A MULTIDIMENSIONAL VERSION OF THE FIRST DARBOUX PROBLEM FOR A SECOND ORDER DEGENERATING HYPERBOLIC EQUATION

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Abstract

A multidimensional version of the first Darboux problem for a second order degenerating hyperbolic equation is considered. Using the a priori estimations method the correct formulation of this problem in the Sobolev weighted space is proved.

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1. *Introduction*

In the space of variables x_1, x_2, t let us consider a second order degenerating hyperbolic equation of the kind

$$Lu \equiv u_{tt} - x_2^m u_{x_1x_1} - u_{x_2x_2} + a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u = F, \qquad (1.1)$$

where $a_i, i = 1, ..., 4, F$ are the given and u is the unknown real functions, $m \in N$ is the positive integer.

Below for equation (1.1) we shall consider a boundary value problem for which data supports are a part of the plane $x_2 = 0$ and a part of characteristic conoid of beams with a vertex at the origin $O(0, 0, 0)$ located in the dihedral angle $x_2 > 0,\ t > 0$. When $m = 0$, i.e., for equation (1.1) with the wave operator in its principal part similar problems have been investigated in [2, 4, 9]. Note that even for $m = 2$ the characteristic conoid of beams with a vertex at the point $O(0, 0, 0)$ of equation (1.1) has geometric structure complicated enough, which in a certain sense makes it difficult to formulate the boundary value problem. Below we consider the case $m = 1$.

2. *Formulation of the Boundary Value Problem.*

When $m = 1$ the characteristic conoid of beams K_O of equation (1.1) composed of bicharacteristic beams, coming out of the origin $O(0,0,0)$, desintegrate on four conical surfaces K_i, $i = 1,...,4$, with a vertex at the point $O(0,0,0)$, each is homeomorphic to the circular cone $t = \sqrt{x_1^2 + x_2^2}$. Two of them K_1, K_2 have the common tangent bicharacteristic beam $x_1 = 0, t + 2x_2 = 0, t \geq 0$, located in the half-space $t - x_2 \geq 0$ and described by the same equation $K_1, K_2 : x_1^2 = \frac{1}{9}(t+2x_2)^2(t-x_2)$ and the two remainder conical surfaces K_3andK_4 have the common tangent bicharacteristic beam $x_1 = 0,\ t-2x_2 = 0,\ t \leq 0$, located in the half-space $t+x_2 \leq 0$ and described by the equation $K_3, K_4 : x_1^2 = -\frac{1}{9}(t-2x_2)^2(t+x_2)$. Note that $K_1(K_2)$ is located in the dihedral angle $t+2x_2 \geq 0, t-x_2 \geq 0$ $(t+2x_2 \leq 0,\ t-x_2 \geq 0)$ and $K_3(K_4)$ is located in the dihedral angle $t+x_2 \leq 0, t-2x_2 \geq 0$ $(t+x_2 \leq 0,\ t-2x_2 \leq 0)$.

Let us denote by $\tilde{S}_1$ the part of the conoid of beams K_0 located in the dihedral angle $x_2 \geq 0,\ t \geq 0$, i.e., $\tilde{S}_1 : x_1^2 = \frac{1}{9}(t+2x_2)^2(t-x_2), x_2 \geq 0,\ t \geq 0$ and denote by $\tilde{S}_2$ the part of the conoid of beams K_p with a vertex at the point $P = (0.0.t_0),\ t_0 > 0$, located in the dihedral angle $x_2 \geq 0, t \leq t_0$, i.e., $\tilde{S}_2 : x_1^2 = \frac{1}{9}(t-t_0-2x_2)^2(t_0-t-x_2), x_2 \geq 0,\ t \leq t_0$. Let D be a domain bounded by the plane $\tilde{S}_0 : x_2 = 0$ and the surfaces $\tilde{S}_1, \tilde{S}_2$ located in the half-space $x_2 > 0$. Let $S_i = \partial D \cap \tilde{S}_i, i = 0,1,2$. It can be verified that $S_i \backslash \{(0,0,0)\} \in C^\infty, i = 1,2$. Below we shall assume that $a_i \in C^2(\bar{D}), i = 1,...,4$, and $m = 1$.

