SOME MATHEMATICAL PROBLEMS RELATED TO THE FIRST APPROXIMATION OF I.VEKUA'S THEORY FOR CUSPED PRISMATIC SHELLS

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Abstract

The bending of a prismatic cusped shell described by the first approximation of I.Vekua's version of the theory of elastic prismatic shells is considered. Mathematically it leads to a Dirichlet type boundary value problem for a strongly elliptic system of differential equations with order degeneration on the boundary. The existence and uniqueness of generalized solutions of the corresponding boundary value problems in the weighted Sobolev spaces are proved.

Key words and phrases: Elliptic systems with order degeneration, weighted Sobolev space, bending of prismatic cusped shells.

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1. *Introduction*

In the fifties of the last century I.Vekua [15–17] suggested a new mathematical method of elastic prismatic shells (i.e., plates of variable thickness) which was based on the expansion of fields of the displacement vector, strain and stress tensors of the three–dimensional theory of linear elasticity into orthogonal Fourier–Legendre series with respect to the variable of plate thickness. By truncating of the corresponding series and preserving only the first $N+1$ terms of the expansions he obtained the so called N-th *approximation* $(N = 0, 1, 2, \cdots)$. Each of these approximations consists of $3N+3$ equations and can be considered as independent mathematical model of plates. The first approximation (for $N = 1$) actually coincides with the classical plate bending theory.

In 1955 I.Vekua [15,16] raised the problem of investigation of cusped shells, i.e., such ones whose thickness vanishes on some part of the boundary of the shell. In the static case the problem mathematically leads to the question of setting and solving of boundary value problems for even order equations and systems of elliptic type with order degeneration.

In the case of the first approximation of I.Vekua's version of the elastic shell theory (i.e., for $N=1$) the general system consists of six second order differential equations. This system is split into two autonomous systems and they are read as follows ([16], p.75):

$$\left.\begin{aligned}
&(\lambda+2\mu)\frac{\partial}{\partial x}\left(h\frac{\partial u_1}{\partial x}\right)+\mu\frac{\partial}{\partial y}\left(h\frac{\partial u_1}{\partial y}\right)+\lambda\frac{\partial}{\partial x}\left(h\frac{\partial u_2}{\partial y}\right)+\mu\frac{\partial}{\partial y}\left(h\frac{\partial u_2}{\partial x}\right)\\
&\qquad\qquad+3\lambda\frac{\partial}{\partial x}(hv_3)+X_1=0,\\
&\mu\frac{\partial}{\partial x}\left(h\frac{\partial u_2}{\partial x}\right)+(\lambda+2\mu)\frac{\partial}{\partial y}\left(h\frac{\partial u_2}{\partial y}\right)+\mu\frac{\partial}{\partial x}\left(h\frac{\partial u_1}{\partial y}\right)+\lambda\frac{\partial}{\partial y}\left(h\frac{\partial u_1}{\partial x}\right)\\
&\qquad\qquad+3\lambda\frac{\partial}{\partial y}(hv_3)+X_2=0,\\
&\mu\frac{\partial}{\partial x}\left(h^3\frac{\partial v_3}{\partial x}\right)+\mu\frac{\partial}{\partial y}\left(h^3\frac{\partial v_3}{\partial y}\right)-\lambda h\frac{\partial u_1}{\partial x}-\lambda h\frac{\partial u_2}{\partial y}\\
&\qquad\qquad-3(\lambda+2\mu)hv_3+X_3=0,
\end{aligned}\right\}\tag{1.1}$$

and

$$\left.\begin{aligned}
&(\lambda+2\mu)\frac{\partial}{\partial x}\left(h^3\frac{\partial v_1}{\partial x}\right)+\mu\frac{\partial}{\partial y}\left(h^3\frac{\partial v_1}{\partial y}\right)+\lambda\frac{\partial}{\partial x}\left(h^3\frac{\partial v_2}{\partial y}\right)+\mu\frac{\partial}{\partial y}\left(h^3\frac{\partial v_2}{\partial x}\right)\\
&\qquad\qquad-\mu h\frac{\partial v_3}{\partial x}-3\mu hv_1+Y_1=0,\\
&\mu\frac{\partial}{\partial x}\left(h^3\frac{\partial v_2}{\partial x}\right)+(\lambda+2\mu)\frac{\partial}{\partial y}\left(h^3\frac{\partial v_2}{\partial y}\right)+\mu\frac{\partial}{\partial x}\left(h^3\frac{\partial v_1}{\partial y}\right)+\lambda\frac{\partial}{\partial y}\left(h^3\frac{\partial v_1}{\partial x}\right)\\
&\qquad\qquad-\mu h\frac{\partial v_3}{\partial y}-3\mu hv_2+Y_2=0,\\
&\mu\frac{\partial}{\partial x}\left(h\frac{\partial u_3}{\partial x}\right)+\mu\frac{\partial}{\partial y}\left(h\frac{\partial u_3}{\partial y}\right)+3\mu h\frac{\partial}{\partial x}(hv_1)+3\mu h\frac{\partial}{\partial y}(hv_2)+Y_3=0,
\end{aligned}\right\}$$

where u_1,u_2,u_3, and v_1,v_2,v_3 are unknown functions and are called *the moments of the displacement vector*, λ and $\mu>0$ are Lamé's constants, X_1,X_2,X_3 and Y_1,Y_2,Y_3 are the Fourier–Legendre moments of a given volume force, h is the thickness of the plate. The system is considered on the so called *middle surface* ω of the shell, which is actually the orthogonal projection of the shell on the plane Oxy.

In this paper we study the case when the middle surface ω is a plane bounded domain with a smooth boundary $\partial\omega \in C^1$, where

$$\partial\omega = \Gamma_0 \cup \Gamma_1$$

$$\Gamma_0 = \{(x,0): \quad a \leq x \leq b, \quad a,b \in I\!R, \quad a < b\},$$

$$\Gamma_1 = \{(x,y): \quad (x,y) \in \partial\omega, \quad y \geq 0\}.$$

$$\Gamma_0 \cap \Gamma_1 = \{(a,0);\ (b,0)\}.$$

Let the thickness h be given by the function

$$h = h(x,y) = y^m, m > 0.$$

In what follows we consider only system (1.1) and investigate the Dirichlet type boundary value problem. Clearly, this system is strongly elliptic on $\overline{\omega} \setminus \Gamma_0$ with order degeneration on Γ_0.

Note that by different approaches the similar problems for the zero ($N = 0$) approximation are considered in the references [8-10] and [2,4]. The systems corresponding to $N = 0$ and $N = 1$ approximations are essentially different by the structure and therefore the methods developed in the above cited papers do not apply to the system (1.1).

2. *Auxiliary material.*

Let $D(\omega)$ be a set of infinitely differentiable compactly supported functions on ω. We define a scalar product and a norm on $D(\omega)$ according to the formulas:

$$(u,v)_m \equiv \int_\omega y^m[\nabla u \nabla v + uv]d\tau, \tag{2.1}$$

$$\|u\|_m \equiv \left(\int_\omega y^m[\nabla u \nabla u + u^2]d\tau\right)^{\frac{1}{2}}, \tag{2.2}$$

where $u, v \in D(\omega)$.

We complete $D(\omega)$ by the norm (2.2) to obtain the Hilbert space $\overset{0}{H}_m(\omega)$, where a scalar product and a norm are defined by formulas (2.1) and (2.2).

