ESTIMATION OF PROBABILITY DENSITY AT A POINT BY THE WEISS-WOLFOWITZ METHOD

R. Absava

Tbilisi State University
Department of Mechanics and Mathematics
2, University Str.
Tbilisi, 380043, Georgia

(Received: 05.02.99; revised: 17.10.99)

Abstract.

One special case of distribution density at a point is considered. The problem is solved by the method of parametric statistics and by the methods of maximal likelihood and moments. A class of densities of high order is considered and asymptotic efficiency of the constructed estimate is established.

Key words and phrases: Likelihood, Density of moments, Asymptotics, Competitive estimate.

AMS subject classification: 62G05.

1. *Introduction*

In the present work we consider one special case of estimation of distribution density at a point. The problem is solved by using the methods of parametric statistics, the method of maximal likelihood and the method of moments. The results obtained by Weiss and Wolfowitz [3] are generalized. We consider classes of densities W of higher order and suppose that there exist derivatives of higher order which are bounded at a point of density estimation.

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables, observable by a statistician, with an unknown density f. f is assumed to belong to a class of densities consisting of more than one element. Let A be an arbitrary point in $R = (-\infty, \infty)$. Let us construct an estimate (more precisely, a sequence of estimates) $f(A)$ under different assumptions for f in the neighborhood of the point A. Assume that the following two assumptions on the acceptability of a class of estimates are fulfilled:

$$(I). \quad \varepsilon_n = n^{-\alpha}, \quad 0 < \alpha < 1. \tag{1.1}$$

(II). All the estimates under consideration belong to the class $V(\varepsilon_n)$. The class $V(\varepsilon_n)$ consists of estimates $f(A)$ which for $n = 1, 2, \ldots$ are the functions of only those X's which found themselves in $(A - \varepsilon_n, A + \varepsilon_n)$.

Definition 1.1. *Density $f(y)$ will be called a function of the class W_s ($s \geq 2$ is a natural number), if it satisfies the following two conditions:*

(1) $0 < a_1' \leq f(A) \leq a_1'' < \infty$;

(2) in the interval $I = (A - h, A + h)$, there exist all derivatives up to the s-th order inclusive, and at the point A they are less in absolute value than some constant $a_2' > 0$, and for all $y \in I$

$$f(y) = f(A) + f'(A)(y - A) + \ldots + \frac{f^{(s)}(A)}{s!}(y - A)^s +$$

$$+\overline{f}(y)|y - A|^{s+a},$$

where $|\overline{f}(y)| \leq a_2'' < \infty$ and $0 < a < 1$.

In the interval $I = (A - h, A + h)$ we write

$$f(y) = f(A)[1 + k(y - A)], \quad y \in I,$$

where

$$k(y) = k_1 y + k_2 y^2 + \ldots + k_s y^s + O(|y|^{s+a}), \tag{1.2}$$

$$k_i = O(1), \quad i = \overline{1, s}.$$

Denote

$$K(\varepsilon_n) = \int_{-\varepsilon_n}^{\varepsilon_n} k(y)dy$$

for n such that $n^{-\alpha} < h$.

Suppose first that $K(\varepsilon_n)$ is known. Let $Y_1, Y_2, ..., Y_N$ be those among $X_1, X_2, ..., X_n$ which lie in the interval $(A - \epsilon_n, A + \epsilon_n)$ The joint function probability of N at m and probability density function of $Y_1, Y_2, \ldots, Y_m$ at $y_1, y_2, \ldots, y_m$ is

$$\frac{n!}{m!(n-m)!}[f(A)(2\varepsilon_n + K(\varepsilon_n))]^m [1 - f(A)(2\varepsilon_n + K(\varepsilon_n))]^{n-m} \times$$

$$\times \prod_{i=1}^{m} \frac{f(A)[1 + k(y_i - A)]}{f(A)[2\varepsilon_n + K(\varepsilon_n)]},$$

from which we obtain the maximal likelihood estimator $\widehat{f}_n$ for $f(A)$:

$$\widehat{f}_n = \frac{N}{n(2\varepsilon_n + K(\varepsilon_n))}. \tag{1.3}$$

It is clear that $E\widehat{f}_n = f(A)$. In view of (1.1) we find that

$$\sigma^2(\widehat{f}_n) = \Omega(n^{\alpha-1}). \tag{1.4}$$

Remark 1.1. *Everywhere in what follows the use will be made of the following notation:* $\Psi = O(u^r)$ *denotes that* $|\Psi n^{-r}|$ *is bounded above uniformly with respect to* n *and to all* f *from the class* W_s. $\Psi = \Omega(n^r)$ *denotes that* $|\Psi n^{-r}|$ *is bounded above and below uniformly with respect to* n *and* f *in* W_s. *Finally,* O_p, Ω_p *denote that* O, Ω *hold respectively with probability which can be chosen arbitrarily close to unity.* According to the Mouavry-Laplace theorem, the distribution

$$[\widehat{f}_n - f(A)] \cdot D(\widehat{f}_n)^{-1/2}$$

tends to the normal distribution with the mean 0 and dispersion 1; note that the normalizing factor $(D\widehat{f}_n)^{1/2} = \Omega(n^{(1-\alpha)/2})$ and the random variable $N = \Omega_p(n^{1-\alpha})$. It follows from Theorem 3.1 [4] that $\widehat{f}_n$ is asymptotically effective in the sense that for all competitive estimates T_n, satisfying both

$$\lim_{n\to\infty}\left[P\Big\{\gamma(n)(T_n-\theta) \le -\frac{r}{2}\,\Big|\,\theta\Big\} - P\Big\{\gamma(n)\Big(T_n-\theta-\frac{r}{\gamma(n)}\Big) \le \right.$$
$$\left. \le -\frac{r}{2}\,\Big|\,\theta+\frac{r}{\gamma(n)}\Big\}\right] = 0, \tag{1.5}$$

with $\gamma(n) = n^{\alpha-1)/2}$ and assumptions (I) and (II), and for any fixed $r > 0$ we have

$$\lim_{n\to\infty} P\Big\{-rn^{\frac{\alpha-1}{2}} < \widehat{f}_n - f(A) < rn^{\frac{\alpha-1}{2}}\Big\} \ge$$
$$\ge \lim_{n\to\infty} P\Big\{-rn^{\frac{\alpha-1}{2}} < T_n - f(A) < rn^{\frac{\alpha-1}{2}}\Big\} \tag{1.6}$$