For equation (1.1) we shall consider a multidimensional version of the Goursat problem formulated as follows: in the domain D find a solution $u(x_1,x_2,t)$ of equation (1.1) satisfying the boundary condition

$$u|_{S_0 \cup S_1} = 0 \,. \tag{2.1}$$

In a similar manner we formulate the problem for the equation

$$L^*v \equiv v_{tt} - x_2^m v_{x_1x_1} - v_{x_2x_2} - (a_1v)_{x_1} - (a_2v)_{x_1} - (a_3v)_t + a_4v = F_1 \tag{2.2}$$

in the domain using the boundary condition

$$v|_{S_0 \cup S_2} = 0 \,, \tag{2.3}$$

where L^* is the formal conjugate operator of L.

3. *Some Functional Spaces and Lemmas*

Denote by E and E^* the classes of functions from the space $C^2(\bar{D})$ satisfying the boundary condition (2.1) or (2.3), respectively. Let $W_+(W_+^*)$ be the

Hilbert space with weight obtained by the closure of the space $E(E^*)$ with respect to the norm

$$||u||_1^2 = \int_D [u^2 + x_2^m u_{x_1}^2 + u_{x_2}^2 + u_t^2] dD.$$

Denote by $W_-(W_-^*)$ the space with negative norm constructed with respect to $L_2(D)$ and $W_+(W_+^*)$ [1]. Since the class of functions from the space $E(E^*)$ vanishing in some (own for every function) three-dimensional neighborhood of the segment $I_0 : x_1 = x_2 = 0,\ 0 \le t \le t_0$ of the axis t , is likewise dense in the space $W_+(W_+^*)$ [10], below as $E(E^*)$ we take the class of functions possessing this property.

Impose on the lower coefficient a_1 in equation (1.1) the following restriction

$$M = \sup_{\bar{D}} |x_2^{-\frac{m}{2}} a_1(x_1, x_2, t)| < +\infty. \tag{3.1}$$

Lemma 3.1. *Let condition* (3.1) *be fulfilled. Then for every* $u \in E,\ v \in E^*$ *we have the inequalities*

$$||Lu||_{W_-^*} \le c_1 ||u||_{W_+}, \tag{3.2}$$

$$||L^* v||_{W_-} \le c_2 ||v||_{W_+^*}, \tag{3.3}$$

where the positive constants c_1 *and* c_2 *do not depend on* u *and* v, *respectively,* $||.||_{W_+} = ||.||_{W_-} = ||.||_1$.

Proof. Let $n = (\nu_1, \nu_2, \nu_0)$ be the unit vector of the outer ∂D normal , i.e., $\nu_1 = cos(\widehat{n, x_1}), \nu_2 = cos(\widehat{n, x_2}), \nu_0 = cos(\widehat{n, t})$. Since for the operator L the derivative with respect to the conormal $\partial / \partial N$ is the internal differential operator on the characteristic surfaces of equation (1.1), by virtue of (2.1) and (2.3) we find for the functions $u \in E$ and $v \in E^*$ that

$$\left.\frac{\partial u}{\partial N}\right|_{S_1} = \left.\frac{\partial v}{\partial N}\right|_{S_2} = 0. \tag{3.4}$$

By the definition of a negative norm, for $u \in E$ with regard to equalities (2.1), (2.3) and (3.4) we have

$$||Lu||_{W_-^*} = \sup_{v \in W_+^*} ||v||_{W_+^*}^{-1} (Lu, v)_{L_2(D)} = \sup_{v \in E^*} ||v||_{W_+^*}^{-1} (Lu, v)_{L_2(D)} =$$

$$= \sup_{v \in E^*} ||v||_{W_+^*}^{-1} \int_{S_0 \cup S_1 \cup S_2} \frac{\partial u}{\partial N} v ds + \sup_{v \in E^*} ||v||_{W_+^*}^{-1} \int_D [-u_t v_t + x_2^m u_{x_1} v_{x_1} +$$

$$+u_{x_2}v_{x_2}+a_1u_{x_1}v+a_2u_{x_2}v+a_3u_tv+a_4uv]dD=\sup_{v\in E^*}||v||^{-1}_{W_+^*}\int_D[-u_tv_t+$$

$$+x_2^mu_{x_1}v_{x_1}+u_{x_2}v_{x_2}+a_1u_{x_1}v+a_2u_{x_2}v+a_3u_tv+a_4uv]dD. \qquad (3.5)$$

