NUMERICAL APPROXIMATION OF EIGENVALUES FOR TRANSVERSE VIBRATIONS OF A WEDGE-SHAPED BEAM

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(Received: December 11, 1997)

Abstract

In this paper a method for computing the eigenvalues of differential problem connected with transverse vibrations of a simply supported wedge-shaped beam is considered. By using an iterative method for computing the eigenvalues of Fredholm second kind equation (see[1]), previous approximations are improved (see[3]).

Key words and phrases: Fredholm integral equations; Eigenvalues; Rayleigh-Ritz method; Inverse powers method.

AMS subject classification: 45C05, 45B05, 65R20.

1. *Introduction*

Many authors, like WARD (1913), NICHOLSON (1917-20), MONONOBE (1921), ONO (1924-25) and SCHWERIN (1926), have studied vibrations of beams whose sections in a mathematically simple way are depending on abscissa; in all this cases the beam has one end built in and other end free.

Many other authors, like PFEIFFER (1928), BRUNELLI (1929), KRALL (1930), HOHENEMSER (1932), FROLA (1933-34), TRICOMI (1936), have studied the same problem with a simply supported beam. During years, many methods connected with computing the eigenvalues of differential operators for vibrations of beams have been introduced, for example by Rayleigh-Ritz, Weyl, Courant, Picone, Carleman, Tricomi, Weinstein, Agmon and Fichera.

As a matter of fact in a paper of G. Fichera the attempts of these authors have been re-engaged.

In this paper we re-engage the Tricomi's method and in the particular case of wedge-shaped beam we write his results and we compare it with Fichera's results got with orthogonal invariants method.

At last, by using an iterative method for computing the eigenvalues of second kind Fredholm integral operators (see [1], [5]),we show that it is possible to obtain best approximations of eigenvalues.

2. *Tricomi's problem and his results in a particular case of a wedge-shaped beam*

Tricomi [7] used a method for computing an approximation of the first critical frequency for a simply supported beam. Given a not too large beam with his barycentric axis coinciding with x axis, the vibrations equation is expressed by

$$\frac{\partial^2}{\partial x^2}\left[EJ(x)\frac{\partial^2 y}{\partial x^2}\right] + \rho\sigma(x)\frac{\partial^2 y}{\partial t^2} = 0 \tag{2.1}$$

where $\sigma(x)$ is the surface of a transverse section, $J(x)$ – the moment of inertia with respect to the barycentric axis normal to measure of the (x, y) plane, ρ and E are respectively the density and Young's modulus of beam's material.

We suppose that the length of beam is equal to one, so that $x \in [0, 1]$.

After suitable transformations (2.1) becomes

$$\frac{d^2}{dx^2}\left[j(x)\frac{d^2 u}{dx^2}\right] = \lambda m(x)u(x) \tag{2.2}$$

with

$$j(x) = \frac{EJ(x)}{A}, \quad m(x) = \frac{\rho\sigma(x)}{B}, \quad \lambda = 4\pi^2\frac{B}{A}\nu^2 \tag{2.3}$$

where A, B are suitable constants and ν is the frequency of characteristic vibrations of the beam.

Suppose we know the first eigenvalue λ_1, by using the last of (2.3) we can find the first eigenfrequency ν_1 of the beam. This is an important information because if during the periodic excitations of beam the frequency is not less than ν_1, dangerous resonance phenomena are possible.

For computing the approximations of eigenvalues, Tricomi, starting from conditions

$$u(0) = u''(0) = u(1) = u''(1) = 0\,, \tag{2.4}$$

changed (2.1) into an integral equation with symmetric kernel

$$U(y) = \lambda\int_0^1 K(x, y)U(x)dx \tag{2.5}$$

where $U(x) = \sqrt{m(x)}u(x)$ and $K(x, y) = \sqrt{m(x)m(y)}G(x, y)$; $G(x, y)$ is Green function of differential selfadjoint equation

$$\frac{d^2}{dx^2}\left[j(x)\frac{d^2 u}{dx^2}\right] = 0 \tag{2.6}$$

given by

$$G(x,y)=\int_0^1 \frac{g(x,z)g(z,y)}{j(z)}dz \tag{2.7}$$

where

$$g(x,y)=\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\sin n\pi x\sin n\pi y}{n^2}=\begin{cases} x\,(1-y)\,, & x\le y\\ y(1-x), & x\ge y.\end{cases} \tag{2.8}$$

In this way the problem (2.2)-(2.4) becomes an integral equation with symmetric kernel (2.5).