Consider now the case, in which $K(\varepsilon_n)$ is unknown. It follows from (1.2) that

$$K(\varepsilon_n) = \begin{array}{l} \frac{2}{3}k_2\varepsilon_n^3 + \frac{2}{5}k_4\varepsilon_n^5 + \ldots + \frac{2}{s}k_{s-1}\varepsilon_n^s + O(\varepsilon_n^{s+a+1}),\ \ if\ \ s\ \ is\ \ odd, \\ \frac{2}{3}k_2\varepsilon_n^3 + \frac{2}{5}k_4\varepsilon_n^5 + \ldots + \frac{2}{s+1}k_s\varepsilon_n^{s+1} + O(\varepsilon_n^{s+a+1}),\ \ if\ \ s\ \ is\ \ even. \end{array}$$

Consider the case, where s is odd, and find estimates of the parameters $k_2, k_4, \ldots, k_{s-1}$.

To obtain estimates of $\widehat{k}_2, \widehat{k}_4, \ldots, \widehat{k}_{s-1}$, we act as follows: let $\overline{J} = (A - n^{-\beta}, A + n^{-\beta})$, $\beta < \alpha$. Let $\underline{\underline{Z}}_1, \underline{\underline{Z}}_2, \ldots, \underline{\underline{Z}}_{M(n)}$ be those of the observed X's which found themselves in $\overline{J}$. It is clear that the conditional density at the point $x = y + A$ of the interval $\overline{J}$ is

$$f^*(y + A/\overline{J}) =$$

$$= \frac{1 + k_1 y + k_2 y^2 + \ldots + k_s y^s + O(y^{s+a})}{2n^{-\beta} + \frac{2}{3}k_2 n^{-3\beta} + \ldots + \frac{2}{s}k_{s-1}n^{-s\beta} + O(n^{-\beta(s+1+a)})}. \tag{1.7}$$

Therefore

$$E|\underline{\underline{Z}}_i - A|^m = \frac{\int_{-n^{-\beta}}^{n^{-\beta}} |y|^m [1 + k_1 y + k_2 y^2 + \ldots + k_s y^s + O(y^{s+a})]dy}{2n^{-\beta} + \frac{2k_2}{3}n^{-3\beta} + \ldots + \frac{2}{s}k_{s-1}n^{-s\beta} + O(n^{-\beta(s+1+a)})},$$

where $1 \leq m \leq \frac{s-1}{2}$, $s > 2$.

It is not difficult to calculate that

$$E|\underline{\underline{Z}}_i - A|^m =$$

$$= \frac{\frac{1}{m+1}n^{-m\beta} + \frac{k_2}{m+3}n^{-(m+2)\beta} + \ldots + \frac{k_{s-1}}{m+s}n^{-\beta(m-1+s)} + O(n^{-\beta(m+s+a)})}{1 + \frac{1}{3}k_2 n^{-2\beta} + \frac{k_4}{5}n^{-4\beta} + \ldots + \frac{1}{s}k_{s-1}n^{-\beta(s-1)} + O(n^{-\beta(s+a)})}.$$

Denote

$$l_{nm}(k_2, \ldots, k_{s-1}) = \frac{1}{m+1}n^{-m\beta} + \frac{k_2}{m+3}n^{-(m+2)\beta} + \ldots +$$

$$+ \frac{k_{s-1}}{m+s}n^{-\beta(m-1+s)} + O(n^{-\beta(m+s+a)})$$

$$\gamma_n(k_2, k_4, \ldots, k_{s-1}) = \frac{1}{3}k_2 n^{-2\beta} + \frac{k_4}{5}n^{-4\beta} + \ldots + \frac{1}{s}k_{s-1}n^{-\beta(s-1)} +$$

$$+ O(n^{-\beta(s+a)}). \tag{1.8}$$

Then

$$E|\underline{\underline{Z}}_i - A|^m = l_{nm}(k_2, \ldots, k_{s-1})[1 - \gamma_n + \gamma_n^2 + \cdots]. \tag{1.9}$$

Denote

$$Q_{nm} = \frac{1}{M(n)} \sum_{j=1}^{M(n)} |\underline{\underline{Z}}_j - A|^m, \quad m = \overline{1, \frac{s-1}{2}}, \quad s > 2.$$

Next, using the method of moments for estimating the unknown parameters, we construct a system of equations

$$l_{n1}(k_2, k_4, \ldots, k_{s-1})[1 - \gamma_n + \gamma_n^2 + \cdots] = Q_{n1}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$l_{n\frac{s-1}{2}}(k_2, k_4, \ldots, k_{s-1})[1 - \gamma_n + \gamma_n^2 + \cdots] = Q_{n\frac{s-1}{2}} \tag{1.10}$$

Retaining in (1.10) the terms up to the order $n^{-\beta(m-1+s)}$ inclusive, $m = \overline{1, \frac{s-1}{2}}$, and solving the system with respect to $k_2, k_4, \dots, k_{s-1}$, we obtain estimates of the parameters $\widehat{k}_2, \widehat{k}_4, \dots, \widehat{k}_{s-1}$.