Lemma 1. *For every $u \in \overset{0}{H}_m(\omega)$ there hold the estimates:*

$$\int_\omega y^{m-2}u^2 d\tau \leq c\int_\omega y^m\left(\frac{\partial u}{\partial y}\right)^2 d\tau \quad \text{for} \quad m \neq 1, \tag{2.3}$$

$$\int\limits_{\omega} y^{-1} \mid \ln ky \mid^{-2-\varepsilon} u^2 d\tau \leq c \int\limits_{\omega} y \left(\frac{\partial u}{\partial y}\right)^2 d\tau \quad \text{for} \quad m = 1, \tag{2.4}$$

where $k > 0$ is some positive constant such that $ky < 1$, for all $(x, y) \in \overline{\omega}$, and $c > 0$ is a constant independent of u.

Proof. Since ω is a bounded domain, there exists $d = const > 0$ such that $\omega \subset [a; b] \times [0, d] \equiv E$. Let $u \in D(\omega)$. Clearly, by extension (preserving the notation) we can assume that $u \in D(E)$. First we consider the case $m \neq 1$:

$$\int\limits_0^d y^{m-2} u^2 dy = \int\limits_0^d \tfrac{1}{m-1} u^2 dy^{m-1} = \tfrac{1}{m-1} y^{m-1} u^2 \mid_0^d$$

$$-\tfrac{1}{m-1} \int\limits_0^d y^{m-1} 2u \tfrac{\partial u}{\partial y} dy = -\tfrac{2}{m-1} \int\limits_0^d y^{m-1} u \tfrac{\partial u}{\partial y} dy$$

$$= -\tfrac{2}{m-1} \int\limits_0^d \tfrac{1}{3} y^{\frac{m-2}{2}} u 3 y^{\frac{m}{2}} \tfrac{\partial u}{\partial y} dy \leq (2ab \leq a^2 + b^2)$$

$$\leq \tfrac{1}{9} \int\limits_0^d y^{m-2} u^2 dy + \tfrac{9}{(m-1)^2} \int\limits_0^d y^m \left(\tfrac{\partial u}{\partial y}\right)^2 dy.$$

Hence,

$$\frac{8}{9} \int\limits_0^d y^{m-2} u^2 dy \leq \frac{9}{(m-1)^2} \int\limits_0^d y^m \left(\frac{\partial u}{\partial y}\right)^2 dy,$$

i.e.,

$$\int\limits_0^d y^{m-2} u^2 dy \leq \frac{81}{8(m-1)^2} \int\limits_0^d y^m \left(\frac{\partial u}{\partial y}\right)^2 dy.$$

If we integrate the last inequality by x over the interval $[a, b]$, we obtain

$$\int\limits_a^b \int\limits_0^d y^{m-2} u^2 dy dx \leq \frac{81}{8(m-1)^2} \int\limits_a^b \int\limits_0^d y^m \left(\frac{\partial u}{\partial y}\right)^2 dy dx.$$

Since $u(x, y) = 0$ for $(x, y) \in E \backslash \omega$ we have

$$\int\limits_{\omega} y^{m-2} u^2 d\tau \leq c \int\limits_{\omega} y^m \left(\frac{\partial u}{\partial y}\right)^2 d\tau. \tag{2.5}$$

Now, let $u \in \overset{0}{H}_m(\omega)$ and $\{u_n\}_{n=1}^{\infty}$ with $u_n \in D(\omega)$, be a sequence such that

$$\lim_{n\to\infty} \|u - u_n\|_m = 0.$$

According to (2.5) the sequence $\left\{y^{\frac{m-2}{2}}u_n\right\}_{n=1}^{\infty}\subset D(\omega)$ converges in $L_2(\omega)$. Due to the completeness of $L_2(\omega)$ there exists a function $v\in L_2(\omega)$ such that

$$\lim_{n\to\infty}\int\limits_{\omega}\left(y^{\frac{m-2}{2}}u_n-v\right)^2d\tau=\lim_{n\to\infty}\int\limits_{\omega}y^{m-2}\left(u_n-y^{\frac{2-m}{2}}v\right)^2d\tau=0.$$

According to (2.2) it follows that

$$u=y^{\frac{2-m}{2}}v\quad\text{on}\quad\omega,$$

i.e.,

$$\lim_{n\to\infty}\int\limits_{\omega}y^{m-2}(u_n-u)^2d\tau=0.$$

If we replace u by u_n in (2.5) and pass to the limit as $n\to\infty$ we obtain that the inequality (2.5) is true for every function $u\in\overset{0}{H}_m(\omega)$, $m\neq 1$.

Now we consider the case $m=1$. Let $u\in D(\omega)$. We have

$$u^2(x,y)=\left(\int\limits_y^d\frac{\partial u}{\partial t}(x,t)dt\right)^2=\left(\int\limits_y^d t^{-\frac{1}{2}}t^{\frac{1}{2}}\frac{\partial u}{\partial t}dt\right)^2$$

$$\leq\int\limits_y^d t^{-1}dt\int\limits_y^d t\left(\frac{\partial u}{\partial t}\right)^2dt=(\ln d-\ln y)\int\limits_0^d t\left(\frac{\partial u}{\partial t}\right)^2dt$$

$$\leq n\mid\ln y\mid\int\limits_0^d t\left(\frac{\partial u}{\partial t}\right)^2dt.$$

If we multiply both sides of the last inequality by $y^{-1}\mid\ln ky\mid^{-2-\varepsilon}$ with k as in the lemma and $\varepsilon>0$, and integrate it first by y and then by x, we obtain

$$\int\limits_0^d y^{-1}\mid\ln ky\mid^{-2-\varepsilon}u^2(x,y)dy\leq n_1\int\limits_0^d y^{-1}\mid\ln ky\mid^{-1-\varepsilon}dy\int\limits_0^d t\left(\frac{\partial u}{\partial t}\right)^2dt$$

$$\leq n_2\int\limits_0^d y\left(\frac{\partial u}{\partial y}\right)^2dy.$$

Since

$$\int\limits_0^d y^{-1}\mid\ln ky\mid^{-1-\varepsilon}dy<\infty,$$

$$\int_a^b dx \int_0^d y^{-1} \mid \ln ky \mid^{-2-\varepsilon} u^2(x,y)dy = \int_\omega y^{-1} \mid \ln ky \mid^{-2-\varepsilon} u^2(x,y)d\tau$$

$$\leq c \int_\omega y \left(\frac{\partial u}{\partial y}\right)^2 d\tau.$$

If we repeat the above reasonings we obtain that the inequality (2.4) is true for every function $u \in \overset{0}{H}_1(\omega)$.

Lemma 1 represents the two–dimensional version of Hardy's inequality (see, e.g., [1,7]). ∎

Now we consider the question of a trace of functions from the space $\overset{0}{H}_m(\omega)$.

Lemma 2. *The trace of a function* $u \in \overset{0}{H}_m(\omega)$, $0 < m < 1$, *on* $\partial\omega$ *is zero.*

Proof. We introduce the distance function on ω

$$\rho(x,y) = \text{dist}\{(x,y),\ \partial\omega\}.$$

By completing $D(\omega)$ with the norm

$$\|u\|_{\rho^m} = \left(\int_\omega \rho^m(x,y)[\nabla u \nabla u + u^2]d\tau\right)^{\frac{1}{2}} \tag{2.6}$$

we obtain the Hilbert space $\overset{0}{W}{}^1_{2,\frac{m}{2}}(\omega)$ whose properties are well known (see [11,12,14]). In particular, the trace of every function $u \in \overset{0}{W}{}^1_{2,\frac{m}{2}}(\omega)$, $-1 < m < 1$, on $\partial\omega$ is zero (see [11], p.393).