This equation, by leaving out the known developments of this theory (see for example [5]), leads to the next formulae:

a) a lower approximated value λ_1' of the first eigenvalue is given by

$$\frac{1}{\lambda_1'}=T-\left(\frac{1}{90}-\frac{1}{\pi^4}\right)\frac{m_0}{j_1}\,, \tag{2.9}$$

where

$$T=\int_0^1\int_o^1 g^2(x,y)\frac{m(x)}{j(y)}\,dx\,dy$$

and m_0, j_1 are respectively minimum of $m(x)$ and maximum of $j(x)$.

b) an upper approximated value λ_1'' of the first eigenvalue is given by

$$\lambda_1''=\pi^4\frac{\displaystyle\int_0^1 j(x)\sin^2 nx\,dx}{\displaystyle\int_0^1 m(x)\sin^2 nx\,dx}\,. \tag{2.10}$$

For the particular case of a wedge-shaped beam, we have:

$$j(x)=(1-\theta x)^3\,,\qquad m(x)=1-\theta x\,, \tag{2.11}$$

where $\theta\in[0,1]$ is the thinning coefficient of the considered beam.

With particular choice $\theta=0{,}5$, equations (2.9), (2.10) give the following estimates for λ_1:

$$49{,}92156735<\lambda_1<57{,}15536862\,. \tag{2.12}$$

3. *Results obtained by orthogonal invariants method for a wedge-shaped beam*

In [3] it is possible to find the numerical results obtained by the orthogonal invariants method for computing eigenvalues of differential problems when the Green function or in any case suitable kernel are known.

In the particular case of wedge-shaped beam, the problem can be written

$$\frac{d^2}{dx^2}\left[(1-\theta x)^3 \frac{d^2 u}{dx^2}\right] = \lambda(1-\theta x)u(x)$$

$$u(0) = u''(0) = u(1) = u''(1) = 0\,, 0 < x < 1\,. \qquad (3.1)$$

This problem with orthogonal invariants method is just studied by M.P. Colautti [2].

We show in Table I the lower and upper approximation of values λ_k $(k = 1, 2, \ldots, 13)$ for problem (3.1) with $\theta = 0{,}5$.

We note that for λ_1 the Tricomi's result (2.12) is considerable improved.

50,71623063	$< \lambda_1$	$<$ 50,71623066
838,2089	$< \lambda_2$	$<$ 838,2091
4222,235	$< \lambda_3$	$<$ 4222,247
13305,47	$< \lambda_4$	$<$ 13305,82
32427	$< \lambda_5$	$<$ 32432
67140	$< \lambda_6$	$<$ 67185
124108	$< \lambda_7$	$<$ 124390
210726	$< \lambda_8$	$<$ 212111
334060	$< \lambda_9$	$<$ 339657
498394	$< \lambda_{10}$	$<$ 517576
701216	$< \lambda_{11}$	$<$ 757963
928773	$< \lambda_{12}$	$<$ 1074337
1166861	$< \lambda_{13}$	$<$ 1504837

4. *An iterative method for computing the Fredholm operator eigenvalues*

The use of the orthogonal invariants method in order to approximate the eigenvalues of differential problem leads sometimes to very cumbersome computations so it is better to use the next method, called the inverse iteration method.