Thus, for the estimate of $K(\varepsilon_n)$ we can take

$$\widehat{K}(\varepsilon_n) = \frac{2}{3}\,\widehat{k}_2\varepsilon_n^3 + \ldots + \frac{2}{s}\,\widehat{k}_{s-1}\varepsilon_n^s,$$

and for the estimate of $f(A)$

$$\widehat{f}^* = \frac{N}{n[2\varepsilon_n + \widehat{K}(\varepsilon_n)]}. \tag{1.11}$$

The case, where s is an even number, is considered analogously.

Consider at greater length a particular case, for example, when $s = 5$, i.e., $f \in W_5$.

From (1.8) we obtain

$$l_{n1}(k_2, k_4) = \frac{1}{2}\,n^{-\beta} + \frac{k_2}{4}\,n^{-3\beta} + \frac{k_4}{6}\,n^{-5\beta} + O(n^{-\beta(6+a)}),$$

$$l_{n2}(k_2, k_4) = \frac{1}{3}\,n^{-2\beta} + \frac{k_2}{5}\,n^{-4\beta} + \frac{k_4}{7}\,n^{-6\beta} + O(n^{-\beta(7+a)}),$$

$$\gamma_n(k_2, k_4) = \frac{1}{3}\,k_2 n^{-2\beta} + \frac{k_4}{5}\,n^{-4\beta} + O(n^{-\beta(5+a)}).$$

Hence the system of equations (1.10) takes the form

$$\frac{1}{2}\,n^{-\beta} + \frac{1}{12}\,k_2 n^{-3\beta} + \frac{12k_4 - 5k_2^2}{180}\,n^{-5\beta} = Q_{n1},$$

$$\frac{1}{3}\,n^{-2\beta} + \frac{4}{45}\,k_2 n^{-4\beta} + \frac{4(18k_4 - 7k_2^2)}{945}\,n^{-6\beta} = Q_{n2},$$

and the solution of that system will be

$$\widehat{k}_2 = \frac{3}{2}\,n^{2\beta}\Big[1 - \sqrt{1 - \frac{4}{3}n^{-2\beta}(\Theta_{n1} - 14n^{-2\beta}\Theta_{n3})}\,\Big],$$

$$\widehat{k}_4 = \frac{35}{2}\,\Theta_{n3},$$

where

$$\Theta_{n1} = 12n^{2\beta}(n^{\beta}Q_{n1} - 1/2),$$

$$\Theta_{n3} = n^{2\beta}(\Theta_{n2} - \Theta_{n1}),$$

$$\Theta_{n2} = \frac{45}{4}\,n^{2\beta}(n^{2\beta}Q_{n2} - 1/3), \tag{1.12}$$

Further, from (1.8) and (1.9) we get

$$EQ_{n1} = \frac{1}{2}\, n^{-\beta}\Big[1 + \frac{k_2}{6}\, n^{-2\beta} + \frac{12k_4 - 5k_2^2}{90}\, n^{-4\beta} + O(n^{-\beta(5+a)})\Big],$$

$$EQ_{n2} = \frac{1}{3}\, n^{2\beta}\Big[1 + \frac{4}{15}\, n^{-2\beta} + \frac{4(18k_4 - 7k_2^2}{315}\, n^{-4\beta} + O(n^{-\beta(5+a)})\Big],$$

$$DQ_{n1} = \Omega(n^{-(1+\beta)}),$$

$$DQ_{n2} = \Omega(n^{-(1+3\beta)}).$$

Therefore

$$Q_{n1} = \frac{1}{2}\, n^{-\beta}\Big[1 + \frac{k_2}{6}\, n^{-2\beta} + \frac{12k_4 - 5k_2^2}{90}\, n^{-4\beta} + O(n^{-\beta(5+a)})\Big] +$$

$$+\Omega_p(n^{-\frac{(1+\beta)}{2}}),$$

$$Q_{n2} = \frac{1}{3}\, n^{-2\beta}\Big[1 + \frac{4}{15}\, k_2 n^{-2\beta} + \frac{4(18k_4 - 7k_2^2)}{315}\, n^{-4\beta} + O(n^{-\beta(5+a)})\Big] +$$

$$+\Omega_p(n^{-\frac{(1+3\beta)}{2}}).$$

This and equations (1.12) obviously imply

$$\Theta_{n1} = k_2 + \frac{12k_4 - 5k_2^2}{15} n^{-2\beta} + O(n^{-\beta(3+a)}) + \Omega_p(n^{-\frac{(1-5\beta)}{2}}),$$

$$\Theta_{n2} = k_2 + \frac{18k_4 - 7k_2^2}{21} n^{-2\beta} + O(n^{-\beta(3+a)}) + \Omega_p(n^{-\frac{(1-5\beta)}{2}})$$

$$\Theta_{n3} = \frac{35}{2} k_4 + O(n^{-\beta(1+a)}) + \Omega_p(n^{-\frac{(1-9\beta)}{2}}).$$

Hence

$$\widehat{k}_2 = k_2 + O(n^{-\beta(2+a)}) + \Omega_p(n^{-\frac{(1-5\beta)}{2}}),$$

$$\widehat{k}_4 = k_4 + O(n^{-\beta(1+a)}) + \Omega_p(n^{-\frac{(1-9\beta)}{2}}).$$

