of Theorem* 4.9 *are satisfied, the following error estimate holds between the solution u of problem* (2.2) *and the solution u_δ of problem* (4.1)*:*

$$\begin{aligned}&\|u - u_\delta\|_{L^2(\Omega)} \\ &\quad\leq c\,\sup\{\varepsilon^\sigma\,N^{-\sigma} + M^{-\sigma},\,\varepsilon^\tau\,N^{-\tau},\,(\varepsilon\,N^{-2} + M^{-2})\,\log(MN\,\varepsilon^{-\frac{1}{2}})\} \\ &\qquad\qquad (\|f\|_{H^\sigma(\Omega)} + \|g\|_{H^\tau(\Gamma_n)}).\end{aligned} \quad (4.22)$$

Proof. We start from the formula

$$\|u - u_\delta\|_{L^2(\Omega)} = \sup_{h\in L^2(\Omega)} \frac{\int_\Omega (u - u_\delta)(\mathbf{x})h(\mathbf{x})\,d\mathbf{x}}{\|h\|_{L^2(\Omega)}}. \quad (4.23)$$

For any h in $L^2(\Omega)$, we solve problem (3.27) and we observe from Lemma 2.6 that its solution w admits the expansion

$$w = w_r + \sum_{j=1}^{2} \nu_j\,\Sigma_j, \quad \text{with} \quad \|w_r\|_{H^2(\Omega)} + \sum_{j=1}^{2} |\nu_j| \leq c\,\|h\|_{L^2(\Omega)}. \quad (4.24)$$

We have the formula, for all w_δ in $X_{\delta-}$,

$$\int_\Omega (u - u_\delta)(\mathbf{x})h(\mathbf{x})\,d\mathbf{x} = a(u - u_\delta, w - w_\delta) + \int_\Omega f(\mathbf{x})w_\delta(\mathbf{x})\,d\mathbf{x} - ((f, w_\delta))_\delta$$

$$+\langle g, w_\delta\rangle - ((g, w_\delta))_{\delta,\Gamma_n}.$$

Since the distance of w_r to $X_{\delta-}$ can be estimated from Lemma 4.5, using (4.21) yields the desired result.

In view of estimates (4.20) and (4.22), the final idea is to take M and N such that

$$\kappa\, \varepsilon^{-\frac{1}{2}}\, N \leq M \leq \kappa'\, \varepsilon^{-\frac{1}{2}}\, N, \tag{4.25}$$

for some positive constants κ and κ' (this of course implies condition (4.13)). With this choice, for smooth data f and g, and up to $\log M$, the errors in $H^1(\Omega)$ and in $L^2(\Omega)$ behave respectively like $c\,M^{-1}$ and $c\,M^{-2}$. So the convergence order which is optimal (and better than for discretizations without domain decomposition) is rather low but this seems unavoidable due to the weak smoothness of the singular functions in this situation.

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