In view of (3.1) and the known inequalities

$$|\int_D \mu fgdD|\le(\int_D\mu f^2dD)^{\frac{1}{2}}(\int_D\mu g^2dD)^{\frac{1}{2}},\mu=\mu(x_1,x_2,t)\ge 0,$$

$$|\sum_{i=1}^k x_iy_i|\le(\sum_{i=1}^k x_i^2)^{\frac{1}{2}}(\sum_{i=1}^k y_i^2)^{\frac{1}{2}}$$

we obtain

$$|\int_D[-u_tv_t+x_2^mu_{x_1}v_{x_1}+u_{x_2}v_{x_2}]dD|\le|\int_D(u_t^2+x_2^mu_{x_1}^2+u_{x_2}^2)dD|^{\frac{1}{2}}\times$$

$$\times|\int_D(v_t^2+x_2^mv_{x_1}^2+v_{x_2}^2)dD|^{\frac{1}{2}}\le||u||_{W_+}||v||_{W_+^*}, \qquad (3.6)$$

$$|\int_D[a_1u_{x_1}v+a_2u_{x_2}v+a_3u_tv+a_4uv]dD|\le[M(\int_D x_2^mu_{x_1}^2dD)^{\frac{1}{2}}+$$

$$+\sup_D|a_2|||u_{x_2}||_{L_2(D)}+\sup_D|a_3|||u_t||_{L_2(D)}+\sup_D|a_4|||u||_{L_2(D)}]||v||_{L_2(D)}\le$$

$$\le(M+\sum_{i=2}^4\sup_D|a_i|)||u||_{W_+}||v||_{W_+^*}. \qquad (3.7)$$

Inequality (3.2) follows directly from (3.5) – (3.7). Since the inequality (3.3) is proved analogously , lemma 3.1 is thereby completely proved.

Remark 3.1. *By virtue of inequality* (3.2) ((3.3)) *the operator* $L: W_+\to W_-^*(L^*: W_+^*\to W_-)$ *with a dense domain of definition* $E(E^*)$ *admits a closure being a continuous operator from the space* $W_+(W_+^*)$ *to the space* $W_-(W_-^*)$. *Retaining for this operator the previous notation* $L(L^*)$, *we note that it is defined on the whole Hilbert space* $W_+(W_+^*)$.

Lemma 3.2. *Problem* (1.1), (2.1) *and* (2.2), (2.3) *are mutually conjugate , i.e., the equality*

$$(Lu,v)=(u,L^*v) \qquad (3.8)$$

holds for any $u\in W_+$ *and* $v\in W_+^*$.

Proof. By remark 3.1 it is enough to prove equality (3.8) when $u \in E$ u $v \in E^*$. We have

$$(Lu, v) = (Lu, v)_{L_2(D)} = \int\limits_{\partial D} [v\frac{\partial u}{\partial N} - u\frac{\partial v}{\partial N} + (a_1\nu_1 + a_2\nu_2 + a_3\nu_0)uv]ds +$$

$$+(u, L^*v)_{L_2(D)}. \tag{3.9}$$

By virtue of (2.1), (2.3) and (3.4) we readily obtain equality (3.8) from (3.9), which proves lemma 3.2.

4. *A Priori Estimations and Proof of the Main Theorem*

Consider the conditions

$$\omega|_{S_2} \leq 0 \ , \ (\lambda\omega + \omega_t)|_D \leq 0, \tag{4.1}$$

where the second inequality is fulfilled for sufficiently large λ , and $\omega = a_{1x_1} + a_{2x_2} + a_{3t} - a_4$.

Lemma 4.1. *Let conditions* (3.1) *and* (4.1) *be fulfilled. Then for any* $u \in W_+$ *we have the inequality*

$$c||u||_{L_2(D)} \leq ||Lu||_{W_-^*}, \tag{4.2}$$

where the positive constant c *does not depend on* u.

Proof. Let us denote by Ω the orthogonal projection $\bar{D}$ on the plane $O_{x_1x_2}$. Then , it is easily verified that the conic characteristic surface S_1 from (2.1) admits the representation $S_1 : t = g_1(x_1, x_2) \in C^\infty(\Omega\backslash\{(0,0)\})$, where

$$g_1(x_1, x_1) = x_2 + \sqrt{-\frac{3}{2}x_1 + \sqrt{\frac{9}{4}x_1^2 + x_2^3}} + \sqrt{-\frac{3}{2}x_1 - \sqrt{\frac{9}{4}x_1^2 + x_2^3}}.$$

Analogously we have $S_2 : t = g_2(x_1, x_2) \in C^\infty(\Omega\backslash\{(0,0)\})$, where $g_2(x_1, x_2) = t_0 - g_1(x_1, x_2)$.