Denote

$$D_n = K(\varepsilon_n) - \widehat{K}(\varepsilon) = \frac{2}{3}(k_2 - \widehat{k}_2)\varepsilon_n^3 + \frac{2}{5}(k_4 - \widehat{k}_4)\varepsilon_n^5 + O(\varepsilon_n^{6+a}) =$$

$$= O(n^{-\beta(2+a)}) + \Omega_p(n^{-3\alpha - \frac{(1-5\beta)}{2}}) + O(n^{-5\alpha - \beta(1+a)}) +$$

$$+\Omega_p(n^{-5\alpha - \frac{(1-9\beta)}{2}}) + O(n^{-\alpha(6+a)}). \tag{1.13}$$

Then

$$\widehat{f}_n{}^* - \widehat{f}_n = \frac{ND_n}{n}[2\varepsilon_n + K(\varepsilon_n)]^{-1}[2\varepsilon + K(\varepsilon_n) - D_n]^{-1}$$

and owing to the fact that $N = \Omega_p(n^{1-\alpha})$, from the latter equation and from (1.13) we obtain

$$\widehat{f_n}^* - \widehat{f_n} = O(n^{-\beta(2+a)-2\alpha}) + \Omega_p(n^{-2\alpha-\frac{(1-5\beta)}{2}}) +$$

$$+ O(n^{-4\alpha-\beta(1+a)}) + \Omega_p(n^{-4\alpha-\frac{(1-9\beta)}{2}}) +$$

$$+ O(n^{-\alpha(5+a)}). \tag{1.14}$$

Consider now the problem dealing with the sampling of α which in the sequel will allow us to calculate $\widehat{f_n}^*$. But for this purpose we have to maintain the validity of the equation

$$\lim_{n\to\infty} P\{-rn^{\frac{\alpha-1}{2}} < \widehat{f_n}^* - f(A) < rn^{\frac{\alpha-1}{2}}\} =$$

$$= \lim_{n\to\infty} P\{-rn^{\frac{\alpha-1}{2}} < \widehat{f_n} - f(A) < rn^{\frac{\alpha-1}{2}}\} \tag{1.15}$$

for any fixed $r > 0$, since in this case from (1.6) we have

$$\lim_{n\to\infty} P\{-rn^{\frac{\alpha-1}{2}} < \widehat{f_n}^* - f(A) < rn^{\frac{\alpha-1}{2}}\} \geq$$

$$\geq \lim_{n\to\infty} P\{-rn^{\frac{\alpha-1}{2}} < T_n - f(A) < rn^{\frac{\alpha-1}{2}}\}$$

for any $r > 0$ and any competitive estimate T_n, such as indicated in (1.5).

In order for (1.15) to be fulfilled, in view of (1.4) it is sufficient that

$$\widehat{f_n}^* - \widehat{f}(A) = o_p(n^{\frac{\alpha-1}{2}}). \tag{1.16}$$

By virtue of (1.14), equation (1.16) is fulfilled if

$$\alpha > \beta$$

$$5\alpha + 4\beta > 1 - 2a\beta$$

$$11\alpha > 1 - 2\alpha a$$

$$9\alpha + 2\beta(1+a) > 1. \tag{1.17}$$

If a is unknown, then a number greater than $\frac{1}{11}$ is considered to be satisfactory sampling of α, while β we choose such that $1/11 < \beta < \alpha$. If a is known, we choose α and β such that conditions (1.17) be fulfilled.

Remark 1.2. *Cases W_1, W_2 and W_3 were considered in [3]. Case W_4 can be found in [1].*

2. *Multivariate Case.*

Let $X_i = (X_1^{(i)}, X_2^{(i)})$ be independent, uniformly distributed two-dimensional random variables with an unknown density $f(y_1, y_2)$. Let $A = (A_1, A_2)$ be an arbitrary fixed point. Construct an estimate of $f(A_1, A_2)$ under different (as in one-dimensional case) assumptions:

(I). $\varepsilon_n = n^{-\alpha}$, $\alpha > 0$,

(II). All the estimates under consideration belong to the class $V(\varepsilon_n)$. The class $V(\varepsilon_n)$ consists of the estimates $f(A_1, A_2)$ which for $n = 1, 2, \ldots$ are the functions of only those X's which found themselves in $(A_1 - \varepsilon_n, A_1 + \varepsilon_n) \times (A_2 - \varepsilon_n, A_2 + \varepsilon_n)$.

Definition 2.1. *Density f is said to be a function of the class W_s $(s \geq 1)$ if it satisfies the following conditions:*

1) $0 < a_1' \leq f(A_1, A_2) \leq a_1'' < \infty$;

2) in the interval $I = (A_1 - h, A_1 + h) \times (A_2 - h, A_2 + h)$ there exist all partial derivatives of the s-th order of density $f(y_1, y_2)$, and at the point $A = (A_1, A_2)$ they all are less in absolute value than a constant $a_2' > 0$, and for any $y \in I$

$$f(y) = f(A) + \sum_{p=1}^{s} \Big((y_1 - A_1)\frac{\partial}{\partial y_1} + (y_2 - A_2)\frac{\partial}{\partial y_2} \Big)^{(p)} f(A) +$$

$$+\overline{f}(y)(|y_1|^{s+a} + |y_2|^{s+a}),$$

where $|\overline{f}(y)| \leq a_2'' < \infty$, $0 < a < 1$.

In the interval $I = (A_1 - h, A_1 + h) \times (A_2 - h, A_2 + h)$ we write

$$f(y_1, y_2) = f(A_1, A_2)[1 + k(y_1 - A_1, y_2 - A_2)],$$

and

$$K(\varepsilon_n) = \int_{-\varepsilon_n}^{\varepsilon_n} \int_{-\varepsilon_n}^{\varepsilon_n} k(y_1, y_2)\, dy_1\, dy_2,$$

for n such that $n^{-\alpha} < h$.