It is easy to show that

$$\|u\|_m \geq \|u\|_{\rho^m}$$

for every $u \in D(\omega)$. Therefore,

$$\overset{0}{H}_m(\omega) \subset \overset{0}{W}{}^1_{2,\frac{m}{2}}(\omega),$$

and $u \mid_{\partial\omega} = 0$ for all $u \in \overset{0}{H}_m(\omega)$. ∎

Consider the case $m \geq 1$. The following assertion describes the behaviour of functions from the space $\overset{0}{H}_m(\omega)$ near the boundary Γ_0.

Lemma 3. *Let φ be a continuous function with piecewise continuous first order partial derivatives on ω which are bounded for $y > \varepsilon\ \forall\varepsilon > 0$. Moreover, let $\varphi\,|_{\Gamma_1} = 0$, $\|\varphi\|_m < \infty$ and*

$$|\,\varphi\,| \le c y^{\frac{1-m}{2}} \text{ for } m > 1, \qquad |\,\varphi\,| \le c\,|\,\ln(ky)\,|^{\frac{1}{2}} \text{ for } m = 1, \tag{2.7}$$

with k as in Lemma 1 and a positive constant c.

Then φ belongs to the space $\overset{0}{H}_m(\omega)$.

Proof. The proof follows the approach of Vishik [8]. Let $m \ge 1$. We introduce the function

$$\psi_\delta(y) = \begin{cases} 0, \ \ 0 < y \le \delta, \\ (\ln\,|\,\ln\delta\,|)^\varepsilon - (\ln\,|\,\ln y\,|)^\varepsilon, \ \ \delta \le y \le \delta_1, \\ 1, \ \ y \ge \delta_1, \end{cases} \tag{2.8}$$

where δ_1 is a constant such that

$$(\ln\,|\,\ln\delta\,|)^\varepsilon - (\ln\,|\,\ln\delta_1\,|)^\varepsilon = 1, \ \ 0 < \varepsilon < \frac{1}{2}. \tag{2.9}$$

Clearly, δ_1 by δ

$$\delta_1 = exp\left[-exp\left((\ln\,|\,\ln\delta\,|)^\varepsilon - 1\right)^{\frac{1}{\varepsilon}}\right]. \tag{2.10}$$

From (2.10) it follows that

$$\lim_{\delta\to 0}\delta_1 = 0. \tag{2.11}$$

Consider the following function

$$\varphi_\delta(x,y) := \varphi(x,y)\cdot\psi_\delta(y).$$

Evidently $\varphi_\delta \in \overset{0}{W}{}^1_2(\omega)$ ($\overset{0}{W}{}^1_2(\omega)$ is the usual Sobolev space), since φ_δ has square integrable generalized partial derivatives of the first order. On the other hand $\overset{0}{W}{}^1_2(\omega) \subset \overset{0}{H}_m(\omega)$ and therefore $\varphi_\delta \in \overset{0}{H}_m(\omega)$.

To complete the proof it is sufficient to show that

$$\lim_{\delta\to 0}\|\varphi - \varphi_\delta\|_m = 0. \tag{2.12}$$

To this end we calculate $\psi'_\delta(y)$ on $]\delta, \ \ \delta_1[$:

$$\frac{d\psi_\delta}{dy} = -\varepsilon(\ln\,|\,\ln y\,|)^{\varepsilon-1}\,|\,\ln y\,|^{-1}\,(-y)^{-1} = \varepsilon y^{-1}\,|\,\ln y\,|^{-1}\,(\ln\,|\,\ln y\,|)^{\varepsilon-1}.$$

Hence

$$\frac{d\psi_\delta}{dy} = \begin{cases} 0 \quad \text{for} \quad 0 \leq y \leq \delta, \\ \\ \varepsilon y^{-1} \mid \ln y \mid^{-1} (\ln \mid \ln y \mid)^{\varepsilon - 1} \quad \text{for} \quad \delta < y < \delta_1, \\ \\ 0 \quad \text{for} \quad y \geq \delta_1. \end{cases} \tag{2.13}$$

Further we derive

$$\int_\omega y^m \left[\left(\frac{\partial}{\partial x}(\varphi - \varphi_\delta) \right)^2 + \left(\frac{\partial}{\partial y}(\varphi - \varphi_\delta) \right)^2 \right] d\tau$$

$$= \int_\omega y^m \left[\left(\frac{\partial}{\partial x}((1-\psi_\delta)\varphi) \right)^2 + \left(\frac{\partial}{\partial y}((1-\psi_\delta)\varphi) \right)^2 \right] d\tau$$

$$= \int_\omega y^m \left[\left((1-\psi_\delta)\frac{\partial \varphi}{\partial x} \right)^2 + \left((1-\psi_\delta)\frac{\partial \varphi}{\partial y} + \varphi \frac{\partial}{\partial y}(1-\psi_\delta) \right)^2 \right] d\tau$$

$$= \int_\omega y^m \left[(1-\psi_\delta)^2 \left(\frac{\partial \varphi}{\partial x} \right)^2 + \left((1-\psi_\delta)\frac{\partial \varphi}{\partial y} - \varphi \frac{\partial \psi_\delta}{\partial y} \right)^2 \right] d\tau$$

$$\leq \int_\omega y^m \left[(1-\psi_\delta)^2 \left(\frac{\partial \varphi}{\partial x} \right)^2 + 2(1-\psi_\delta)^2 \left(\frac{\partial \varphi}{\partial y} \right)^2 + 2\varphi^2 \left(\frac{\partial \psi_\delta}{\partial y} \right)^2 \right] d\tau$$

$$\leq 2(I_1^\delta + I_2^\delta),$$

where

$$I_1^\delta := \int_\omega y^m (1-\psi_\delta)^2 \left[\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 \right] d\tau,$$

$$I_2^\delta := \int_\omega y^m \varphi^2 \left(\frac{\partial \psi_\delta}{\partial y} \right)^2 d\tau.$$

Let

$$\omega_\delta := \{(x,y) : \ (x,y) \in \omega, \ \ y \leq \delta\},$$

$$\omega^\delta := \{(x,y) : \ (x,y) \in \omega, \ \ y \geq \delta\}.$$

Let us estimate I_1^δ:

$$I_1^\delta = \int_\omega y^m (1-\psi_\delta)^2 \left[\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 \right] d\tau$$

$$= \int_{\omega_{\delta_1}} y^m (1-\psi_\delta)^2 \left[\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 \right] d\tau.$$

Since $(1-\psi_\delta)^2 \leq 1$ we get

$$I_1^\delta \leq \int\limits_{\omega_{\delta_1}} y^m \left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2\right] d\tau.$$

From the condition of Lemma 3

$$\|\varphi\|_m < \infty$$

and the equality

$$\lim_{\delta\to 0} mes\omega_{\delta_1} = 0$$

which follows from (2.11), we have

$$\lim_{\delta\to 0} I_1^\delta = 0. \qquad (2.14)$$

Now we estimate I_2^δ. Let $m > 1$. If we use the condition of Lemma 3

$$|\ \varphi\ | < ky^{\frac{1-m}{2}}$$

with the help of (2.13) and (2.9), we get

$$I_2^\delta \leq \int\limits_a^b \int\limits_0^d y^m k^2 y^{1-m} \left(\frac{d\psi_\delta}{dy}\right)^2 dydx$$

$$= k^2 \int\limits_a^b \int\limits_\delta^{\delta_1} y\varepsilon^2 y^{-2}\ |\ \ln y\ |^{-2} (\ln\ |\ \ln y\ |)^{2\varepsilon-2} dydx$$
$$= k^2\varepsilon^2(b-a) \int\limits_\delta^{\delta_1} y^{-1} |\ln y|^{-2} (\ln |\ln y|)^{2\varepsilon-2} dy.$$