We premise that the integral equation (2.5) is a homogeneous Fredholm equation of second kind

$$\varphi(x) - \lambda \int_0^1 K(x,y)\varphi(y)dy = 0 \tag{4.1}$$

this can be written

$$(J - \lambda\ \mathcal{K})\varphi = 0 \tag{4.2}$$

with operator $\mathcal{K} : L^2[0,1] \to L^2[0,1]$ associated to the kernel $K(x,y)$ given by

$$\mathcal{K}\varphi = (\ \mathcal{K}\varphi)(x) = \int_0^1 K(x,y)\varphi(y)dy$$

and the identity operator $\mathcal{I}$.

We explain now the inverse iterations method.

By writing (4.2) in the form

$$(\ \mathcal{K} - \mu\ \mathcal{I})\varphi = 0\,, \quad with \quad \mu = \lambda^{-1} \tag{4.3}$$

that is

$$\mathcal{K}\varphi = \mu\varphi \tag{4.4}$$

the eigenvalues of operator $\mathcal{K}$ can be put in order as a decreasing sequence with regard to their moduli

$$0 < \ldots \leq |\mu_3| \leq |\mu_2| \leq |\mu_1|\,. \tag{4.5}$$

In the particular case of $K(x,y)$ symmetric $K(x,y) = K(y,x)$, hermitian positive $K(x,y) = \overline{K(y,x)}$, $(K\varphi,\varphi) > 0$ if $\varphi \neq 0$ in $L^2[0,1]$ in the last formula the modulus signs can be avoided:

$$0 < \ldots \leq \mu_3 \leq \mu_2 \leq \mu_1\,.$$

Suppose we know an initial approximation $\tilde{\mu}$ of the searched eigenvalue μ_j, $j \geq 2$, such that

$$|\tilde{\mu} - \mu_j| < \frac{1}{2} \min_{\mu_k \neq \mu_j k=1,2,\ldots,v} |\mu_k - \mu_j| \tag{4.6}$$

for a suitable choice of the integer ν. In practice in this condition the eigenvalues will be replaced by their Rayleigh-Ritz approximations, for sufficiently large ν:

$$|\tilde{\mu} - \mu_j^{(\nu)}| < \frac{1}{2} \min_{\mu_k^{(\nu)} \neq \mu_j^{(\nu)} k=1,2,\ldots,\nu} |\mu_k^{(\nu)} - \mu_j^{(\nu)}|\,. \tag{4.7}$$

From equation (4.2) we get:

$$(\mathcal{K} - \tilde{\mu}\mathcal{I})\phi = (\mu - \tilde{\mu})\phi \tag{4.8}$$

Consequently, if μ_j is an eigenvalue of $\mathcal{K}$ with eigenfunction ϕ_j, then $\mu_j - \tilde{\mu}$ is an eigenvalue of $\mathcal{K} - \tilde{\mu}\mathcal{I}$ with eigenfunction ϕ_j. By writing (4.8) in the form

$$(\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}\phi = (\mu - \tilde{\mu})^{-1}\phi \tag{4.9}$$

it follows that $(\mu_j - \tilde{\mu})^{-1}$ is an eigenvalue of $(\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}$ with the same eigenfunction ϕ_j.

By using condition (4.7), for ν sufficiently large, the eigenvalue $(\mu_j - \tilde{\mu})^{-1}$ becomes the (unique) eigenvalues of maximum modulus for the operator $(\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}$. This leads to the possibility to apply the Kellog method (see [4]) in order to approximate $(\mu_j - \tilde{\mu})^{-1}$ and a corresponding eigenfunction. This can be done in the usual way, starting from an arbitrary function ω_0 (which theoretically should not be orthogonal to the eigenspace associated with $(\mu_j - \tilde{\mu})^{-1}$), and defining the sequence

$$\omega_{n+1} := (\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}\omega_n\,, \qquad (n = 0, 1, 2, \ldots). \tag{4.10}$$