Suppose first that $K(\varepsilon_n)$ is known. Denote by $Y_1, Y_2, \ldots, Y_N$ those of X_i $i = \overline{1, n}$ which found themselves in $I = (A_1 - \varepsilon_n, A_1 + \varepsilon_n) \times (A_2 - \varepsilon_n, A_2 + \varepsilon_n)$. The joint function of probabilities N in m and the density function of probabilities $Y_1, Y_2, \ldots, Y_m$ at $y_1, y_2, \ldots, y_m$ is

$$\frac{n!}{m!(n-m)!}[f(A)(4\varepsilon_n^2 + K(\varepsilon_n))]^m [1 - f(A)(4\varepsilon_n^2 + K(\varepsilon_n))]^{n-m} \times$$

$$\times \prod \frac{f(A)[1 + k(y_1^{(i)} - A_1, y_2^{(i)} - A_2)]}{f(A)[4\varepsilon_n^2 + K(\varepsilon_n)]}.$$

From this we obtain the estimate of maximal likelihood of $f(A)$,

$$\widehat{f}_n = \frac{N}{n[4\varepsilon_n^2 + K(\varepsilon_n)]}.$$

Obviously, $E\widehat{f}_n = f(A_1, A_2)$, $D(\widehat{f}_n) = \Omega(n^{2\alpha-1})$ and the random variable $N = \Omega_p(n^{1-2\alpha})$, $0 < \alpha < \frac{1}{2}$. Here again, by virtue of Theorem 3.1 [4], it follows that $\widehat{f}_n$ is asymptotically effective in the sense that

$$\lim_{n\to\infty} P\{-r\cdot n^{\frac{2\alpha-1}{2}} < \widehat{f}_n - f(A) < r\cdot n^{\frac{2\alpha-1}{2}}\} \geq$$

$$\geq \lim_{n\to\infty} P\{-r\cdot n^{\frac{2\alpha-1}{2}} < T_n - f(A) < r\cdot n^{\frac{2\alpha-1}{2}}\} \tag{2.1}$$

for all competitive estimates T_n satisfying the condition (1.5) and the assumptions (I) and (II).

Remark 2.1. *Cases W_1, W_2 and W_3 were considered in [1].*

Consider now the problem arising in the case, where $K(\varepsilon_n)$ is unknown and $f \in W_4$. Since $f \in W_4$, for $y \in I$ we have

$$k(y_1, y_2) = k_1y_1 + k_2y_2 + l_1y_1^2 + l_2y_2^2 +$$

$$+ m_{11}y_1y_2 + m_{30}y_1^3 + m_{21}y_1^2y_2 + m_{12}y_1y_2^2 +$$

$$+ m_{03}y_2^3 + \overline{M}_4y_1^4 + M_{31}y_1^3y_2 + M_{22}y_1^2y_2^2 +$$

$$+ M_{13}y_1y_2^3 + \overline{\overline{M}}_4y_4^4 + O(|y_1|^{4+a} + |y_2|^{4+a}).$$

Then

$$K(\varepsilon_n) = \frac{4}{3}(l_1 + l_2)\varepsilon_n^4 + \Big[\frac{4}{5}(\overline{M}_4 + \overline{\overline{M}}_4) + \frac{4}{9}M_{22}\Big]\varepsilon_n^6 + O(\varepsilon_n^{6+a}).$$

In order to obtain estimates $\widehat{l_1 + l_2}$, $\widehat{\overline{M}}_4$, $\widehat{\overline{\overline{M}}}_4$ and $\widehat{M}_{22}$ of parameters $l_1 + l_2$, $\overline{M}_4$, $\overline{\overline{M}}_4$ and M_{22} we again apply the method of moments and consider the interval $J^* = (A_1 - n^{-\beta}, A_1 + n^{-\beta}) \times (A_2 - n^{-\beta}, A_2 + n^{-\beta})$, $\beta < \alpha$. Let $\underline{\underline{Z}}_1, \underline{\underline{Z}}_2, \ldots, \underline{\underline{Z}}_{M(n)}$ be those of observed X's which found themselves in J^*. A conditional density at the point $(y_1 + A_1, y_2 + A_2)$ of the interval J^* is

$$f^*(y + A/J^*) =$$

$$= \frac{1 + k(y_1, y_2)}{4n^{-2\beta} + \frac{4}{3}(l_1 + l_2)n^{-4\beta} + [\frac{4}{5}(\overline{M}_4 + \overline{\overline{M}}_4) + \frac{4}{9}M_{22}]n^{-6\beta} + O(n^{-\beta(6+a)})}.$$

Therefore

$$E|\underline{\underline{Z}}_1^{(i)} - A_1| = \frac{1}{2}n^{-\beta} + \frac{l_1}{12}n^{-3\beta} - \Big[\frac{1}{36}(l_1 + l_2)l_1 - \frac{1}{15}\overline{M}_4 -$$

$$-\frac{1}{36}M_{22}\Big]n^{-5\beta}+O(n^{-\beta(5+a)}),$$

$$E|\underline{\underline{Z}}_2^{(i)}-A_2|=\frac{1}{2}n^{-\beta}+\frac{l_2}{12}n^{-3\beta}-\Big[\frac{1}{36}(l_1+l_2)l_2-\frac{1}{15}\overline{\overline{M}}_4-$$

$$-\frac{1}{36}M_{22}\Big]n^{-5\beta}+O(n^{-\beta(5+a)}),$$

$$E|\underline{\underline{Z}}_1^{(i)}-A_1|^2=\frac{1}{3}n^{-2\beta}+\frac{4}{45}l_1n^{-4\beta}-\Big[\frac{4}{135}(l_1+l_2)l_1-\frac{8}{105}\overline{M}_4-$$