Since

$$\int\limits_0^d y^{-1} |\ln y|^{-2}\, dy < \infty,$$

we get

$$\lim_{\delta\to 0} I_2^\delta = 0. \qquad (2.15)$$

Taking into account (2.7) and (2.13), for $m = 1$ we derive

$$I_2^\delta \leq \int\limits_a^b \int\limits_0^d yk^2\ |\ \ln y\ |\ \left(\frac{d\psi_\delta}{dy}\right)^2 dydx$$
$$= k^2 \int\limits_a^b \int\limits_\delta^{\delta_1} y\ |\ \ln y\ |\ \varepsilon^2 y^{-2}\ |\ \ln y\ |^{-2} (\ln\ |\ \ln y\ |)^{2\varepsilon-2} dydx$$
$$= k^2\varepsilon^2(b-a) \int\limits_\delta^{\delta_1} y^{-1} |\ln y|^{-1} (\ln |\ln y|)^{2\varepsilon-2} dy.$$

Due to the inequality

$$\int\limits_0^{\delta_1} y^{-1} |\ln y|^{-1} (\ln|\ln y|)^{2\varepsilon-2} dy < \infty$$

for every $0 < \varepsilon < \frac{1}{2}$. Hence (2.13) follows.

Thus, according to (2.14) and (2.15), we have (2.12). ∎

It is easy to see that $\forall u \in \overset{0}{H}_m(\omega)$ the trace $u\,|_{\Gamma_1}= 0$.

Now, with the help of Lemma 3, we can construct functions belonging to $\overset{0}{H}_m(\omega)$ for $m \geq 1$, which have not traces on Γ_0. To this end let us introduce the function $\psi(x,y) \in C^\infty(\overline{\omega})$,

$$\psi(x,y) \equiv 0 \;\text{ for }\; (x,y) \in \{(x,y):\; (x,y)\in\omega,\; dist[(x,y),\Gamma_1] < \delta\},$$

$$\psi(x,y) \equiv 1 \;\text{ for }\; (x,y) \in \{(x,y):\; (x,y)\in\omega,\; dist[(x,y),\Gamma_1] > 2\delta\},$$

$([16], p.89)$. Then the function

$$\varphi(x,y) := \begin{cases} \psi(x,y) y^{\frac{1-m+\varepsilon}{2}}, & m > 1, \;\; 0 < \varepsilon < m-1, \\ \psi(x,y)\mid \ln y \mid^{\frac{1-\varepsilon}{2}}, & m = 1, \;\; 0 < \varepsilon < 1, \end{cases}$$

belongs to $\overset{0}{H}_m(\omega)$ and has not a trace on Γ_0

In what follows we derive a Korn's type weighted inequality in a special functional space which will be employed later on. To this end let us define the vector space

$$\overset{\overset{\rightarrow}{0}}{H}_{m_1,m_2}(\omega) = \overset{0}{H}_{m_1}(\omega) \times \overset{0}{H}_{m_2}(\omega)$$

with the norm:

$$\left\|\vec{u}\right\|^2_{m_1,m_2} = \|u_1\|^2_{m_1} + \|u_1\|^2_{m_2} \;\;\text{for}\;\;\; \vec{u}= (u_1,u_2) \in \overset{\overset{\rightarrow}{0}}{H}_{m_1,m_2}(\omega).$$

Clearly, $\overset{\overset{\rightarrow}{0}}{H}_{m_1,m_2}(\omega)$ is a Hilbert space.

Lemma 4. (Korn's weighted inequality). *Let* $\vec{u}= (u_1,u_2) \in\overset{\overset{\rightarrow}{0}}{H}_{m,m}(\omega)$, $m \neq 1$. *Then the inequality holds true*

$$\int\limits_\omega y^m [\nabla u_1 \nabla u_1 + \nabla u_2 \nabla u_2] d\tau \leq$$

$$\leq c_1 \int_\omega y^m \left[\left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 + \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right)^2 \right] d\tau,$$

where c_1 is a positive constant independent of $\vec{u}$.

Proof. First we prove the lemma for a function $\vec{u} = (u_1, u_2) \in [D(\omega)]^2$.

We have

$$\int_\omega y^m \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right)^2 d\tau = \int_\omega y^m \left[\left(\frac{\partial u_1}{\partial y} \right)^2 + \left(\frac{\partial u_2}{\partial x} \right)^2 + 2 \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial x} \right] d\tau.$$

Let us estimate the last summand. With the help of Green's formula we have

$$\left| 2 \int_\omega y^m \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial x} d\tau \right| = \left| 2 \int_\omega y^m \frac{\partial^2 u_1}{\partial x \partial y} u_2 d\tau \right|$$

$$= \left| 2 \int_\omega y^m \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial y} d\tau + 2m \int_\omega y^{m-1} \frac{\partial u_1}{\partial x} u_2 d\tau \right|$$

$$\leq 2 \left| \int_\omega y^m \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial y} d\tau \right| + 2m \left| \int_\omega y^{m-1} \frac{\partial u_1}{\partial x} u_2 d\tau \right|.$$

We proceed as follows

$$\left| 2 \int_\omega y^m \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial y} d\tau \right| \leq \left| 2 \int_\omega y^{\frac{m}{2}} \frac{\partial u_1}{\partial x} y^{\frac{m}{2}} \frac{\partial u_2}{\partial y} d\tau \right|$$

$$\leq \int_\omega \left[y^m \left(\frac{\partial u_1}{\partial x} \right)^2 + y^m \left(\frac{\partial u_2}{\partial y} \right)^2 \right] d\tau$$

$$= \int_\omega y^m \left[\left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 \right] d\tau. \qquad (2.16)$$

Using the inequality (2.3) we have

$$\left| 2m \int_\omega y^{m-1} \frac{\partial u_1}{\partial x} u_2 d\tau \right| \leq \left| 2m \int_\omega y^{\frac{m}{2}} \frac{\partial u_1}{\partial x} y^{\frac{m-2}{2}} u_2 d\tau \right|$$

$$\leq m \int_\omega y^m \left(\frac{\partial u_1}{\partial x} \right)^2 d\tau + m \int_\omega y^{m-2} u_2^2 d\tau$$

$$\leq m \int_\omega y^m \left(\frac{\partial u_1}{\partial x} \right)^2 d\tau + m\, c \int_\omega y^m \left(\frac{\partial u_2}{\partial y} \right)^2 d\tau, \qquad (2.17)$$

where c is the constant involved in (2.3).

From (2.16) and (2.17) we get

$$\left|2\int\limits_\omega y^m \frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}d\tau\right| \leq \int\limits_\omega y^m\left[(1+m)\left(\frac{\partial u_1}{\partial x}\right)^2+(1+mc)\left(\frac{\partial u_2}{\partial y}\right)^2\right]d\tau$$

$$\leq \alpha\int\limits_\omega y^m\left[\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial y}\right)^2\right]d\tau, \qquad (2.18)$$

where $\alpha=\max(1+m,\ \ 1+mc)$.