By remark 3.1 it is enough to show that inequality (4.2) is fulfilled when $u \in E$. If $u \in E$ and thus vanishes in some neighborhood of the segment $I_0 : x_1 = x_2 = 0 \ , \ 0 \leq t \leq t_0$ of the axis t , then one can easily verify that the function

$$v(x_1, x_2, t) = \int\limits_t^{g_2(x_1,x_2)} e^{-\lambda\tau}u(x_1, x_2, \tau)d\tau, \ \lambda = const > 0,$$

belongs to the space E^* and the equalities

$$v_t(x_1,x_2,t)=-e^{-\lambda t}u(x_1,x_2,t),\ u(x_1,x_2,t)=-e^{\lambda t}v_t(x_1,x_2,t) \qquad (4.3)$$

are fulfilled.

In view of (2.1), (2.3), (3.4) and (4.3) we have

$$(Lu,v)_{L_2(D)}=\int\limits_{\partial D}[v\frac{\partial u}{\partial N}+(a_1\nu_1+a_2\nu_2+a_3\nu_0)uv]ds+\int\limits_{\partial D}[-u_tv_t+x_2^m u_{x_1}v_{x_1}+$$

$$+u_{x_2}v_{x_2}-ua_{1x_1}v-ua_1v_{x_1}-ua_{2x_2}v-ua_2v_{x_2}-ua_{3t}v-ua_3v_t+$$

$$+a_4uv]dD=\int\limits_{\partial D}e^{-\lambda t}u_tudD+\int\limits_{\partial D}e^{\lambda t}[-x_2^m v_{x_1t}v_{x_1}-v_{x_2t}v_{x_2}+a_{1x_1}v_tv+$$

$$+a_1v_tv_{x_1}+a_{2x_2}v_tv+a_2v_tv_{x_2}+a_{3t}v_tv+a_3v_t^2-a_4v_tv]dD, \qquad (4.4)$$

$$\int\limits_{\partial D}e^{-\lambda t}u_tudD=\frac{1}{2}\int\limits_{\partial D}e^{-\lambda t}u^2\nu_0ds+\frac{1}{2}\int\limits_{D}e^{-\lambda t}\lambda u^2dD=\frac{1}{2}\int\limits_{S_2}e^{-\lambda t}u^2\nu_0ds+$$

$$+\frac{1}{2}\int\limits_{D}e^{\lambda t}\lambda v_t^2dD=\frac{1}{2}\int\limits_{S_2}e^{\lambda t}v_t^2\nu_0ds+\frac{1}{2}\int\limits_{D}e^{\lambda t}\lambda v_t^2dD, \qquad (4.5)$$

$$\int\limits_{D}e^{\lambda t}[-x_2^m v_{x_1t}v_{x_1}-v_{x_2t}v_{x_2}]dD=-\frac{1}{2}\int\limits_{dD}e^{\lambda t}[x_2^m v_{x_1}^2+v_{x_2}^2]\nu_0ds+$$

$$+\frac{1}{2}\int\limits_{D}e^{\lambda t}\lambda[x_2^m v_{x_1}^2+v_{x_2}^2]dD. \qquad (4.6)$$

Since $v|_{S_2}=0$, the gradient $\nabla v=(v_{x_1},v_{x_2},v_t)$ is proportional to the unit vector of the outer to S_2 normal , i.e., for some α we have $v_{x_1}=\alpha\nu_1,\ v_{x_2}=\alpha\nu_2,\ v_t=\alpha\nu_0$ on S_2. Therefore , recalling that the surface S_2 is characteristic, we obtain

$$(v_t^2-x_2^m v_{x_1}^2-v_{x_2}^2)|_{S_2}=\alpha^2(\nu_0^2-x_2^m\nu_1^2-\nu_2^2)|_{S_2}=0. \qquad (4.7)$$