$$-\frac{4}{105}M_{22}\Big]n^{-6\beta}+O(n^{-\beta(6+a)}),$$

$$E|\underline{\underline{Z}}_2^{(i)}-A_2|^2=\frac{1}{3}n^{-2\beta}+\frac{4}{45}l_2n^{-4\beta}-\Big[\frac{4}{135}(l_1+l_2)l_2+$$

$$+2cm-\frac{8}{105}\overline{\overline{M}}_4-\frac{4}{105}M_{22}\Big]n^{-6\beta}+O(n^{-\beta(6+a)}),$$

$$E|\underline{\underline{Z}}_1^{(i)}-A_1||\underline{\underline{Z}}_2^{(i)}-A_2|=\frac{1}{4}n^{-2\beta}+\frac{1}{24}(l_1+l_2)n^{-4\beta}-\Big[\frac{1}{72}(l_1+l_2)^2-$$

$$-\frac{1}{30}(\overline{M}_4+\overline{\overline{M}}_4)-\frac{5}{4\cdot 36}M_{22}\Big]n^{-6\beta}+O(n^{-\beta(6+a)}),$$

$$E|\underline{\underline{Z}}_j^{(i)}-A_j|^4=\frac{n^{-4\beta}}{5}(1+O(n^{-\beta})),\quad j=1,2.$$

Denote

$$Q_{n10}=\frac{1}{M(n)}\sum_{i=1}^{M(n)}|\underline{\underline{Z}}_1^{(i)}-A_1|,\quad Q_{n01}=\frac{1}{M(n)}\sum_{i=1}^{M(n)}|\underline{\underline{Z}}_2^{(i)}-A_2|$$

$$Q_{n20}=\frac{1}{M(n)}\sum_{i=1}^{M(n)}|\underline{\underline{Z}}_1^{(i)}-A_1|^2,\quad Q_{n02}=\frac{1}{M(n)}\sum_{i=1}^{M(n)}|\underline{\underline{Z}}_2^{(i)}-A_2|^2$$

$$Q_{n11}=\frac{1}{M(n)}\sum_{i=1}^{M(n)}|\underline{\underline{Z}}_1^{(i)}-A_1||\underline{\underline{Z}}_2^{(i)}-A_2|.$$

Then

$$EQ_{n10}=\frac{1}{2}n^{-\beta}+\frac{l_1}{12}n^{-3\beta}-\Big[\frac{1}{36}(l_1+l_2)l_1-\frac{1}{15}\overline{M}_4-\frac{1}{36}M_{22}\Big]n^{-5\beta}+$$

$$+O(n^{-\beta(5+a)}),$$

$$EQ_{n01}=\frac{1}{2}n^{-\beta}+\frac{l_2}{12}n^{-3\beta}-\Big[\frac{1}{36}(l_1+l_2)l_2-\frac{1}{15}\overline{\overline{M}}_4-\frac{1}{36}M_{22}\Big]n^{-5\beta}+$$

$$+O(n^{-\beta(5+a)}),$$

$$EQ_{n20} = \frac{1}{3}n^{-2\beta} + \frac{4}{45}l_1 n^{-4\beta} - \Big[\frac{4}{135}(l_1 + l_2)l_1 - \frac{8}{105}\overline{M}_4 -$$
$$-\frac{4}{105}M_{22}\Big]n^{-6\beta} + O(n^{-\beta(6+a)}),$$
$$EQ_{n02} = \frac{1}{3}n^{-2\beta} + \frac{4}{45}l_2 n^{-4\beta} - \Big[\frac{4}{135}(l_1 + l_2)l_2 -$$
$$-\frac{8}{105}\overline{\overline{M}}_4 - \frac{4}{105}M_{22}\Big]n^{-6\beta} + O(n^{-\beta(6+a)}),$$
$$EQ_{n11} = \frac{1}{4}n^{-2\beta} + \frac{1}{24}(l_1 + l_2)n^{-4\beta} - \Big[\frac{1}{72}(l_1 + l_2)^2 -$$
$$-\frac{1}{30}(\overline{M}_4 + \overline{\overline{M}}_4) - \frac{5}{4\cdot 36}M_{22}\Big]n^{-6\beta} + O(n^{-\beta(6+a)}),$$
$$DQ_{n10} = O\Big(\frac{1}{n}\Big), \quad DQ_{n01} = O\Big(\frac{1}{n}\Big),$$
$$DQ_{n20} = O(n^{-(1+2\beta)}), \quad DQ_{n02} = O(n^{-(1+2\beta)}),$$
$$DQ_{n11} = O(n^{-(1+2\beta)}).$$

Therefore

$$Q_{n10} = EQ_{n10} + \Omega_p(n^{1/2}),$$
$$Q_{n01} = EQ_{n01} + \Omega_p(n^{-1/2}),$$
$$Q_{n20} = EQ_{n20} + \Omega_p(n^{-(1/2+\beta)}),$$
$$Q_{n02} = EQ_{n02} + \Omega_p(n^{-(1/2+\beta)}),$$
$$Q_{n11} = EQ_{n11} + \Omega_p(n^{-(1/2+\beta)}).$$

Denote

$$T_{n10} = 12n^{2\beta}(n^{\beta}Q_{n10} - 1/2),$$
$$T_{n01} = 12n^{2\beta}(n^{\beta}Q_{n01} - 1/2),$$
$$T_{n02} = \frac{45}{4}n^{2\beta}(n^{2\beta}Q_{n20} - 1/3),$$
$$T_{n02} = \frac{45}{4}n^{2\beta}(n^{2\beta}Q_{n02} - 1/3),$$
$$T_{n11} = 24n^{2\beta}(n^{2\beta}Q_{n11} - 1/4).$$