Further for $0<\delta<1$

$$\int\limits_\omega y^m\left[\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial y}\right)^2+\left(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\right)^2\right]d\tau$$

$$=\int\limits_\omega y^m\left[\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial y}\right)^2+(1-\delta)\left(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\right)^2+\delta\left(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\right)^2\right]d\tau$$

$$\geq\int\limits_\omega y^m\left[\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial y}\right)^2+\delta\left(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\right)^2\right]d\tau$$

$$=\int\limits_\omega y^m\left[\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial y}\right)^2+\delta\left(\frac{\partial u_1}{\partial y}\right)^2+\delta\left(\frac{\partial u_2}{\partial x}\right)^2+2\delta\frac{\partial u_1}{\partial y}\frac{\partial u_2}{\partial x}\right]d\tau.$$

With the help of the estimate (2.18) we derive

$$\int\limits_\omega y^m\left[\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial y}\right)^2+\left(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\right)^2\right]d\tau$$

$$\geq\int\limits_\omega y^m\left[\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial y}\right)^2+\delta\left(\frac{\partial u_1}{\partial y}\right)^2+\delta\left(\frac{\partial u_2}{\partial x}\right)^2-\delta\alpha\left(\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial y}\right)^2\right)\right]d\tau$$

$$=\int\limits_\omega y^m\left[(1-\delta\alpha)\left(\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial y}\right)^2\right)+\delta\left(\left(\frac{\partial u_1}{\partial y}\right)^2+\left(\frac{\partial u_2}{\partial x}\right)^2\right)\right]d\tau.$$

We choose δ as follows $\delta=(1+\alpha)^{-1}$. From the previous relation then we have

$$\int\limits_\omega y^m\left[\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial y}\right)^2+\left(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\right)^2\right]d\tau$$

$$\geq\int\limits_\omega y^m\left[\left(1-\frac{\alpha}{1+\alpha}\right)\left(\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial y}\right)^2\right)+\frac{1}{1+\alpha}\left(\left(\frac{\partial u_1}{\partial y}\right)^2+\left(\frac{\partial u_2}{\partial x}\right)^2\right)\right]d\tau$$

$$=\frac{1}{1+\alpha}\int\limits_\omega y^m\left[\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial y}\right)^2+\left(\frac{\partial u_1}{\partial y}\right)^2+\left(\frac{\partial u_2}{\partial y}\right)^2\right]d\tau.$$

If we "close" this inequality in the space $\overset{\rightarrow}{\overset{0}{H}}_{m,m}(\omega)$ we obtain that Korn's weighted inequality is true for every function $\vec{u}=(u_1,u_2)\in\overset{\rightarrow}{\overset{0}{H}}_{m,m}(\omega)$. ∎

Concerning Korn's weighted inequalities in various functional spaces see also in [2-5].

Let us complete $[D(\omega)]^2$ by the norm:

$$\| \vec{u} \|_{1,m} = \left(\int_\omega y^m \left[\left(\frac{\partial u_1}{\partial x}\right)^2 + \left(\frac{\partial u_2}{\partial y}\right)^2 + \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)^2 + u_1^2 + u_2^2 \right] d\tau \right)^{\frac{1}{2}}$$

for $\vec{u}=(u_1,u_2)\in[D(\omega)]^2$.

We obtain the Hilbert space $\overset{\rightarrow}{\overset{0}{K}}_m(\omega)$ with the scalar product:

$$(\vec{u},\vec{v})_{1,m} = \int_\omega y^m \left[\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x} + \frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y} + \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) + \left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) + \right.$$

$$\left. + u_1 v_1 + u_2 v_2 \right] d\tau.$$

for $\vec{u}=(u_1,u_2)$, $\vec{v}=(v_1,v_2)\in\overset{\rightarrow}{\overset{0}{K}}_m(\omega)$.

According to Korn's weighted inequality we conclude for $m\neq 1$ that the above introduced norms in the spaces $\overset{\rightarrow}{\overset{0}{K}}_m(\omega)$ and $\overset{\rightarrow}{\overset{0}{H}}_{m,m}(\omega)$ are equivalent and, moreover, these spaces coincide as sets of vector functions. To indicate this coincidence we will use the notation:

$$\overset{\rightarrow}{\overset{0}{K}}_m(\omega) \equiv \overset{\rightarrow}{\overset{0}{H}}_{m,m}(\omega). \tag{2.19}$$

In the case $m=1$

$$\overset{\rightarrow}{\overset{0}{H}}_{1,1}(\omega) \subset \overset{\rightarrow}{\overset{0}{K}}_1(\omega) \subset \overset{\rightarrow}{\overset{0}{H}}_{1+\varepsilon,1+\varepsilon}(\omega) \quad \forall\varepsilon>0. \tag{2.20}$$

The first embedding is trivial. As to the second one, it follows from the inequality

$$\int_\omega y \left[\left(\tfrac{\partial u_1}{\partial x}\right)^2 + \left(\tfrac{\partial u_2}{\partial y}\right)^2 + \left(\tfrac{\partial u_1}{\partial y} + \tfrac{\partial u_2}{\partial x}\right)^2 \right]^2 d\tau$$

$$\geq c_1 \int_\omega y^{1+\varepsilon} \left[\left(\tfrac{\partial u_1}{\partial x}\right)^2 + \left(\tfrac{\partial u_2}{\partial y}\right)^2 + \left(\tfrac{\partial u_1}{\partial y} + \tfrac{\partial u_2}{\partial x}\right)^2 \right]^2 d\tau$$

$$\geq c_2 \int_\omega y^{1+\varepsilon} \left[\left(\tfrac{\partial u_1}{\partial x}\right)^2 + \left(\tfrac{\partial u_1}{\partial y}\right)^2 + \left(\tfrac{\partial u_2}{\partial x}\right)^2 + \left(\tfrac{\partial u_2}{\partial y}\right)^2 \right] d\tau$$

for $\vec{u} = (u_1, u_2) \in [D(\omega)]^2$.

We introduce the functional space $\overset{*}{H}_m(\omega)$, which is obtained by completing $D(\omega)$ with the norm

$$\|u\|_{2,m} = \left(\int_\omega y^m \nabla u \cdot \nabla u d\tau + \int_\omega y^{\frac{m}{3}} u^2 d\tau \right)^{\frac{1}{2}}.$$

Since the inequality

$$y^{\frac{m}{3}} \leq y^{m-2}$$

considered in the neighborhood of the point $y = 0$ implies

$$\frac{m}{3} \geq m - 2 \Rightarrow m \leq 3,$$

we have

$$\overset{0}{H}_m(\omega) \equiv \overset{*}{H}_m(\omega) \tag{2.21}$$

for $m \leq 3$.

Further, we introduce the following vector spaces

$$\overset{\overset{\rightarrow}{0}}{H}_{m_1,m_2,m_3}(\omega) := \overset{0}{H}_{m_1}(\omega) \times \overset{0}{H}_{m_2}(\omega) \times \overset{0}{H}_{m_3}(\omega)$$

and

$$\overset{\overset{\rightarrow}{*}}{K}_{m_1,m_2}(\omega) := \overset{\overset{\rightarrow}{0}}{K}_{m_1}(\omega) \times \overset{*}{H}_{m_2}(\omega)$$

with norms

$$\| \vec{u} \|^2_{m_1,m_2,m_3} = \|u_1\|^2_{m_1} + \|u_2\|^2_{m_2} + \|u_3\|^2_{m_3}$$

for $\vec{u} = (u_1, u_2, u_3) \in \overset{\overset{\rightarrow}{0}}{H}_{m_1,m_2,m_3}(\omega)$, and

$$\| \vec{u} \|^2_{1,(m_1,m_2)} = \|(u_1, u_2)\|^2_{1,m_1} + \|u_3\|^2_{2,m_2}$$

for $\vec{u} = (u_1, u_2, u_3) \in \overset{\overset{\rightarrow}{*}}{K}_{m_1,m_2}(\omega)$.