Let $S_i^{'}=S_i\backslash O$, $i=1,2$. It is easily seen that

$$\nu_0|_{S_0}=0\ ,\ \nu_0|_{S_1^{'}}<0\ ,\ \nu_0|_{S_2^{'}}>0. \qquad (4.8)$$

By virtue of (2.3), (4.7) and (4.8) we have

$$\frac{1}{2}\int\limits_{S_1}e^{\lambda t}v_t^2\nu_0ds-\frac{1}{2}\int\limits_{\partial D}e^{\lambda t}[x_2^m v_{x_1}^2+v_{x_2}^2]\nu_0ds=\frac{1}{2}\int\limits_{S_2}e^{\lambda t}v_t^2\nu_0ds-$$

$$-\frac{1}{2}\int\limits_{S_1} e^{\lambda t}[x_2^m v_{x_1}^2 + v_{x_2}^2]\nu_0 ds - \frac{1}{2}\int\limits_{S_2} e^{\lambda t}[x_2^m v_{x_1}^2 + v_{x_2}^2]\nu_0 ds \geq$$

$$\geq \frac{1}{2}\int\limits_{S_2} e^{\lambda t} v_t^2 \nu_0 ds - \frac{1}{2}\int\limits_{S_2} e^{\lambda t}[x_2^m v_{x_1}^2 + v_{x_2}^2]\nu_0 ds = \frac{1}{2}\int\limits_{S_2} e^{\lambda t}(v_t^2 -$$

$$- x_2^m v_{x_1}^2 - v_{x_2}^2)\nu_0 ds = 0. \qquad (4.9)$$

Taking into account (4.5), (4.6) and (4.9), we obtain from (4.4)

$$(Lu, v)_{L_2(D)} = \frac{1}{2}\int\limits_{S_2} e^{\lambda t} v_t^2 \nu_0 ds + \frac{1}{2}\int\limits_{D} e^{\lambda t}\lambda v_t^2 dD - \frac{1}{2}\int\limits_{\partial D} e^{\lambda t}[x_2^m v_{x_1}^2 + v_{x_2}^2]\nu_0 ds +$$

$$+\frac{1}{2}\int\limits_{D} e^{\lambda t}\lambda[x_2^m v_{x_1}^2 + v_{x_2}^2] dD + \int\limits_{D} e^{\lambda t}[a_1 v_t v_{x_1} + a_2 v_t v_{x_2} + a_3 v_t^2 +$$

$$+(a_{1x_1} + a_{2x_2} + a_{3t} - a_4) v_t v] dD \geq \frac{\lambda}{2}\int\limits_{D} e^{\lambda t}[v_t^2 + x_2^m v_{x_1}^2 + v_{x_2}^2] dD +$$

$$+\int\limits_{D} e^{\lambda t}(a_{1x_1} + a_{2x_2} + a_{3t} - a_4) v_t v dD - \left|\int\limits_{D} e^{\lambda t}[a_1 v_1 v_{x_1} + a_2 v_t v_{x_2} + a_3 v_t^2] dD\right|. \qquad (4.10)$$

By virtue of (3.1) we easily find that

$$\left|\int\limits_{D} e^{\lambda t}[a_1 v_t v_{x_1} + a_2 v_t v_{x_2} + a_3 v_t^2] dD\right| \leq \int\limits_{D} e^{\lambda t}[M\frac{1}{2}(x_2^m v_{x_1}^2 + v_t^2) +$$

$$+\frac{\gamma}{2}(v_{x_2}^2 + v_t^2) + \gamma v_t^2] dD \leq (\frac{1}{2}M + \frac{3}{2}\gamma)\int\limits_{D} e^{\lambda t}[v_t^2 + x_2^m v_{x_1}^2 + v_{x_2}^2] dD, \qquad (4.11)$$

where $\gamma = \max(\sup\limits_{D}|a_2|, \sup\limits_{D}|a_3|)$.

In view of (2.3), (4.1) and (4.8) and integrating them by parts, we obtain

$$\int\limits_{D} e^{\lambda t}(a_{1x_1} + a_{2x_2} + a_{3t} - a_4) v_t v dD = \frac{1}{2}\int\limits_{\partial D} e^{\lambda t}(a_{1x_1} + a_{2x_2} + a_{3t} - a_4) v^2 \nu_0 ds -$$

$$-\frac{1}{2}\int\limits_{D} e^{\lambda t}[\lambda(a_{1x_1} + a_{2x_2} + a_{3t} - a_4) + (a_{1x_1} + a_{2x_2} + a_{3t} - a_4)_t] v^2 dD \geq 0, \qquad (4.12)$$

where λ is a sufficiently large positive number.

