ON A METHOD FOR COMPUTING THE EIGENVALUES OF SECOND KIND FREDHOLM OPERATORS

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Abstract

We derive a general method for computing eigenvalues of second kind Fredholm operators, which is an analog of the inverse iteration method for finite dimensional operators.

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1 *Introduction*

Let $k(x,y) \in L^2(Q), Q := I \times I$(e.g.$I = [0,1]$), $f(x) \in L^2(I)$, and consider the second kind homogeneous Fredholm integral equation

$$\varphi(x) - \lambda \int_I k(x,y)\,\varphi(y)\,dy = 0, \tag{1.1}$$

where $\lambda \in \mathbb{C}$ is a complex parameter.

By introducing the operator $\mathcal{K} : L^2(I) \to L^2(I)$, defined by

$$(\mathcal{K}\varphi)(x) := \int_I k(x,y)\,\varphi(y)\,dy$$

and the identity operator $\mathcal{I}$, eq. (1.1) becomes

$$(\mathcal{I} - \lambda\mathcal{K})\,\varphi = 0. \tag{1.1}'$$

By Fredholm theorems, it is well known that eq. (1.1) admits at most a denumerable set of non vanishing characteristic values which does not accumulate to finite points.

Writing (1.1)' in the form

$$(\mathcal{K} - \mu\mathcal{I})\,\varphi = 0, \qquad \left(\mu = \lambda^{-1}\right), \tag{1.2}$$

i.e.

$$\mathcal{K}\varphi = \mu\varphi, \tag{1.3}$$

the eigenvalues of the operator $\mathcal{K}$ can be ordered with respect to their modulus in a decreasing sequence:

$$0 < ... \leq |\mu_3| \leq |\mu_2| \leq |\mu_1| .$$

In the particular case of symmetric or hermitian positive operators $k(x,y) = k(y,x)$ or $k(x,y) = \overline{k(y,x)}$, $(\mathcal{K}\varphi, \varphi) > 0$, if $\varphi \neq 0 \in L^2(I)$, in the last formula the modulus signs can be avoided, since the eigenvalues are real and positive:

$$0 < ... \leq \mu_3 \leq \mu_2 \leq \mu_1. \tag{1.4}$$

We will limit ourselves to this last case, since this is the most important case in physical applications.

In this case it is well known that by using the Rayleigh-Ritz method (cfr. e.g. G. FICHERA [4]) , for every we can find lower bounds approximations of the eigenvalues (1.4). By using the orthogonal invariants method, due to G. FICHERA [3]-[4], corresponding upper bounds for the same eigenvalues can always be computed theoretically. These two methods can be used in a more general framework, namely they can be applied to every positive definite compact operator.

Anyway, the orthogonal invariants method is sometimes computationally very expensive, so that we are interested to present here a more simple, iterative method, in order to approximate the eigenvalues when the Rayleigh-Ritz approximation can be computed.

In our opinion this method could be applied to more general situations, but for the sake of simplicity, we will limit ourselves to the above mentioned case of a second kind Fredholm operator.

In section 2 we will give, for completeness, a simple sketch of the results of Rayleigh-Ritz and Fichera theory. In section 3 we will present an inverse iteration method.

Some numerical experiments and graphs will be shown in the Applications at the end of the paper.

2 *The Rayleigh-Ritz and orthogonal invariants methods*

Let $\{v_k\}_{k\in\mathbb{N}}$ be a complete system of linearly independent vectors in a Hilbert space $\mathcal{H}$, put $\mathcal{V}_\nu = span\{v_1, v_2, ..., v_\nu\}$, denote by P_ν the orthogonal projector, $P_\nu : \mathcal{H} \to \mathcal{V}_\nu$, and consider a positive definite hermitian operator

$T : \mathcal{H} \to \mathcal{H}$ and the corresponding eigenvalue problem

$$T\varphi = \mu\varphi. \tag{2.1}$$

Proposition 1 *(Rayleigh-Ritz). - Consider the positive eigenvalues*

$$\mu_1^{(\nu)} \geq \mu_2^{(\nu)} \geq ... \geq \mu_\nu^{(\nu)}, \tag{2.2}$$

of the operator $P_\nu T P_\nu$. Then

i) the positive eigenvalues (2.2) of $P_\nu T P_\nu$ are obtained by solving the equation:

$$\det\left\{(Tv_j, v_h) - \mu\,(v_j, v_h)\right\} = 0, \qquad (j, h = 1, ...\nu)$$

($\mu = 0$ is always an eigenvalue of $P_\nu T P_\nu$).

ii) For any fixed k and for any $\nu \geq k$ the following inequality holds true

$$\mu_k^{(\nu)} \leq \mu_k^{(\nu+1)} \leq \mu_k.$$

iii) Furthermore, the limit condition is valid

$$\lim_{\nu\to\infty} \mu_k^{(\nu)} = \mu_k.$$

i.e. the Rayleigh-Ritz method always gives lower bounds for the first ν eigenvalues of the operator T.