Then

$$T_{n10} = l_1 - \Big[\frac{1}{3}(l_1 + l_2)l_1 - \frac{4}{5}\overline{M}_4 - \frac{1}{3}M_{22}\Big]n^{-2\beta} +$$
$$+O(n^{-\beta(2+a)}) + \Omega_p(n^{-1/2+3\beta}),$$
$$T_{n01} = l_2 - \Big[\frac{1}{3}(l_1 + l_2)l_2 - \frac{4}{5}\overline{\overline{M}}_4 - \frac{1}{3}M_{22}\Big]n^{-2\beta} +$$

$$+O(n^{-\beta(2+a)})+\Omega_p(n^{-1/2+3\beta}),$$

$$T_{n20}=l_1-\Big[\frac{1}{3}(l_1+l_2)l_1-\frac{6}{7}\overline{M}_4-\frac{3}{7}M_{22}\Big]n^{-2\beta}+$$
$$+O(n^{-\beta(2+a)})+\Omega_p(n^{-1/2+3\beta}),$$

$$T_{n02}=l_2-\Big[\frac{1}{3}(l_1+l_2)l_2-\frac{6}{7}\overline{\overline{M}}_4-\frac{3}{7}M_{22}\Big]n^{-2\beta}+$$
$$+O(n^{-\beta(2+a)})+\Omega_p(n^{-1/2+3\beta}),$$

$$T_{n11}=l_1+l_2-\Big[\frac{1}{3}(l_1+l_2)^2-\frac{4}{5}(\overline{M}_4+\overline{\overline{M}}_4)-\frac{5}{6}M_{22}\Big]n^{-2\beta}+$$
$$+O(n^{-\beta(2+a)})+\Omega_p(n^{-1/2+3\beta}).$$

Denote

$$L_n=T_{n11}-(T_{n01}+T_{n10}),$$
$$Q_n^{(1)}=7T_{n20}-3T_{n10}-4(T_{n11}-T_{n01}),$$
$$Q_n^{(2)}=7T_{n02}-3T_{n01}-4(T_{n11}-T_{n10}).$$

and

$$\widehat{M}_{22}=6n^{2\beta}L_n$$
$$\widehat{\overline{M}}_4=\frac{5}{2}n^{2\beta}Q_n^{(1)}$$
$$\widehat{\overline{\overline{M}}}_4=\frac{5}{2}n^{2\beta}Q_n^{(2)}.$$

Then

$$\widehat{M}_{22}=M_{22}+O(n^{-\beta a})+\Omega_p(n^{-1/2+5\beta}),$$
$$\widehat{\overline{M}}_4=\overline{M}_4+O(n^{-\beta a})+\Omega_p(n^{-1/2+5\beta}),$$
$$\widehat{\overline{\overline{M}}}_4=\overline{\overline{M}}_4+O(n^{-\beta a})+\Omega_p(n^{-1/2+5\beta}). \tag{2.2}$$

Suppose

$$l_1\widehat{+}l_2=\frac{3}{2}n^{2\beta}\Big(1-\sqrt{1-\frac{4}{3}[T_{n11}-2(Q_n^{(1)}+Q_n^{(2)})-5L_n]n^{-2\beta}}\Big).$$

Then

$$l_1\widehat{+}l_2=l_1+l_2+O(n^{-\beta(2+a)})+\Omega_p(n^{-1/2+3\beta}). \tag{2.3}$$

In the capacity of the estimate $K(\varepsilon_n)$ we consider

$$\widehat{K}(\varepsilon_n)=\frac{4}{3}(l_1\widehat{+}l_2)\varepsilon_n^4+\Big[\frac{4}{5}(\widehat{\overline{M}}_4+\widehat{\overline{\overline{M}}}_4)+\frac{4}{9}\widehat{M}_{22}\Big]\varepsilon_n^6.$$

Denote

$$D_n = \widehat{K}(\varepsilon_n) - K(\varepsilon_n) = \frac{4}{3}[l_1\widehat{+}l_2 - (l_1 + l_2)]\varepsilon_n^4 +$$

$$+\frac{4}{5}(\widehat{\overline{M}}_4 - \overline{M}_4 + \widehat{\overline{\overline{M}}}_4 - \overline{\overline{M}}_4)\varepsilon_n^6 + \frac{4}{9}(\widehat{M}_{22} - M_{22})\varepsilon_n^6.$$

from which and from (2.1) and (2.2) it follows that

$$D_n = O(n^{-\beta(2+a)-4\alpha}) + \Omega_p(n^{-1/2+3\beta-4\alpha}) +$$

$$+O(n^{-\beta a-6\alpha}) + \Omega_p(n^{-1/2+5\beta-6\alpha}) + O(n^{-\alpha(6+a)}). \qquad (2.4)$$

Since $N = \Omega_p(n^{1-2\alpha})$ and

$$\widehat{f}_n{}^* - \widehat{f}_n = \frac{ND_n}{n}[4\varepsilon_n^2 + K(\varepsilon_n)]^{-1}[4\varepsilon_n^2 + K(\varepsilon_n) - D_n]^{-1},$$

from (2.3) we find that

$$\widehat{f}_n{}^* - \widehat{f}_n = \Omega_p(n^{-\beta(2+a)-2\alpha}) + \Omega_p(n^{-1/2+3\beta-4\alpha}) +$$

$$+O(n^{-\beta a-4\alpha}) + \Omega_p(n^{-1/2+5\beta-4\alpha}) + O(n^{-\alpha(4+a)}). \qquad (2.5)$$

It is desirable to maintain the validity of the equality

$$\lim_{n\to\infty} P\{-rn^{2\alpha-1} < \widehat{f}_n{}^* - f(A) < rn^{\frac{2\alpha-1}{2}}\} =$$

$$= \lim_{n\to\infty} P\{-rn^{\frac{2\alpha-1}{2}} < \widehat{f} - f(A) < rn^{\frac{2\alpha-1}{2}}\}, \qquad (2.6)$$

since in this case from (1.18) we have

$$\lim_{n\to\infty} P\{-rn^{2\alpha-1} < \widehat{f}_n{}^* - f(A) < rn^{\frac{2\alpha-1}{2}}\} \geq$$

$$\geq \lim_{n\to\infty} P\{-rn^{2\alpha-1} < T_n - f(A) < rn^{\frac{2\alpha-1}{2}}\}$$

for any fixed $r > 0$ and any competitive estimate T_n, such as indicated above.