Then (see [4]):

$$\lim_{n\to\infty} \frac{\|\omega_{n+1}\|_2}{\|\omega_n\|_2} = (\mu_j - \tilde{\mu})^{-1}\,, \lim_{n\to\infty} \frac{\omega_{2n}}{\|\omega_{2n}\|_2} = \pm\phi_j\,. \tag{4.11}$$

After computing with prescribed accuracy the eigenvalue

$$\xi_j := (\mu_j - \tilde{\mu})^{-1}\,,$$

one finds

$$\mu_j = \frac{1}{\xi_j} + \tilde{\mu}\,,$$

so that, by recalling $\mu = \lambda^{-1}$ ($\tilde{\mu} =: \tilde{\lambda}^{-1}$), we obtain for the characteristic values of the kernel the expression

$$\lambda_j = \frac{\tilde{\lambda}\xi_j}{\tilde{\lambda} + \xi_j}\,.$$

It is important to note that (as in the finite dimensional case) we can avoid the determination of the inverse operator $(\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}$, since the equation (4.10) is equivalent to

$$(\mathcal{K} - \tilde{\mu}\mathcal{I})\omega_{n+1} = \omega_n\,, \qquad n = 0, 1, 2, \ldots\,. \tag{4.12}$$

However, this leads to the necessity to solve numerically, at each step, a Fredholm integral equation of the first kind. This can be done by using different methods, namely we could use, e.g., the Fast Galerkin method, or the Nyström method. The latter method was used, since it turned out to be very simple and efficient both with respect to time and number of iterations.

The rate of convergence of the method is given by the formula:

$$\frac{\|\omega_n\|_2}{\|\omega_0\|_2} = \mathcal{O}[(\mu'(\mu_j - \tilde{\mu}))^n] \,,$$

where $\mu' \neq (\mu_j - \tilde{\mu})^{-1}$ denotes a suitable eigenvalue of $(\mathcal{K} - \tilde{\mu}\mathcal{I})^{-1}$ (see [6]).

As a matter of fact, by the numerical point of view, the use of Nyström method in the solution of equation (4.13) is substantially equivalent to the substitution of the original kernel $K(x, y)$ by an approximating kernel $\widetilde{K}(x, y)$ given by a suitably defined two-dimensional step function (i.e. instead of the original operator, we consider an approximating finite dimensional operator given by a suitable matrix).

In order to define this finite dimensional operator, and to discuss the accuracy of our approximation we introduce some notations.

Let n be the number of nodes in the application of the Nyström method, and denote by $x_1, x_2, \ldots, x_n$ (or $y_1, y_2, \ldots, y_n$) the knots of the modified Gauss-Legendre quadrature formula on the x (or y) axis, and by $w_1, w_2, \ldots, w_n$ the corresponding Christoffel constants.

In the sequel we suppose that the kernel is sufficiently regular in $Q := [0, 1] \times [0, 1]$, and precisely such that the Peano-Jordan measure of the eventual singularities of K in Q is zero. This assumption is natural, dealing with a compact operator.

Divide Q into the sub-squares $Q_{i,j}$ defined by $Q_{i,j} := \{(x, y) | \sum_{l=1}^{i-1} w_l \leq x \leq \sum_{l=1}^{i} w_l; \sum_{k=1}^{j-1} w_k \leq y \leq \sum_{k=1}^{j} w_k\}$, assuming $\sum_{s=1}^{i-1} w_s := 0$, if $i = 1$, and recalling that obviously $\sum_{s=1}^{n} w_s = 1$. Denote by $Q_{i,j}^*$ those particular sub-squares in which $K(x, y)$ is not bounded, then define

$$\widetilde{K}(x, y) = K(x_i, y_j), if \quad (x, y) \in Q_{i,j} K_{i,j} \,, if \quad (x, y) \in Q_{i,j}^* \tag{4.13}$$

where $K_{i,j}$ are such constants that

$$\|K(x, y) - \widetilde{K}(x, y)\|_{L^2(\cup_{i,j} Q_{i,j}^*)} < eps \,,$$

where *eps* denotes the smallest positive number used by the computer (i.e. the *machine epsilon*). This condition can always be satisfied provided that n is sufficiently large.