Now by virtue of (4.11) and (4.12), we obtain from (4.10)

$$(Lu,v)_{L_2(D)} \geq \frac{1}{2}(\lambda - M - 3\gamma)\int\limits_D e^{\lambda t}[v_t^2 + x_2^m v_{x_1}^2 + v_{x_2}^2]dD \geq$$

$$\geq \mu[\int\limits_D e^{\lambda t} v_t^2 dD]^{\frac{1}{2}}[\int\limits_D e^{\lambda t}(v_t^2 + x_2^m v_{x_1}^2 + v_{x_2}^2)dD]^{\frac{1}{2}} =$$

$$= \mu[\int\limits_D e^{-\lambda t} u^2 dD]^{\frac{1}{2}}[\int\limits_D e^{\lambda t}(v_t^2 + x_2^m v_{x_1}^2 + v_{x_2}^2)dD]^{\frac{1}{2}} \geq$$

$$\geq \mu e^{-\frac{1}{2}\lambda t_0}[\int\limits_D e^{\lambda t}(v_t^2 + x_2^m v_{x_1}^2 + v_{x_2}^2)dD]^{\frac{1}{2}}, \quad (4.13)$$

where $\mu = \frac{1}{2}(\lambda - M - 3\gamma) > 0$ for sufficiently large λ, and

$e^{-\frac{1}{2}\lambda t_0} = (\inf\limits_D e^{-\lambda t})^{\frac{1}{2}} > 0$ by the structure of the domain$D$.

Since $v|_{S_2} = 0$ $(u|_{S_1} = 0)$, using the standard arguments we can easily prove the validity of the inequality

$$\int\limits_D v^2 dD \leq c_0 \int\limits_D v_t^2 dD \quad (\int\limits_D u^2 dD \leq c_0 \int\limits_D u_t^2 dD)$$

for which $c_0 = const > 0$ does not depend on $v \in E^*(u \in E)$. Thus we conclude that , in the space $W_+(W_+^*)$, the norm

$$||u||^2_{W_+(W_+^*)} = \int\limits_D (u^2 + x_2^m u_{x_1}^2 + u_{x_2}^2 + u_t^2)dD$$

is equivalent to the norm

$$||u||^2 = \int\limits_D (u_t^2 + x_2^m u_{x_1}^2 + u_{x_2}^2)dD. \quad (4.14)$$

Therefore, retaining for norm (4.14) the previous designation $||u||_{W_+(W_+^*)}$ from (4.13) we have

$$(Lu,v)_{L_2(D)} \geq \mu e^{-\frac{1}{2}\lambda t_0}||u||_{L_2(D)}||v||_{W_+^*}. \quad (4.15)$$

If now we apply the generalized Schwarz inequality

$$(Lu,v) \leq ||Lu||_{W_-^*}||v||_{W_+^*}$$

to the left-hand side (4.15), then after reducing by $||v||_{W_+^*}$ we get inequality (4.2) in which $c = \sigma e^{-\frac{1}{2}\lambda t_0} = const > 0$. Lemma 4.1 is thereby completely proved.

Consider the conditions

$$a_4|_{S_2} \geq 0 \ , \ (\lambda a_4 - a_{4t})|_D \geq 0, \tag{4.16}$$

of which the second one takes place for sufficiently large λ.

Lemma 4.2. *Let conditions* (3.1) *and* (4.16) *be fulfilled . Then for any* $v \in W_+^*$ *the inequality*

$$c||v||_{L_2(D)} \leq ||L^* v||_{W_-} \tag{4.17}$$

is valid for some $c = const > 0$ *independent of* $v \in W_+^*$.

Proof. Just as in lemma 4.1 and because of remark 3.1 , it suffices to prove the validity of inequality (4.17) for $v \in E^*$. Let $v \in E^*$ and let us take into consideration the function

$$u(x_1, x_2, t) = \int\limits_{g_1(x_1,x_2)}^{t} e^{\lambda\tau} v(x_1, x_2, \tau) d\tau, \ \lambda = const > 0,$$

where $t = g_1(x_1, x_2)$ is the equation of the characteristic surface S_1. The function $u(x_1, x_2, t)$ belongs to the space E and the equalities

$$u_t(x_1, x_2, t) = e^{\lambda t} v(x_1, x_2, t), \ v(x_1, x_2, t) = e^{-\lambda t} u_t(x_1, x_2, t) \tag{4.18}$$

hold.