According to the above mentioned relations (see (2.19), (2.20), (2.21))

$$\overset{\overset{\rightarrow}{0}}{H}_{m,m,m_1}(\omega) \equiv \overset{\overset{\rightarrow}{*}}{K}_{m,m_1}(\omega).$$

for $m \neq 1$ and $m_1 \leq 3$, and

$$\overset{\overset{\rightarrow}{0}}{H}_{1,1,m_1}(\omega) \subset \overset{\overset{\rightarrow}{*}}{K}_{1,m_1}(\omega) \subset \overset{\overset{\rightarrow}{0}}{H}_{1+\varepsilon,1+\varepsilon,m_1}(\omega)$$

for $m_1 \leq 3$ and $\forall \varepsilon > 0$.

3. *Existence and uniqueness results.*

Now we are in the position to introduce the definition of a generalized solution to the system (1.1).

We say that $\vec{u} = (u_1, u_2, v_3) \in \overset{\rightarrow}{\overset{*}{K}}_{m,3m}(\omega)$ is a generalized solution of the system (1.1) if

$$B(\vec{u},\vec{w}) := \int\limits_{\omega} y^m \Big[(\lambda+2\mu)\tfrac{\partial u_1}{\partial x}\tfrac{\partial w_1}{\partial x} + \mu\tfrac{\partial u_1}{\partial y}\tfrac{\partial w_1}{\partial y} + \lambda\tfrac{\partial u_2}{\partial y}\tfrac{\partial w_1}{\partial x} + \mu\tfrac{\partial u_2}{\partial x}\tfrac{\partial w_1}{\partial y}$$

$$+3\lambda v_3\tfrac{\partial w_1}{\partial x} + \mu\tfrac{\partial u_2}{\partial x}\tfrac{\partial w_2}{\partial x} + (\lambda+2\mu)\tfrac{\partial u_2}{\partial y}\tfrac{\partial w_2}{\partial y} + \mu\tfrac{\partial u_1}{\partial y}\tfrac{\partial w_2}{\partial x} + \lambda\tfrac{\partial u_1}{\partial x}\tfrac{\partial w_2}{\partial y}$$

$$+3\lambda v_3\tfrac{\partial w_2}{\partial y} + y^{2m}3\mu\left(\tfrac{\partial v_3}{\partial x}\tfrac{\partial w_3}{\partial x} + \tfrac{\partial v_3}{\partial y}\tfrac{\partial w_3}{\partial y}\right)$$

$$+3\lambda\tfrac{\partial u_1}{\partial x}w_3 + 3\lambda\tfrac{\partial u_2}{\partial y}w_3 + 9(\lambda+2\mu)v_3w_3\Big]\,d\tau =$$

$$= \int\limits_{\omega}(X_1w_1 + X_2w_2 + 3X_3w_3)\,d\tau$$

for all $\vec{w} = (w_1, w_2, w_3) \in [D(\omega)]^3$.

For our further aims we need the space of such vector–functions $\vec{X} = (X_1, X_2, X_3)$ for which the right–hand side in the last equality defines a bounded functional.

Let $L_{2,\sigma_m}(\omega)$ be a Hilbert space of measurable functions ψ such that the norm

$$\|\psi\|_{L_{2,\sigma_m}} = \left(\int\limits_{\omega}\sigma_m(y)\psi^2(x,y)d\tau\right)^{\frac{1}{2}}$$

is finite, where

$$\sigma_m(y) = \begin{cases} y^{2-m}, & m \neq 1, \\ y^{1-\varepsilon}, & \varepsilon > 0,\ m = 1. \end{cases}$$

Here ε is an arbitrary positive number.

Further, let

$$\sigma_m^*(y) = \begin{cases} y^{2-m}, & m \le 3,\ m \neq 1, \\ y^{1-\varepsilon}, & \varepsilon > 0,\ m = 1, \\ y^{-\frac{m}{3}}, & m > 3, \end{cases}$$

and

$$\vec{L}_{2,(\sigma_m,\sigma_m,\sigma_m^*)}(\omega) := L_{2,\sigma_m}(\omega) \times L_{2,\sigma_m}(\omega) \times L_{2,\sigma_m^*}(\omega)$$

with the norm

$$\| \vec{u} \|^2_{L_{2,(\sigma_m,\sigma_m,\sigma^*_m)}} = \|u_1\|^2_{L_{2,\sigma_m}} + \|u_2\|^2_{L_{2,\sigma_m}} + \|u_3\|^2_{L_{2,\sigma^*_m}},$$

where $\vec{u} = (u_1, u_2, u_3) \in \vec{L}_{2,(\sigma_m,\sigma_m,\sigma^*_m)}(\omega)$.

Theorem. *If* $\vec{X} = (X_1, X_2, X_3) \in \vec{L}_{2,(\sigma_m,\sigma_m,\sigma^*_{3m})}(\omega)$ *, then the system (1.1) has a unique generalized solution* $\vec{u} = (u_1, u_2, u_3) \in \vec{\overset{*}{K}}_{m,3m}(\omega)$ *and there holds the estimate*

$$\| \vec{u} \|_{1,(m,3m)} \le c \| \vec{X} \|_{L_{2,(\sigma_m,\sigma_m,\sigma^*_{3m})}},$$

where c *is a positive constant independent of* $\vec{X}$ *and* $\vec{u}$.

Proof. Let us show that the form $B(\vec{u}, \vec{w})$ is coercive on $\vec{\overset{*}{K}}_{m,3m}(\omega)$. First we estimate $B(\vec{u}, \vec{u})$ for $\vec{u} = (u_1, u_2, u_3) \in \vec{\overset{*}{K}}_{m,3m}(\omega)$:

$$B(\vec{u}, \vec{u}) = \int\limits_\omega y^m \left[(\lambda + 2\mu) \left(\frac{\partial u_1}{\partial x} \right)^2 + \mu \left(\frac{\partial u_1}{\partial y} \right)^2 + \lambda \frac{\partial u_2}{\partial y} \frac{\partial u_1}{\partial x} + \mu \frac{\partial u_2}{\partial x} \frac{\partial u_1}{\partial y} \right.$$

$$+ 3\lambda u_3 \frac{\partial u_1}{\partial x} + \mu \left(\frac{\partial u_2}{\partial x} \right)^2 + (\lambda + 2\mu) \left(\frac{\partial u_2}{\partial y} \right)^2 + \mu \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial x} + \lambda \frac{\partial u_2}{\partial y} \frac{\partial u_1}{\partial x}$$

$$+ 3\lambda u_3 \frac{\partial u_2}{\partial y} + y^{2m} 3\mu \left(\left(\frac{\partial u_3}{\partial x} \right)^2 + \left(\frac{\partial u_3}{\partial y} \right)^2 \right) + 3\lambda \frac{\partial u_1}{\partial x} u_3 + 3\lambda \frac{\partial u_2}{\partial y} u_3$$

$$\left. + 9(\lambda + 2\mu) u_3^2 \right] d\tau = \int\limits_\omega y^m \left\{ \left[\lambda \left[\left(\frac{\partial u_1}{\partial x} \right)^2 + 2 \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial y} + \left(\frac{\partial u_2}{\partial y} \right)^2 \right. \right. \right.$$

$$\left. + 6 \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \right) u_3 + 9 u_3^2 \right] + \mu \left[2 \left(\frac{\partial u_1}{\partial x} \right)^2 + 2 \frac{\partial u_2}{\partial x} \frac{\partial u_1}{\partial y} + \left(\frac{\partial u_1}{\partial y} \right)^2 \right.$$

$$\left. \left. + \left(\frac{\partial u_2}{\partial x} \right)^2 + 2 \left(\frac{\partial u_2}{\partial y} \right)^2 + 18 u_3^2 \right] + y^{2m} 3\mu \left[\left(\frac{\partial u_3}{\partial x} \right)^2 + \left(\frac{\partial u_3}{\partial y} \right)^2 \right] \right\} d\tau.$$