Then the numerical computation by using the inverse iteration method yields to approximating the exact eigenvalues $\tilde{\mu}_j$, $(j = 1, 2, \ldots, n)$ of the kernel $\widetilde{K}(x, y)$.

Anyway, by using the well known Aronszajn Theorem (see e.g. [3], it is possible to find an upper bound for the absolute error $|\mu_j - \tilde{\mu}_j|$, which is given simply, for every j, by the estimate

$$|\mu_j - \tilde{\mu}_j| \leq \|K(x, y) - \widetilde{K}(x, y)\|_{L^2(Q)}.$$

Then, in order to find an approximation $\tilde{\mu}_j$, which is exact, with respect to the corresponding μ_j, up to the p-th digit, it is sufficient to increase n (and eventually to use adaptive composite quadrature formulas, increasing the number of knots close to the singularities) in such a way that the further inequality $\|K(x, y) - \widetilde{K}(x, y)\|_{L^2(Q)} < 0.5 \times 10^{-p}$ holds true.

This can always be done, and permits to control the error of our approximation, independently by the use of the orthogonal invariants method.

5. *Numerical results*

By using for $\mathcal{K}$ operator the Rayleigh-Ritz method for the lower bounds and the iterative method described in Section 4 for the upper bounds, and putting $\theta = 0.5$, we have obtained the μ_k approximations of Table II.

1,971755367 $E - 002$	$< \mu_1$	$<$ 1,971755368 $E - 002$
1,193019737 $E - 003$	$< \mu_2$	$<$ 1,193020022 $E - 003$
2,368407154 $E - 004$	$< \mu_3$	$<$ 2,368413885 $E - 004$
7,515508251 $E - 005$	$< \mu_4$	$<$ 7,515705947 $E - 005$
3,083374445 $E - 005$	$< \mu_5$	$<$ 3,083849878 $E - 005$
1,488427476 $E - 005$	$< \mu_6$	$<$ 1,489425082 $E - 005$
8,039231449 $E - 006$	$< \mu_7$	$<$ 8,057498303 $E - 006$
4,714512684 $E - 006$	$< \mu_8$	$<$ 4,745498894 $E - 006$
2,944146595 $E - 006$	$< \mu_9$	$<$ 2,993474226 $E - 006$
1,932083404 $E - 006$	$< \mu_{10}$	$<$ 2,0064447 $E - 006$
1,319325614 $E - 006$	$< \mu_{11}$	$<$ 1,426094099 $E - 006$
9,308066277 $E - 007$	$< \mu_{12}$	$<$ 10,76689353 $E - 007$
6,645237989 $E - 007$	$< \mu_{13}$	$<$ 8,570001054 $E - 007$

Consequently we have found for the eigenvalues λ_k the following approximations

50,7162243109	$< \lambda_1$	$< 50{,}71623066$
837,990222744	$< \lambda_2$	$< 838{,}2091$
4222,18086328	$< \lambda_3$	$< 4222{,}247$
13211,24	$< \lambda_4$	$< 13305{,}82$
32426	$< \lambda_5$	< 32432
64914	$< \lambda_6$	< 67185
124311	$< \lambda_7$	< 124390
185084	$< \lambda_8$	< 212111
339236	$< \lambda_9$	< 339657
413523	$< \lambda_{10}$	< 517576
756935	$< \lambda_{11}$	< 757963
1058922	$< \lambda_{12}$	< 1074337
1465741	$< \lambda_{13}$	< 1504837

Remark 5.1. *The inverse iteration method has been implemented by using an algorithm written in Fortran by P. Natalini and C. Falcone.*

After some attempts we can see that the accuracy of eigenvalue approximation increases when the number of nodes and iterations is increased. Furthermore, the convergence is monotonic. Table III has been computed by using 35 nodes and 60 iterations.

Acknowledgements

This work was supported by the funds of the M.U.R.S.T.

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