The method of the orthogonal invariants have been introduced by G. FICHERA [3]-[4], in order to provide upper bounds for the same eigenvalues.

A complete orthogonal invariants system is a complete system of numbers which is invariant under the unitary equivalence for operators. Such a system must depend only on the eigenvalues of the operator. Theoretically we could consider the system

$$\mathcal{I}_s^n(T) = \sum_{k_1<k_2<...<k_s} \left[\mu_{k_1}\mu_{k_2}...\mu_{k_s}\right]^n,$$

for any fixed s (order of the invariant) and $n = 1, 2, 3, ...$ (degree of the invariant), provided that all these numbers can be computed independently by the knowledge of the eigenvalues of T.

Let

$$v_1^{(\nu)}, v_2^{(\nu)}, ..., v_\nu^{(\nu)},$$

be normalized eigenvectors of $P_\nu T P_\nu$ corresponding to the eigenvalues (2.2) and denote by

$$\mathcal{V}_\nu^{(k)} := span\left\{v_1^{(\nu)}, ..., v_{k-1}^{(\nu)}, v_{k+1}^{(\nu)}, ..., v_\nu^{(\nu)}\right\}$$

and by $P_\nu^{(k)}$ the orthogonal projector $P_\nu^{(k)} : \mathcal{H} \to \mathcal{V}_\nu^{(k)}$.

Proposition 2 *(Fichera) - If $\mathcal{I}_s^n(T) < \infty$, for any fixed $n, s \in N$ and $\forall k : k \leq \nu$, put*

$$\sigma_k^{(\nu)} := \left[\frac{\mathcal{I}_s^n(T) - \mathcal{I}_s^n(P_\nu T P_\nu)}{\mathcal{I}_{s-1}^n\left(P_\nu^{(k)} T P_\nu^{(k)}\right)} + \left[\mu_k^{(\nu)}\right]^n \right]^{\frac{1}{n}}.$$

Then:

$$\sigma_k^{(\nu)} \geq \sigma_k^{(\nu+1)} \geq \mu_k$$

and the limit condition

$$\lim_{\nu \to \infty} \sigma_k^{(\nu)} = \mu_k$$

holds true.

In the particular case we have considered in section 1, namely, the case of a Hilbert space $\mathcal{H} = L^2(I)$,

$$\forall n, s \in N, \qquad \mathcal{I}_s^n(T) < \infty \qquad iff \qquad T^n \varphi = \int_I k(x,y)\,\varphi(y)\,dy,$$

where $k(x,y) = \int_I h(x,z)\,h(z,y)\,dz, \qquad h(x,y) = \overline{h(y,x)} \in L^2[Q]$.

Then the orthogonal invariants can be expressed (see [3]) by the multiple integral

$$\mathcal{I}_s^n(T) = \frac{1}{s!} \int_I ... \int_I f(x_1, ..., x_s)\,dx_1 ... dx_s,$$

where $f(x_1, ..., x_s)$ denotes the Fredholm determinant

$$f(x_1, x_2, ..., x_s) := \begin{vmatrix} k(x_1,x_1) & k(x_1,x_2) & & k(x_1,x_s) \\ k(x_2,x_1) & k(x_2,x_2) & & k(x_2,x_s) \\ ... & ... & ... & ... \\ k(x_s,x_1) & k(x_s,x_2) & & k(x_s,x_s) \end{vmatrix}.$$

In particular, for $s = 1$:

$$\mathcal{I}_1^n(T) = \int_I k(x,x)\,dx = \int\int_Q |h(x,y)|^2\,dxdy,$$

and, for $s = 2$:

$$\mathcal{I}_2^n(T) = \frac{1}{2} \int\int_Q \left[k(x,x)\,k(y,y) - |k(x,y)|^2\right] dxdy.$$

Remark *The above mentioned methods provide theoretically precise tools for approximating the eigenvalues of a positive definite compact operator in a Hilbert space $L^2(I)$, since they permit to control the error of the*

obtained approximation of the eigenvalue μ_k by evaluating the difference $\sigma_k^{(\nu)} - \mu_k^{(\nu)}$.