Since $\lambda, \mu > 0$, we have

$$B(\vec{u}, \vec{u}) \ge c \left[\int\limits_\omega y^m \left[\left(\frac{\partial u_1}{\partial x} \right)^2 + \left(\frac{\partial u_2}{\partial y} \right)^2 + \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right)^2 \right] d\tau \right.$$

$$\left. + \int\limits_\omega \left\{ y^{3m} \left[\left(\frac{\partial u_3}{\partial x} \right)^2 + \left(\frac{\partial u_3}{\partial y} \right)^2 \right] + y^m u_3^2 \right\} d\tau \right] = c \| \vec{u} \|^2_{1,(m,3m)}.$$

Further we estimate $B(\vec{u},\vec{w})$ for $\vec{u}=(u_1,u_2,u_3)$, $\vec{w}=(w_1,w_2,w_3)\in \overset{\rightarrow}{\overset{*}{K}}_{m,3m}(\omega)$. We have

$$B(\vec{u},\vec{w})=\int\limits_{\omega} y^m\Big[(\lambda+2\mu)\tfrac{\partial u_1}{\partial x}\tfrac{\partial w_1}{\partial x}+\mu\tfrac{\partial u_1}{\partial y}\left(\tfrac{\partial w_1}{\partial y}+\tfrac{\partial w_2}{\partial x}\right)+\lambda\tfrac{\partial u_2}{\partial y}\tfrac{\partial w_1}{\partial x}$$

$$+\mu\tfrac{\partial u_2}{\partial x}\left(\tfrac{\partial w_1}{\partial y}+\tfrac{\partial w_2}{\partial x}\right)+3\lambda\left(\tfrac{\partial w_1}{\partial x}+\tfrac{\partial w_2}{\partial y}\right)u_3+(\lambda+2\mu)\tfrac{\partial u_2}{\partial y}\tfrac{\partial w_2}{\partial y}$$

$$+\lambda\tfrac{\partial u_1}{\partial x}\tfrac{\partial w_2}{\partial y}+3\lambda\left(\tfrac{\partial u_1}{\partial x}+\tfrac{\partial u_2}{\partial y}\right)w_3+9(\lambda+2\mu)u_3w_3$$

$$+y^{2m}3\mu\left(\tfrac{\partial u_3}{\partial x}\tfrac{\partial w_3}{\partial x}+\tfrac{\partial u_3}{\partial y}\tfrac{\partial w_3}{\partial y}\right)\Big]d\tau=\int\limits_{\omega} y^m\Big[(\lambda+2\mu)\tfrac{\partial u_1}{\partial x}\tfrac{\partial w_1}{\partial x}$$

$$+\mu\left(\tfrac{\partial u_2}{\partial x}+\tfrac{\partial u_1}{\partial y}\right)\left(\tfrac{\partial w_1}{\partial y}+\tfrac{\partial w_2}{\partial x}\right)+\lambda\tfrac{\partial u_2}{\partial y}\tfrac{\partial w_1}{\partial x}+3\lambda\left(\tfrac{\partial w_1}{\partial x}+\tfrac{\partial w_2}{\partial y}\right)u_3$$

$$+(\lambda+2\mu)\tfrac{\partial u_2}{\partial y}\tfrac{\partial w_2}{\partial y}+\lambda\tfrac{\partial u_1}{\partial x}\tfrac{\partial w_2}{\partial y}+3\lambda\left(\tfrac{\partial u_1}{\partial x}+\tfrac{\partial u_2}{\partial y}\right)w_3$$

$$+9(\lambda+2\mu)u_3w_3+y^{2m}3\mu\left(\tfrac{\partial u_3}{\partial x}\tfrac{\partial w_3}{\partial x}+\tfrac{\partial u_3}{\partial y}\tfrac{\partial w_3}{\partial y}\right)\Big]d\tau.$$

By the Hölder inequality we can estimate each term of the last sum separately:

$$\left|\int\limits_{\omega} y^m\tfrac{\partial u_1}{\partial x}\tfrac{\partial w_1}{\partial y}d\tau\right|\le\left(\int\limits_{\omega} y^m\left(\tfrac{\partial u_1}{\partial x}\right)^2d\tau\right)^{\frac12}\left(\int\limits_{\omega} y^m\left(\tfrac{\partial w_1}{\partial x}\right)^2d\tau\right)^{\frac12}$$

$$\le\|(u_1,u_2)\|_{1,m}\|(w_1,w_2)\|_{1,m};$$

$$\left|\int\limits_{\omega} y^m\left(\tfrac{\partial u_2}{\partial x}+\tfrac{\partial u_1}{\partial y}\right)\left(\tfrac{\partial w_1}{\partial y}+\tfrac{\partial w_2}{\partial x}\right)d\tau\right|\le\left(\int\limits_{\omega} y^m\left(\tfrac{\partial u_2}{\partial x}+\tfrac{\partial u_1}{\partial y}\right)^2d\tau\right)^{\frac12}\times$$

$$\times\left(\int\limits_{\omega} y^m\left(\tfrac{\partial w_1}{\partial y}+\tfrac{\partial w_2}{\partial x}\right)^2d\tau\right)^{\frac12}\le\|(u_1,u_2)\|_{1,m}\|(w_1,w_2)\|_{1,m};$$

$$\left|\int\limits_{\omega} y^m\tfrac{\partial u_2}{\partial y}\tfrac{\partial w_1}{\partial x}d\tau\right|\le\left(\int\limits_{\omega} y^m\left(\tfrac{\partial u_2}{\partial y}\right)^2d\tau\right)^{\frac12}\left(\int\limits_{\omega} y^m\left(\tfrac{\partial w_1}{\partial x}\right)^2d\tau\right)^{\frac12}$$

$$\le\|(u_1,u_2)\|_{1,m}\|(w_1,w_2)\|_{1,m};$$

$$\left|\int\limits_{\omega} y^m\left(\tfrac{\partial w_1}{\partial x}+\tfrac{\partial w_2}{\partial y}\right)u_3d\tau\right|\le\left(\int\limits_{\omega} y^m\left(\tfrac{\partial w_1}{\partial x}+\tfrac{\partial w_2}{\partial y}\right)^2d\tau\right)^{\frac12}\times$$

$$\times\left(\int\limits_{\omega} y^m u_3^2d\tau\right)^{\frac12}\le\left(\int\limits_{\omega} 2y^m\left[\left(\tfrac{\partial w_1}{\partial x}\right)^2+\left(\tfrac{\partial w_2}{\partial y}\right)^2\right]d\tau\right)^{\frac12}\times$$

$$\times\left(\int\limits_\omega y^m u_3^2 d\tau\right)^{\frac{1}{2}} \leq \sqrt{2}\|(w_1,w_2)\|_{1,m}\|u_3\|_{2,3m};$$

$$\left|\int\limits_\omega y^m \frac{\partial u_2}{\partial y}\frac{\partial w_2}{\partial y}d\tau\right| \leq \left(\int\limits_\omega y^m\left(\frac{\partial u_2}{\partial y}\right)^2 d\tau\right)^{\frac{1}{2}}\left(\int\limits_\omega y^m\left(\frac{\partial w_2}{\partial y}\right)^2 d\tau\right)^{\frac{1}{2}}$$