By virtue of (2.1), (2.3), (3.4) and (4.18), analogously to (4.4) – (4.9) we have

$$(L^* v, u)_{L_2(D)} = -\int\limits_D e^{\lambda t} v_t v dD + \int\limits_D e^{-\lambda t} [x_2^m u_{x_1 t} u_{x_1} + u_{x_2 t} u_{x_2}] dD +$$

$$+ \int\limits_D e^{-\lambda t} [a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t + a_4 u] u_t dD, \tag{4.19}$$

$$-\int\limits_D e^{\lambda t} v_t v dD = -\frac{1}{2} \int\limits_{\partial D} e^{\lambda t} v^2 \nu_0 ds + \frac{1}{2} \int\limits_D e^{\lambda t} \lambda v^2 dD =$$

$$= -\frac{1}{2} \int\limits_{S_1} e^{-\lambda t} v^2 \nu_0 ds + \frac{1}{2} \int\limits_D e^{-\lambda t} \lambda u_t^2 dD =$$

$$= -\frac{1}{2}\int\limits_{S_1} e^{-\lambda t} u_t^2 \nu_0 ds + \frac{1}{2}\int\limits_D e^{-\lambda t}\lambda u_t^2 dD \, , \tag{4.20}$$

$$\int\limits_D e^{-\lambda t}[x_2^m u_{x_1 t} u_{x_1} + u_{x_2 t} u_{x_2}] dD = \frac{1}{2}\int\limits_D e^{-\lambda t}[x_2^m u_{x_1}^2 + u_{x_2}^2]\nu_0 ds +$$

$$+\frac{1}{2}\int\limits_D e^{-\lambda t}\lambda[x_2^m u_{x_1}^2 + u_{x_2}^2] dD, \tag{4.21}$$

$$(u_t^2 - x_2^m u_{x_1}^2 - u_{x_2}^2|_{S_1} = 0 \, , \tag{4.22}$$

$$-\frac{1}{2}\int\limits_{S_1} e^{-\lambda t} u_t^2 \nu_0 ds + \frac{1}{2}\int\limits_D e^{-\lambda t}[x_2^m u_{x_1}^2 + u_{x_2}^2]\nu_0 ds = -\frac{1}{2}\int\limits_{S_1} e^{-\lambda t} u_t^2 \nu_0 ds +$$

$$+\frac{1}{2}\int\limits_{S_1} e^{-\lambda t}[x_2^m u_{x_1}^2 + u_{x_2}^2]\nu_0 ds + \frac{1}{2}\int\limits_{S_2} e^{-\lambda t}[x_2^m u_{x_1}^2 + u_{x_2}^2]\nu_0 ds \geq$$

$$-\frac{1}{2}\int\limits_{S_1} e^{-\lambda t}[u_t^2 - x_2^m u_{x_1}^2 - u_{x_2}^2]\nu_0 ds = 0. \tag{4.23}$$

In view of (4.20) – (4.23) from (4.19) we find that

$$(L^* v, u)_{L_2(D)} \geq \frac{\lambda}{2}\int\limits_D e^{-\lambda t}[u_t^2 + x_2^m u_{x_1}^2 + u_{x_2}^2] dD + \int\limits_D e^{-\lambda t} a_4 u u_t dD -$$

$$-|\int\limits_D e^{-\lambda t}[a_1 u_{x_1} + a_2 u_{x_2} + a_3 u_t] u_t dD|,$$

whence as in obtained inequality (4.2) from (4.10) in lemma 4.1, we arrive at inequality (4.17).

Definition. For $F \in L_2(D)(W_-^*)$ the function u will be called a strongly generalized solution of problem (1.1), (2.1) from the class $W_+(L_2)$ provided that $u \in W_+(L_2(D))$ and there exists a sequence of functions $u_n \in E_0$ such that $u_n \to u$ in the space $W_+(L_2(D))$ and $Lu_n \to F$ in the space $W_-^*(W_-^*)$, i.e.

$$\lim_{n\to\infty} ||u_n - u||_{W_+} = \lim_{n\to\infty} ||Lu_n - F||_{W_-^*} = 0.$$

By the results of $[1, 8, 10]$ Lemmas $1 - 4$ give rise to the following theorem.

Theorem. *Let conditions* (3.1), (4.1) *and* (4.16) *be fulfilled. Then for any* $F \in L_2(D)(W_-^*)$ *there exists a unique strong generalized solution* u *of problem* (1.1), (2.1) *of the class* $W_+(L_2)$ *for which the estimate*

$$||u||_{L_2(D)} \leq c||F||_{W_-^*}$$

with the positive constant c independent of F holds.

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