However, the use of the orthogonal invariants method leads sometimes to very cumbersome computations. In this case, we suggest the use of the iterative method described in section 3.

3 *The inverse iteration method*

This method will be called the "inverse iteration method" since it reduces to this classical method (Wielandt's method) in the case of a finite dimensional operator (see e.g. [6]-[7]).

Suppose we know an initial approximation $\widetilde{\mu}$ of the sought for eigenvalue μ_j, such that

$$|\widetilde{\mu} - \mu_j| < \frac{1}{2} \min_{\substack{\mu_k \neq \mu_j \\ k=1,2,...\nu}} |\mu_k - \mu_j|$$

for a suitable choice of the integer ν. In practice in this condition the eigenvalues will be replaced by their Rayleigh-Ritz approximations, for sufficiently large ν:

$$\left|\widetilde{\mu} - \mu_j^{(\nu)}\right| < \frac{1}{2} \min_{\substack{\mu_k \neq \mu_j \\ k=1,2,...\nu}} \left|\mu_k^{(\nu)} - \mu_j^{(\nu)}\right|. \tag{3.1}$$

From (1.3) we get:

$$(\mathcal{K} - \widetilde{\mu}\mathcal{I})\,\varphi = (\mu - \widetilde{\mu})\,\varphi. \tag{3.2}$$

Consequently, if μ_j is an eigenvalue of $\mathcal{K}$ with eigenfunction φ_j, then $(\mu_j - \widetilde{\mu})$ is an eigenvalue of $\mathcal{K} - \widetilde{\mu}\mathcal{I}$ with eigenfunction φ_j. By writing (3.2) in the form

$$(\mathcal{K} - \widetilde{\mu}\mathcal{I})^{-1}\,\varphi = (\mu - \widetilde{\mu})^{-1}\,\varphi. \tag{3.3}$$

it follows that $(\mu_j - \widetilde{\mu})^{-1}$ is an eigenvalue of $(\mathcal{K} - \widetilde{\mu}\mathcal{I})^{-1}$ with the same eigenfunction φ_j.

By using condition (3.1), for ν sufficiently large, the eigenvalue $(\mu_j - \widetilde{\mu})^{-1}$ becomes the (unique) eigenvalue of maximum modulus for the operator $(\mathcal{K} - \widetilde{\mu}\mathcal{I})^{-1}$. This leads to the possibility to apply the Kellog method (see [5]) in order to approximate $(\mu_j - \widetilde{\mu})^{-1}$, and a corres- ponding eigenfunction. This can be done in the usual way, starting from an arbitrary function ω_0 (which theoretically should not be orthogonal to the eigenspace associated with $(\mu_j - \widetilde{\mu})^{-1}$), and defining the sequence

$$\omega_{n+1} := (\mathcal{K} - \widetilde{\mu}\mathcal{I})^{-1}\,\omega_n, \qquad (n = 0, 1, 2, ...)\,.$$

Then (see [5]):

$$\lim_{n\to\infty}\frac{\|\omega_{n+1}\|_2}{\|\omega_n\|_2}=(\mu_j-\widetilde{\mu})^{-1},$$

$$\lim_{n\to\infty}\frac{\omega_{2n}}{\|\omega_n\|_2}=\pm\varphi_j.$$

More precisely, the rate of convergence of the method is given by the formula:

$$\frac{\|\omega_n\|_2}{\|\omega_0\|_2}=\mathcal{O}\left(\left(\mu'\left(\mu_j-\widetilde{\mu}\right)\right)^n\right), \tag{3.4}$$

where $\mu'\neq(\mu_j-\widetilde{\mu})^{-1}$ denotes a suitable eigenvalue of $(\mathcal{K}-\widetilde{\mu}\mathcal{I})^{-1}$ (see [7]).