$$\leq \|(u_1,u_2)\|_{1,m}\|(w_1,w_2)\|_{1,m};$$

$$\left|\int\limits_\omega y^m\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\right)w_3 d\tau\right| \leq \left(\int\limits_\omega y^m\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\right)^2 d\tau\right)^{\frac{1}{2}}\times$$

$$\times\left(\int\limits_\omega y^m w_3^2 d\tau\right)^{\frac{1}{2}} \leq \left(\int\limits_\omega 2y^m\left[\left(\frac{\partial u_1}{\partial x}\right)^2+\left(\frac{\partial u_2}{\partial y}\right)^2\right]d\tau\right)^{\frac{1}{2}}\|w_3\|_{2,3m}^2$$

$$\leq \sqrt{2}\|(u_1,u_2)\|_{1,m}\|w_3\|_{2,3m};$$

$$\left|\int\limits_\omega y^m u_3 w_3 d\tau\right| \leq \left(\int\limits_\omega y^m u_3^2 d\tau\right)^{\frac{1}{2}}\left(\int\limits_\omega y^m w_3^2 d\tau\right)^{\frac{1}{2}} \leq \|u_3\|_{2,3m}\|w_3\|_{2,3m};$$

$$\left|\int\limits_\omega y^{3m}\left(\frac{\partial u_3}{\partial x}\frac{\partial w_3}{\partial x}+\frac{\partial u_3}{\partial y}\frac{\partial w_3}{\partial y}\right)d\tau\right| \leq \left|\int\limits_\omega y^{3m}\frac{\partial u_3}{\partial x}\frac{\partial w_3}{\partial x}d\tau\right|$$

$$+\left|\int\limits_\omega y^{3m}\frac{\partial u_3}{\partial y}\frac{\partial w_3}{\partial y}d\tau\right| \leq \left(\int\limits_\omega y^{3m}\left(\frac{\partial u_3}{\partial x}\right)^2 d\tau\right)^{\frac{1}{2}}\left(\int\limits_\omega y^{3m}\left(\frac{\partial w_3}{\partial x}\right)^2 d\tau\right)^{\frac{1}{2}}$$

$$+\left(\int\limits_\omega y^{3m}\left(\frac{\partial u_3}{\partial y}\right)^2 d\tau\right)^{\frac{1}{2}}\left(\int\limits_\omega y^{3m}\left(\frac{\partial w_3}{\partial y}\right)^2 d\tau\right)^{\frac{1}{2}} \leq 2\|u_3\|_{2,3m}^2\|w_3\|_{2,3m}^2.$$

With the help of these bounds we easily obtain that

$$|\,B(\vec{u},\vec{w})\,| \leq c_1\|\,\vec{u}\,\|_{1,(m,3m)}\|\,\vec{w}\,\|_{1,(m,3m)}.$$

Finally we note that if $\vec{X}=(X_1,X_2,X_3)\in \vec{L}_{2,(\sigma_m,\sigma_m,\sigma_{3m}^*)}(\omega)$, then there holds the following estimate:

$$\left|\int\limits_\omega \vec{X}\,\vec{w}\,d\tau\right| \leq \left|\int\limits_\omega (X_1w_1+X_2w_2+X_3w_3)\,d\tau\right|$$

$$\leq \left|\int\limits_\omega X_1w_1 d\tau\right|+\left|\int\limits_\omega X_2w_2 d\tau\right|+\left|\int\limits_\omega X_3w_3 d\tau\right|$$

$$=\left|\int\limits_\omega \sigma_m^{\frac{1}{2}}X_1\times\sigma_m^{-\frac{1}{2}}w_1 d\tau\right|+\left|\int\limits_\omega \sigma_m^{\frac{1}{2}}X_2\times\sigma_m^{-\frac{1}{2}}w_2 d\tau\right|+\left|\int\limits_\omega {\sigma_m^*}^{\frac{1}{2}}X_3\times{\sigma_m^*}^{-\frac{1}{2}}w_3 d\tau\right|$$

$$\le \left(\int_\omega \sigma_m X_1^2 d\tau\right)^{\frac{1}{2}} \left(\int_\omega \sigma_m^{-1} w_1^2 d\tau\right)^{\frac{1}{2}} + \left(\int_\omega \sigma_m X_2^2 d\tau\right)^{\frac{1}{2}} \left(\int_\omega \sigma_m^{-1} w_2^2 d\tau\right)^{\frac{1}{2}}$$

$$+ \left(\int_\omega \sigma_m^* X_3^2 d\tau\right)^{\frac{1}{2}} \left(\int_\omega {\sigma_m^*}^{-1} w_3^2 d\tau\right)^{\frac{1}{2}}$$

$$\le 2\| \vec{X} \|_{L_2,(\sigma_m,\sigma_m,\sigma^*_{3m})} \|(w_1, w_2)\|_{1,m} + \| \vec{X} \|_{L_2,(\sigma_m,\sigma_m,\sigma^*_{3m})} \|w_3\|_{2,3m}$$

$$\le c_2 \| \vec{X} \|_{L_2,(\sigma_m,\sigma_m,\sigma^*_{3m})} \| \vec{w} \|_{1,(m,3m)}.$$

Thus we have shown that $B(\vec{u}, \vec{w})$ is coercive and the functional $\int\limits_\omega \vec{X}\,\vec{w}\,d\tau$ is bounded on $\overset{\rightarrow}{\overset{*}{K}}_{m,3m}(\omega)$. Now, the Lax-Milgram theorem (see, e.g., [6,13]) completes the proof. ■

In what follows we make some remarks concerning Theorem 5.

When $0 \le m < \frac{1}{3}$, the generalized solution $\vec{u} = (u_1, u_2, v_3)$ of system (1.1) belongs to the space $\overset{\rightarrow}{\overset{*}{K}}_{m,3m}(\omega) \equiv \overset{\rightarrow}{\overset{0}{H}}_{m,m,3m}(\omega)$, and therefore, according to Lemma 2,

$$u_1 \,|_{\partial\omega} = u_2 \,|_{\partial\omega} = v_3 \,|_{\partial\omega} = 0$$

in the trace sense.

Consequently, in this case $\vec{u}$ must be given on the whole boundary.

If $\frac{1}{3} \le m < 1$, then

$$\vec{u} = (u_1, u_2, v_3) \in \overset{\rightarrow}{\overset{*}{K}}_{m,3m}(\omega) \equiv \overset{\rightarrow}{\overset{0}{H}}_{m,m,3m}(\omega).$$

Thus

$$u_1 \,|_{\partial\omega} = u_2 \,|_{\partial\omega} = 0, \quad v_3 \,|_{\Gamma_1} = 0,$$

and, according to Lemma 3, v_3 has not a trace on Γ_0, in general. In this case u_1 and u_2 must be given on the whole boundary $\partial\omega$, while v_3 must be given only on Γ_1.

If $m \ge 1$, then $\vec{u} = (u_1, u_2, v_3) \in \overset{\rightarrow}{\overset{*}{K}}_{m,3m}(\omega)$ and

$$u_1 \,|_{\Gamma_1} = u_2 \,|_{\Gamma_1} = v_3 \,|_{\Gamma_1} = 0;$$

and u_1, u_2 and v_3 do not have traces on Γ_0. Therefore we must give u_1, u_2, and v_3 only on Γ_0, and leave them free on Γ_1.

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