In order for (2.5) to be fulfilled, owing to $D\widehat{f}_n = \Omega(n^{2\alpha-1})$ it is sufficient that

$$\widehat{f}_n{}^* - \widehat{f}_n = o_p(n^{\frac{2\alpha-1}{2}}). \qquad (2.7)$$

Therefore from (2.4) we find that

$$\beta < \alpha \qquad (2.8)$$

$$6\alpha + 2\beta(2+a) > 1$$

$$\beta a + 5\alpha > \frac{1}{2}$$
$$\alpha(5 + a) > \frac{1}{2}.$$

If a is unknown, then a number nearly greater than $\frac{1}{10}$ will be satisfactory sampling of α, and we choose β such that $1/10 < \beta < \alpha < \frac{1}{2}$. If a is known, we choose α and β such that conditions (2.7) be fulfilled. Condition (2.6) is fulfilled for such α's. This means that $\widehat{f_n}^*$ is the asymptotically effective estimate for $f(A)$.

Remark 2.2. *Compare the obtained by us estimate of* $\widehat{f_n}^*$ *with the estimate of the type [2]:*

$$\overline{f}_n = \frac{N}{4n\overline{\varepsilon}_n^2}.$$

It is not difficult to verify that

$$E\overline{f}_n = f(A)\Big\{1 + \frac{1}{4}n^{-2\overline{\alpha}}\Big[\frac{4}{3}(l_1 + l_2) + o(1)\Big]\Big\},$$

and

$$D\overline{f}_n = f(A)\frac{1}{4}n^{2\overline{\alpha}-1}[1 + 2^{-2}\overline{\varepsilon}_n^{-2}K(\overline{\varepsilon}_n)]\{1 - f(A)[4\overline{\varepsilon}_n^2 + K(\overline{\varepsilon}_n)]\}.$$

To avoid confusion in expressions involving α, we write everywhere $\overline{\alpha}$ instead of α.

It is easily seen that $\overline{f}_n$ is asymptotically normal with the mean

$$f(A)\Big\{1 + 2^{-2}n^{-2\overline{\alpha}}\Big(\frac{4}{3}(l_1 + l_2 + o(1)\Big)\Big\}$$

and with the standard deviation

$$\sqrt{2^{-2}f(A)}n^{\frac{2\overline{\alpha}-1}{2}}(1 + o(1))$$

For the fixed $r > 0$ we denote

$$P_n(f, r) = P\{-rn^{-1/3} < \overline{f}_n - f(A) < rn^{-1/3}\}.$$

Compare $\overline{f}_n$ with $\widehat{f_n}^*$ by means of calculated $\alpha = \frac{1}{6}$ and $\beta = \frac{1}{12}$ (note that $\overline{\alpha} = 1/6$ for $\overline{f}_n$ is optimal in the mean square sense [2]). It follows from (2.6) that $\widehat{f_n}^*$ is distributed asymptotically normally with the mean $f(A)$ and with the standard deviation $\sqrt{2^{-2}f(A)}n^{-1/3}(1 + o(1))$.

Denote

$$q_n(f, r) = P\{-rn^{-1/3} < \widehat{f_n}^* - f(A) < rn^{-1/3}\}.$$

Then we easily obtain

$$\lim_{n\to\infty} q_n(f,r) = \frac{1}{\sqrt{2\pi}} \int_{-r\left(\sqrt{2^{-2}f(A)}\right)^{-1}}^{r\left(\sqrt{2^{-2}f(A)}\right)^{-1}} \exp\left\{-\frac{t^2}{2}\right\} dt \equiv L(f,r).$$

For $\overline{\alpha} > \frac{1}{6}$ or $\overline{\alpha} < \frac{1}{6}$, $l_1 + l_2 \neq 0$, it can be easily shown that

$$\lim_{n\to\infty} P_n(f,r) = 0.$$

Then for $\overline{\alpha} = \frac{1}{6}$ and $l_1 + l_2 \neq 0$ we have

$$\lim_{n\to\infty} P_n(f,r) = \frac{1}{\sqrt{2\pi}} \int_{-r\left\{\sqrt{2^{-2}f(A)}\right\}^{-1} - \frac{4}{3}(l_1+l_2)\sqrt{2^{-2}f(A)}}^{r\left\{\sqrt{2^{-2}f(A)}\right\}^{-1} - \frac{4}{3}(l_1+l_2)\sqrt{2^{-2}f(A)}} \exp\left\{-\frac{t^2}{2}\right\} dt \leq$$

$$\leq L(f,r).$$

For $\overline{\alpha} = \frac{1}{6}$, $l_1 + l_2 = 0$, $\lim_{n\to\infty} P_n(f,r) = L(f,r)$.

Since the coefficients l_1 and l_2 are unknown and $\overline{f}_{\underset{\frown}{n}}$ fails to estimate them, we can say that $\overline{f}_n$ is in many cases worse than $\hat{f_n}^*$ is.

Remark 2.3. *Cases $W_5, W_6, \cdots$ can be investigated by means of the same method, but calculations will become more and more cumbersome.*

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