After computing with all possible accuracy the eigenvalue

$$\xi_j:=(\mu_j-\widetilde{\mu})^{-1},$$

one finds

$$\mu_j=\frac{1}{\xi_j}+\widetilde{\mu},$$

so that, by recalling $\mu=\lambda^{-1}\quad\left(\widetilde{\mu}=:\widetilde{\lambda}^{-1}\right)$, we obtain for the characteristic values of the kernel the expressions

$$\lambda_j=\frac{\widetilde{\lambda}\xi_j}{\widetilde{\lambda}+\xi_j}.$$

It is important to note that (as in the finite dimensional case) we can avoid the determination of the inverse operator $(\mathcal{K}-\widetilde{\mu}\mathcal{I})^{-1}$, since the equation

$$\omega_{n+1}=(\mathcal{K}-\widetilde{\mu}\mathcal{I})^{-1}\omega_n$$

is equivalent to

$$(\mathcal{K}-\widetilde{\mu}\mathcal{I})^{-1}\omega_{n+1}=\omega_n.$$

However, this leads to the necessity to solve numerically, at each step, a Fredholm integral equation of the first kind.

This can be done by using different methods (see [1]-[2]), namely we could use, e.g., the method of iterations, the Fast Galerkin method, or the Nyström method.

The numerical experiments performed by using a Turbo C++ program written by the first Author of this article show that the Nyström method is more efficient both with respect to time and number of iterations. In applying the Nyström method we have used usually 20 or at most 40 nodes.

The numerical results applied for computing the eigenvalues of some well known kernels have been implemented by using the computer algebra

system MATHEMATICA in order to visualize the logarithmically scaled behavior of the relative error (see Applications).

All the graphs showing the logarithmically scaled relative error versus the number of iterations exhibit a more or less pronounced linear behavior, confirming the theoretical rate of convergence of the method expressed by formula (3.4).

Applications

The following examples show the obtained results using the method considered above. We consider $k(x, y)$ to be the Kernel of the integral equation and ln are the exact (known) solutions of the eigenvalue problem referred to $k(x, y)$.

Example 1

$$k\,(x,y) = \begin{cases} \sin(x)\cos(y) & x \le y \\ \sin(y)\cos(x) & y < x \end{cases} \qquad a = 0, \quad b = \pi/2; \quad \lambda_n = 4n^2 - 1.$$

Fig. 1 : Kernel

Fig. 2 : Log scale error to approximate the first 10 eigenvalues (with 2 iterations).

Fig. 3 : Solution behavior and log scale error to approximate $\lambda_1 = 3$.

Example 2

$$k(x,y) = \begin{cases} \frac{1}{2}\ln\frac{1+x}{1-y} & x \le y \\ \frac{1}{2}\ln\frac{1+y}{1-x} & y < x \end{cases} \qquad a = -1, \quad b = 1; \quad \lambda_n = n(n+1).$$

Fig. 1 : Kernel

Fig. 2 : Log scale error to approximate the first 10 eigenvalues (with 2 iterations).

Fig. 3 : Solution behavior and log scale error to approximate $\lambda_3 = 12$.

Example 3

$$k(x,y) = x + y \qquad a = 0, \quad b = 1; \quad \lambda_{1,2} = \frac{1}{2} \pm \frac{1}{\sqrt{3}}$$

exact largest module eigenvalue	*error after 0 iterations*	*error after 1 iterations*	*error after 2 iterations*	*error after 5 iterations*
1.077350269	0.9×10^{-5}	0.7×10^{-8}	0.9×10^{-9}	0.4×10^{-14}
-0.077350269	0.2×10^{-4}	0.3×10^{-6}	0.9×10^{-8}	0.5×10^{-14}

Example 4

$$k(x,y) = e^{xy} \qquad a = 0, \quad b = \pi.$$

Estimate for the largest module eigenvalue $1,3527 < l < 1,3534$ (The exact value is unknown).

Fig. 1 : Solution behavior with 2 iterations to approximate λ (with 20 and 40 nodes in the Nyström method).
The obtained result is $\lambda = 1,35315478$.

Example 5

$$k(x,y) = \begin{cases} \frac{1}{2\nu}\left(\frac{x}{y}\right)^{\nu} & x \le y \\ \frac{1}{2\nu}\left(\frac{y}{x}\right)^{\nu} & y < x \end{cases} \qquad a = 0, \quad b = \infty; \quad \lambda_n = \alpha_n^2,$$

where the α_n are the zeros of the Bessel function $J_n(x)$

Fig. 1 : Kernel

Fig. 2 : Log scale error to approximate the first 20 eigenvalues (with 2 iterations).

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